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UNIVERSALLY OPTIMAL MATRICES AND FIELD INDEPENDENCE OF THE MINIMUM RANK OF A GRAPH*

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Abstract. The minimum rank of a simple graph G over a field F is the smallest possible rank among all symmetric matrices over F whose (i,j)th entry (for $i \neq j$) is nonzero whenever $\{i,j\}$ is an edge in G and is zero otherwise. A universally optimal matrix is defined to be an integer matrix A such that every off-diagonal entry of A is 0, 1, or -1, and for all fields F, the rank of A is the minimum rank over F of its graph. Universally optimal matrices are used to establish field independence of minimum rank for numerous graphs. Examples are also provided verifying lack of field independence for other graphs.

Key words. Minimum rank, Universally optimal matrix, Field independent, Symmetric matrix, Rank, Graph, Matrix.

AMS subject classifications. 05C50, 15A03.

1. Introduction. The *minimum rank problem* is, for a given graph G and field F, to determine the smallest possible rank among symmetric matrices over F whose off-diagonal pattern of zero-nonzero entries is described by G. Most work on minimum rank has been on the real minimum rank problem. See [10] for a survey of known results and discussion of the motivation for the minimum rank problem; an extensive bibliography is also provided there. Catalogs of minimum rank and other parameters for families of graphs [15] and small graphs [16] were developed at the AIM workshop "Spectra of families of matrices described by graphs, digraphs, and sign patterns" [2] and are available on-line; these catalogs are updated routinely. The study of minimum rank over fields other than the real numbers was initiated in [5].

The minimum rank of a graph G is *field independent* if the minimum rank of G is the same for all fields. In this paper, we establish the field independence or dependence of minimum rank for most of the families of graphs listed in the AIM on-line minimum

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rank graph catalog and establish the minimum rank of several additional families. For almost every graph discussed that has field independent minimum rank, we exhibit a single integer matrix that over every field has the given graph and has rank in that field equal to the minimum rank over the field (what we call a universally optimal matrix, see Section 2). Note that an integer matrix can be viewed as a matrix over $\mathbb{Q}$ or $\mathbb{Z}_p$, where p is a prime. The results are summarized in Table 1.1. The result number(s) in the first column refer the reader to location(s) within this paper that justify field independence and existence of a universally optimal matrix or lack thereof (a "no" in the field independence column or universally optimal matrix column means that not every member of the family has the property). Note that the assertion that a given graph does not have a universally optimal matrix can be justified by Observation 2.6 and a result showing minimum rank is higher over a specific finite field. The stated minimum rank can be found in either the numbered result (with justification or a reference) or in [1] or [10].

A *graph* is a pair $G = (V_G, E_G)$, where V_G is the (finite, nonempty) set of vertices (usually $\{1, \ldots, n\}$ or a subset thereof) and E_G is the set of edges (an edge is a two-element subset of vertices); what we call a graph is sometimes called a simple undirected graph. Throughout this paper, G will denote a graph. The *order* of a graph G, denoted $|G|$, is the number of vertices of G.

The set of $n \times n$ symmetric matrices over F will be denoted by S_n^F. For $A \in S_n^F$, the *graph* of A, denoted $\mathcal{G}^F(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : \ a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}^F(A)$. The superscript F is used because the graph of an integer matrix may vary depending on the field in which the matrix is viewed. The *minimum rank over field* F of a graph G is

$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) : \ A \in S_n^F, \ \mathcal{G}^F(A) = G\},$$

and the *maximum nullity over* F of a graph G is defined to be

$$M^F(G) = \max\{\mathrm{null}(A) : \ A \in S_n^F, \ \mathcal{G}^F(A) = G\}.$$

In the case $F = \mathbb{R}$, the superscript $\mathbb{R}$ may be omitted, so we write $\mathrm{mr}(G)$ for $\mathrm{mr}^{\mathbb{R}}(G)$, etc. Clearly,

$$\mathrm{mr}^F(G) + M^F(G) = |G|.$$

The adjacency matrix of G, $\mathcal{A}(G) = [a_{ij}]$, is a $(0,1)$-matrix such that $a_{ij} = 1$ if and only if $\{i, j\} \in E_G$. The *complement* of G is the graph $\overline{G} = (V_G, \overline{E_G})$, where $\overline{E_G}$ consists of all two-element sets from V_G that are not in E_G.

The subgraph $G[R]$ of G *induced* by $R \subseteq V_G$ is the subgraph with vertex set R and edge set $\{\{i, j\} \in E_G \mid i, j \in R\}$. The subgraph induced by $\overline{R}$ is also denoted by

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ELA

TABLE 1.1
Summary of field dependence of minimum rank for families of graphs.

result #	G	$\|G\|$	$\mathrm{mr}(G)$	universally optimal matrix	field independent
1.1, 2.4	P_n	n	$n-1$	yes	yes
1.1, 2.5	C_n	n	$n-2$	yes	yes
1.1, 2.4	K_n	n	1	yes	yes
1.1, 2.4	$K_{p,q}$	$p+q$	2	yes	yes
3.10	$K_{n_1,\ldots,n_r}, r \geq 3$	$\sum_{i=1}^{r} n_i$	≤ 3	no	no
3.14	Q_n (hypercube)	2^n	2^{n-1}	?	?
2.11	T_n (supertriangle)	$\frac{1}{2}n(n+1)$	$\frac{1}{2}n(n-1)$	yes	yes
3.1	W_n (wheel)	n	$n-3$	no	no
2.12	N_s (necklace)	$4s$	$3s-2$	yes	yes
2.24	$P_{m,k}$ (pineapple), $m \geq 3, k \geq 2$	$m+k$	3	yes	yes
3.2	M_s (Möbius ladder)	$2s$	$2s-4$	no	no
3.3, 3.4	H_s (half-graph)	$2s$	s	no	no
1.1, 2.4	T (tree)			yes	yes
2.21	unicyclic			yes	yes
1.2, 2.14	polygonal path	n	$n-2$	yes	yes
2.11	claw-free block-clique (i.e., line graph of tree)		# of blocks	yes	yes
3.5	$\overline{C_n}, n \geq 5$	n	3	no	no
3.6	$\overline{T}$ (T a tree)		≤ 3	no	no
3.7	complement of 2-tree		≤ 4	no	no
3.8	$L(K_n)$	$\frac{1}{2}n(n-1)$	$n-2$	no	no
2.17	$P_s \square P_s$	s^2	s^2-s	yes	yes
	$P_s \square P_t,\ s > t \geq 3$	st	$st-t$	?	?
3.5	$K_s \square P_t$	st	$st-s$	no	no
3.5	$C_s \square P_t$	st	$st-\min\{s,2t\}$	no	no
2.16	$K_s \square K_s$	s^2	$2s-2$	yes	yes
3.5	$K_s \square K_t, s > t$	st	$s+t-2$	no	no
3.9	$C_s \square K_t, s \geq 4$	st	$st-2t$	no	no
2.18	$C_s \square C_s$	s^2	$s^2-(s+2\lfloor\frac{s}{2}\rfloor)$	yes	yes
3.9	$C_s \square C_t,\ s > t$	st		no	no
3.12	$P_s \boxtimes P_t$	st	$(s-1)(t-1)$	no	no
2.22	$K_t \circ K_s$	$st+t$	$t+1$	yes	yes
2.22	$C_t \circ K_1,\ t \geq 4$	$2t$	$2t-\lfloor\frac{t}{2}\rfloor$	yes	yes
2.22, 2.23	$C_t \circ K_s,\ s \geq 2$	$st+t$	$2t-2$	yes	yes

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$G - R$, or in the case R is a single vertex v, by $G - v$. If A is an $n \times n$ matrix and $R \subseteq \{1, \ldots, n\}$, the *principal submatrix* $A[R]$ is the matrix consisting of the entries in the rows and columns indexed by R. If $A \in S_n^F$ and $\mathcal{G}^F(A) = G$, then by a slight abuse of notation $\mathcal{G}^F(A[R])$ can be identified with $G[R]$.

A subgraph G' of a graph G is a *clique* if G' has an edge between every pair of vertices of G'. A set of subgraphs of G, each of which is a clique and such that every edge of G is contained in at least one of these cliques, is called a *clique covering* of G. The *clique covering number* of G, denoted by $\mathrm{cc}(G)$, is the smallest number of cliques in a clique covering of G.

A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected. More generally, v is a cut-vertex of a graph G if v is a cut-vertex of a connected component of G. A graph is *nonseparable* if it is connected and has no cut-vertices. A *block* of a graph is a maximal nonseparable induced subgraph. A *block-clique graph* is a graph in which every block is a clique (this type of graph is also called *1-chordal*). A *block-cycle-clique graph* is a graph in which every block is either a cycle or a clique (this type of graph is also called a *block-graph*). A graph is *claw-free* if it does not contain an induced $K_{1,3}$.

A 2-*tree* is a graph built from K_3 by adding to it one vertex at a time adjacent to exactly a pair of existing adjacent vertices. A *polygonal path* is a "path" of cycles built from cycles $C_{m_1}, \ldots, C_{m_k}$ constructed so that for $i = 2, \ldots, k$ and $j < i - 1$, $C_{m_{i-1}} \cap C_{m_i}$ has exactly one edge and $C_{m_j} \cap C_{m_i}$ has no edges. A polygonal path has been called an LSEAC graph, a 2-connected partial linear 2-tree, a 2-connected partial 2-path, or a linear 2-tree by some authors (the last of these terms is unfortunate, since a polygonal path need not be a 2-tree).

The *line graph* of a graph G, denoted $L(G)$, is the graph having vertex set E_G, with two vertices in $L(G)$ adjacent if and only if the corresponding edges share an endpoint in G. Since we require a graph to have a nonempty set of vertices, the line graph $L(G)$ is defined only for a graph G that has at least one edge.

The *Cartesian product* of two graphs G and H, denoted $G \square H$, is the graph with vertex set $V_G \times V_H$ such that (u, v) is adjacent to (u', v') if and only if (1) $u = u'$ and $\{v, v'\} \in E_H$, or (2) $v = v'$ and $\{u, u'\} \in E_G$.

The *strong product* of two graphs G and H, denoted $G \boxtimes H$, is the graph with vertex set $V_G \times V_H$ such that (u, v) is adjacent to (u', v') if and only if (1) $\{u, u'\} \in E_G$ and $\{v, v'\} \in E_H$, or (2) $u = u'$ and $\{v, v'\} \in E_H$, or (3) $v = v'$ and $\{u, u'\} \in E_G$.

The *corona* of G with H, denoted $G \circ H$, is the graph of order $|G||H| + |G|$ obtained by taking one copy of G and $|G|$ copies of H, and joining all the vertices in the ith copy of H to the ith vertex of G.

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ELA

The nth *hypercube*, Q_n, is defined inductively by $Q_1 = K_2$ and $Q_{n+1} = Q_n \Box K_2$. Clearly, $|Q_n| = 2^n$. The nth *supertriangle*, T_n, is an equilateral triangular grid with n vertices on each side. The order of T_n is $\frac{1}{2}n(n+1)$. The *Möbius ladder* is obtained from $C_s \Box P_2$ by replacing one pair of parallel cycle edges with a crossed pair.

Illustrations of these graphs and constructions can be found in [15], and some illustrations can be found in Section 3.

An upper bound for $M^F(G)$, which yields an associated lower bound for $\mathrm{mr}^F(G)$, is the parameter $\mathrm{Z}(G)$ introduced in [1]. If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black (this is called the *color-change rule*). Given a coloring of G, the *derived coloring* is the (unique) result of applying the color-change rule until no more changes are possible. A *zero forcing set* for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black. The *zero forcing number* $\mathrm{Z}(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V_G$.

Observation 1.1. It is known that the following graphs have field independent minimum rank:

1. the complete graph K_n,
2. the path P_n,
3. the cycle C_n,
4. the complete bipartite graph $K_{p,q}$,
5. every tree [8].

Proposition 1.2. *Every polygonal path has field independent minimum rank.*

Proof. Note that for any graph G, $\mathrm{mr}^F(G) = |G|-1$ implies G is a path [7, 11, 19]. The paper [17] addresses only minimum rank over the real numbers, but the proof there shows that if H is a polygonal path, then $\mathrm{Z}(H) = 2$, so $\mathrm{mr}^F(H) \geq |H| - 2$. Since H is not a path, $\mathrm{mr}^F(H) \leq |H| - 2$. □

2. Universally optimal matrices. A matrix $A \in S_n^F$ is *optimal* for a graph G over a field F if $\mathcal{G}^F(A) = G$ and $\mathrm{rank}^F(A) = \mathrm{mr}^F(G)$. Recall that when A is an integer matrix and p is prime, A can be viewed as a matrix over $\mathbb{Z}_p$; the rank of A over $\mathbb{Z}_p$ will be denoted $\mathrm{rank}^{\mathbb{Z}_p}(A)$.

Definition 2.1. A *universally optimal matrix* is an integer matrix A such that every off-diagonal entry of A is 0, 1, or -1, and for all fields F, $\mathrm{rank}^F(A) = \mathrm{mr}^F(\mathcal{G}(A))$.

Note that if A is a universally optimal matrix, then $\mathcal{G}^F(A) = \mathcal{G}(A)$ is independent

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Volume 18, pp. 403-419, July 2009

of the field. The next two results are basic linear algebra.

PROPOSITION 2.2. *Let $S \subset \mathbb{Z}^n$ be a linearly dependent set of vectors over $\mathbb{Q}$. Then for every prime number p, S is linearly dependent over $\mathbb{Z}_p$.*

COROLLARY 2.3. *If $A \in \mathbb{Z}^{n\times n}$, then for every prime p,* $\operatorname{rank}^{\mathbb{Z}_p}(A) \leq \operatorname{rank}(A)$.

OBSERVATION 2.4. Each of the following graphs has a universally optimal matrix of the form $\mathcal{A}(G) + D$ where D is a diagonal $(0,1)$-matrix:

1. the complete graph K_n,
2. the complete bipartite graph $K_{p,q}$,
3. every tree [13].

PROPOSITION 2.5. *Every cycle has a universally optimal matrix.*

Proof. In [13], it was shown that the cycle $C_n, n \neq 5$, has an optimal matrix of the form $\mathcal{A}(G) + D$ where D is a diagonal $(0,1)$-matrix. The matrix $A_5 = \mathcal{A}(C_5) + \operatorname{diag}(0,0,-1,-1,-1)$ is a universally optimal matrix for C_5, because for every field F, $3 = \operatorname{mr}^F(C_5) \leq \operatorname{rank}^F(A_5) \leq \operatorname{rank}(A_5) = 3$. □

OBSERVATION 2.6. The existence of a universally optimal matrix A for the graph G implies $\operatorname{mr}^F(G) \leq \operatorname{mr}(G)$ for all fields F, or equivalently, the existence of a field F such that $\operatorname{mr}^F(G) > \operatorname{mr}(G)$ implies that G cannot have a universally optimal matrix.

Note that the existence of a universally optimal matrix A for the graph G does not imply field independence of minimum rank for G, because the rank of A could be lower over $\mathbb{Z}_p$ for some prime p, as in the next example.

EXAMPLE 2.7. [5] Let $J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & J & J \\ J & 0 & J \\ J & J & 0 \end{bmatrix}$. The matrix A is a universally optimal matrix because if $\operatorname{char} F \neq 2$, then $\operatorname{rank}^F(A) = 3 = \operatorname{mr}^F(K_{3,3,3})$, and if $\operatorname{char} F = 2$, then $\operatorname{rank}^F(A) = 2 = \operatorname{mr}^F(K_{3,3,3})$.

The graphs in Observation 2.4 and Proposition 2.5 are known to have field independent minimum rank. To use a universally optimal matrix to establish field independence of minimum rank, we typically use another bound such as $\mathrm{Z}(G)$ to show that $\operatorname{mr}(G) \leq \operatorname{mr}^F(G)$ for all F.

PROPOSITION 2.8. *The minimum rank of Petersen graph P is field independent, and $\mathcal{A}(P) - I$ is a universally optimal matrix for P.*

Proof. In [1], it was shown that $\operatorname{mr}(P) = 5 = \mathrm{Z}(P)$. So, $5 = |P| - \mathrm{Z}(P) \leq \operatorname{mr}^F(P) \leq \operatorname{rank}^F(\mathcal{A}(P) - I) \leq \operatorname{rank}(\mathcal{A}(P) - I) = 5$. □

Electronic Journal of Linear Algebra ISSN 1081-3810
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We use the idea of covering the edges of a graph with subgraphs to construct optimal matrices. An (edge) *covering* of a graph G is a set of subgraphs $\mathcal{C} = \{G_i, i = 1, \ldots, h\}$ such that G is the (nondisjoint) union $G = \cup_{i=1}^{h} G_i$. A graph has many possible coverings, but some, such as clique coverings, are more useful than others. For a given covering $\mathcal{C}$, $c_{\mathcal{C}}(e)$ will denote the number of subgraphs that have edge e as a member.

PROPOSITION 2.9. *Let F be a field and let G be a graph. Suppose $\mathcal{C} = \{G_i, i = 1, \ldots, h\}$ is a covering of G such that for each G_i there is an optimal matrix of the form $\mathcal{A}(G_i) + D_i$, where D_i is diagonal. If* $\operatorname{char} F = 0$ *or if* $\operatorname{char} F = p$ *and* $c_{\mathcal{C}}(e) \not\equiv 0 \mod p$ *for every edge $e \in E_G$, then*

$$\operatorname{mr}^F(G) \leq \sum_{i=1}^{h} \operatorname{mr}^F(G_i).$$

In particular, if $c_{\mathcal{C}}(e) = 1$ for every edge $e \in E_G$ and $\operatorname{mr}(G) = \sum_{i=1}^{h} \operatorname{mr}(G_i)$, then there is an integer diagonal matrix D such that $\mathcal{A}(G) + D$ is an optimal matrix over $\mathbb{R}$.

Proof. Let A_i be constructed by embedding an optimal matrix $\mathcal{A}(G_i) + D_i$ for G_i in the appropriate place in a $|G| \times |G|$ matrix. For $A = \sum_{i=1}^{h} A_i$, $\mathcal{G}^F(A) = G$ and $\operatorname{rank}^F(A) \leq \sum_{i=1}^{h} \operatorname{rank}^F(A_i) = \sum_{i=1}^{h} \operatorname{mr}^F(G_i)$. □

COROLLARY 2.10. *If $\mathrm{Z}(G) + \operatorname{cc}(G) = |G|$ and G has an optimal clique-covering with cliques intersecting only at the vertices, then the matrix $\mathcal{A}(G) + D$ obtained from the optimal clique-covering as in Proposition 2.9 is a universally optimal matrix for G and the minimum rank of G is field independent.*

COROLLARY 2.11. *Minimum rank is field independent for supertriangles and claw-free block-clique graphs and these graphs have universally optimal matrices.*

Proof. This follows from Corollary 2.10 and [1]. □

The *necklace* with s diamonds, denoted N_s, is a 3-regular graph that can be constructed from a $3s$-cycle by appending s extra vertices, with each "extra" vertex adjacent to 3 sequential cycle vertices; N_3 is shown in Figure 2.1.

PROPOSITION 2.12. *The necklace N_s has a universally optimal matrix $\mathcal{A}(N_s) + I$, has field independent minimum rank, and $\operatorname{mr}^F(N_s) = 3s - 2$ for every field F.*

Proof. By [13], $\operatorname{rank}(\mathcal{A}(C_{3s}) + I) = 3s - 2$. The matrix $\mathcal{A}(N_s) + I$ has s duplicate rows and columns that can be deleted to leave $\mathcal{A}(C_{3s}) + I$ without changing the rank. Since $\operatorname{mr}^F(N_s) \geq \operatorname{mr}^F(C_{3s}) = 3s - 2$, $\mathcal{A}(N_s) + I$ is a universally optimal matrix. □

Next we show that every polygonal path has a universally optimal matrix. We begin with a lemma.

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ELA

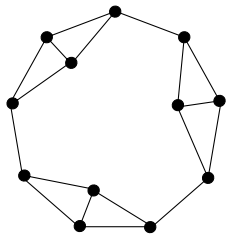


FIG. 2.1. *The necklace N_3.*

LEMMA 2.13. *Let G be a polygonal path built from two cycles. Then G has a universally optimal matrix.*

Proof. Let one induced cycle of G have n vertices and the other m vertices. Then $n + m = |G| + 2$. If $m = n = 3$, then G is $K_4 - e$; label G so that the missing edge is $\{1, 3\}$. Then G has a universally optimal matrix of the form $A_4 = \mathcal{A}(G) + \mathrm{diag}(0, 1, 0, 1)$.

If at least one of m, n is greater than three, we "cover" G with a $K_4 - e$ where the two cycles overlap and $n-3+m-3$ other triangles. Note that this involves extra edges not in G (so is not a covering in the sense of Proposition 2.9). Let J be the 3×3 all 1's matrix. Embed A_4 in the appropriate place in a matrix of order $m + n - 2$, and for each additional triangle used in the "covering," embed $-J$ or J (with signs alternating). Let B be the sum of all these matrices. The entries corresponding to the unwanted edges of the triangles covering G will be zero (by the choice of sign of J), so that $\mathcal{G}^F(B) = G$, and every off-diagonal entry of B is 0, 1, or -1. Since $|G| - 2 = \mathrm{mr}(G) \leq \mathrm{rank}\, B \leq m - 3 + n - 3 + 2 = |G| - 2$, B is a universally optimal matrix for G. □

THEOREM 2.14. *Every polygonal path G has a universally optimal matrix.*

Proof. We proceed by induction on the number of cycles used to build G. If G is built from one or two cycles, then G has a universally optimal matrix by Proposition 2.5 and Lemma 2.13. Now assume that every polygonal path built from $n-1$ or fewer cycles has a universally optimal matrix and G is built from $n \geq 3$ cycles.

Let the nth cycle of G, which is an end cycle, be called C. Let H be an induced subgraph of G formed by deleting the parts of C and its neighbor cycle (the $(n-1)$st cycle) that do not overlap with the other cycles in G. Then H is a polygonal path built from $n-2$ cycles. By assumption H has a universally optimal matrix A_1. Recall that C has a universally optimal matrix A_2. There are two cases. In G, either C shares a vertex with H or it does not.

If H and C share a vertex in G, then the portion of G not covered by H and C

Electronic Journal of Linear Algebra ISSN 1081-3810
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is a path P on at least two vertices. Recall that P has a universally optimal matrix A_3. Then, embedding A_1, A_2, and A_3 in the appropriate places in $n \times n$ matrices and summing, we get a matrix B where $\mathcal{G}(B) = G$. Then we have $|G| - 2 = \mathrm{mr}(G) \leq \mathrm{rank}(B) \leq \mathrm{rank}(A_1) + \mathrm{rank}(A_2) + \mathrm{rank}(A_3) = |H| - 2 + |P| - 1 + |C| - 2 = |G| + 3 - 5 = |G| - 2$.

If H and C do not share a vertex in G, the portion of G not covered by H and C consists of two disjoint paths P_1 and P_2, each on at least two vertices. Recall that both P_1 and P_2 have universally optimal matrices A_4 and A_5. Embedding A_1, A_2, A_4, and A_5 in the appropriate places in $n \times n$ matrices and summing, we obtain a matrix B where $\mathcal{G}(B) = G$. Now we have $|G| - 2 = \mathrm{mr}(G) \leq \mathrm{rank}(B) \leq \mathrm{rank}(A_1) + \mathrm{rank}(A_4) + \mathrm{rank}(A_5) + \mathrm{rank}(A_2) = |H| - 2 + |P_1| - 1 + |P_2| - 1 + |C| - 2 = |G| + 4 - 6 = |G| - 2$.

Thus, in both cases, G has a universally optimal matrix. □

For Cartesian products, $G \square G$ is a special case. For example, we show in Proposition 2.16 that $K_s \square K_s$ has a universally optimal matrix and is field independent, but this need not be true for $K_s \square K_t$ (Proposition 3.5).

In [1], a technique involving Kronecker products was used to construct optimal matrices. If A is an $s \times s$ real matrix and B is a $t \times t$ real matrix, then $A \otimes B$ is the $s \times s$ block matrix whose (i, j)th block is the $t \times t$ matrix $a_{ij}B$. The following results are standard (cf. [12, §9.7]).

Observation 2.15. Let G and H be graphs of order s and t, respectively, and let A and B be matrices over a field F such that $\mathcal{G}^F(A) = G$ and $\mathcal{G}^F(B) = H$. Then $\mathcal{G}^F(A \otimes I_t + I_s \otimes B) = G \square H$.

If $\mathbf{x}$ is an eigenvector of A for eigenvalue λ and $\mathbf{y}$ is an eigenvector of B for eigenvalue μ, then $\mathbf{x} \otimes \mathbf{y}$ is an eigenvector of $A \otimes I_t + I_s \otimes B$ for eigenvalue $\lambda + \mu$.

Proposition 2.16. *The graph $K_s \square K_s$ has a universally optimal matrix and has field independent minimum rank.*

Proof. Let J_s be the $s \times s$ matrix having each entry equal to 1. Over the field $\mathbb{R}$, the multiplicity of eigenvalue 0 for $J_s \otimes I_s + I_s \otimes (-J_s)$ is $(s-1)(s-1) + 1 = s^2 - 2s + 2 = \mathrm{Z}(K_s \square K_s)$, so $J_s \otimes I_s + I_s \otimes (-J_s)$ is a universally optimal matrix for $K_s \square K_s$ and the minimum rank of $K_s \square K_s$ is field independent. □

Proposition 2.17. *The graph $P_s \square P_s$ has a universally optimal matrix and has field independent minimum rank.*

Proof. Let $A_s = \mathcal{A}(P_s)$ (or any $(0, 1, -1)$-matrix having graph P_s). Over the field $\mathbb{R}$, the multiplicity of eigenvalue 0 for $A_s \otimes I_s + I_s \otimes (-A_s)$ is $s = \mathrm{Z}(P_s \square P_s)$ so $A_s \otimes I_s + I_s \otimes (-A_s)$ is universally optimal for $P_s \square P_s$, and the minimum rank of

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 403-419, July 2009

ELA

$P_s \Box P_s$ is field independent. □

THEOREM 2.18. *The graph $C_s \Box C_s$ has a universally optimal matrix, has field independent minimum rank, and $M^F(C_s \Box C_s) = \mathrm{Z}(C_s \Box C_s) = s + 2\lfloor \frac{s}{2} \rfloor$ for every field F.*

Proof. Let A_s be obtained from $\mathcal{A}(C_s)$ by changing the sign of a pair of symmetrically placed ones. Let $k = \lceil \frac{s}{2} \rceil$. Then over the field $\mathbb{R}$, the (distinct) eigenvalues of A are $\lambda_i = 2\cos\frac{\pi(2i-1)}{s}$, $i = 1, \ldots, k$, each with multiplicity 2, except that if s is odd, $\lambda_k = -2$ has multiplicity 1. Therefore over $\mathbb{R}$ the multiplicity of eigenvalue 0 for $A_s \otimes I_s + I_s \otimes (-A_s)$ is at least $4\left(\frac{s}{2}\right) = 2s = s + 2\lfloor \frac{s}{2} \rfloor$ if s is even and $4\left(\frac{s-1}{2}\right) + 1 = 2s - 1 = s + 2\lfloor \frac{s}{2} \rfloor$ if s is odd. Furthermore, $A_s \otimes I_s + I_s \otimes (-A_s)$ is a $(0, 1, -1)$-matrix.

If s is even, $\mathrm{Z}(C_s \Box C_s) \leq 2s$ because the vertices in two successive cycles form a zero forcing set (cf. [1, Corollary 2.8]). So assume s is odd. The graph $C_s \Box C_s$ is a cycle of cycles. Number the cycles sequentially as $C_s^{(i)}$, $i = 1, \ldots, s$ and within cycle $C_s^{(i)}$ denote the vertices sequentially by $v_j^{(i)}$, $j = 1, \ldots, s$. We claim that $\{v_1^{(s)}, \ldots, v_s^{(s)}, v_1^{(1)}, \ldots, v_{s-1}^{(1)}\}$ is a zero forcing set, and so $\mathrm{Z}(C_s \Box C_s) \leq 2s - 1$. Forcing across cycles $C_s^{(i)}$, vertices $v_1^{(s)}, \ldots, v_{s-1}^{(s)}$ can force vertices $v_1^{(s-1)}, \ldots, v_{s-1}^{(s-1)}$, so we have one cycle of s black vertices surrounded by two cycles each having $s - 1$ black vertices. Then vertices $v_2^{(1)}, \ldots, v_{s-2}^{(1)}$ can force vertices $v_2^{(2)}, \ldots, v_{s-2}^{(2)}$ and vertices $v_2^{(s-1)}, \ldots, v_{s-2}^{(s-1)}$ can force vertices $v_2^{(s-2)}, \ldots, v_{s-2}^{(s-2)}$. Repeating this process, we obtain one completely black cycle, two cycles with all but 1 vertex colored black, two cycles with all but 3 vertices colored black, ..., two cycles with all but $s - 2$ vertices colored black, i.e., two cycles with 2 black vertices. Note that all cycles now have at least 2 black vertices. We can then force the remaining vertices along the cycles $C_s^{(i)}$, starting with $C_s^{((s-1)/2)}$ and $C_s^{((s+1)/2)}$, each of which has black vertices in the two positions $\frac{s-1}{2}$ and $\frac{s+1}{2}$ when we start the process of forcing along cycles $C_s^{(i)}$.

Therefore, $M(C_s \Box C_s) \geq \mathrm{null}(A_s \otimes I_s + I_s \otimes (-A_s)) \geq s + 2\lfloor \frac{s}{2} \rfloor \geq \mathrm{Z}(C_s \Box C_s) \geq M(C_s \Box C_s)$, so we have equality throughout, $A_s \otimes I_s + I_s \otimes (-A_s)$ is a universally optimal matrix for $C_s \Box C_s$, and the minimum rank of $C_s \Box C_s$ is field independent. □

THEOREM 2.19. *Let v be a cut-vertex of graph G. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the ith component of $G - v$ and let $V_i = \{v\} \cup W_i$. If the minimum rank of $G[V_i]$ and $G[W_i]$ is field independent for all $i = 1, \ldots, h$, then the minimum rank of G is field independent.*

Suppose in addition that for all $i = 1, \ldots, h$, $G[V_i]$ and $G[W_i]$ have universally optimal matrices of the form $A_i = \mathcal{A}(G[V_i]) + D_i$ and $\widetilde{A_i} = \mathcal{A}(G[W_i]) + \widetilde{D_i}$, respectively, where $D_i, \widetilde{D_i}$ are integer diagonal matrices. Then G has a universally optimal matrix of the form $\mathcal{A}(G) + D$, where D is an integer diagonal matrix. The analogous result

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 403-419, July 2009

ELA

is true if the adjacency matrices are replaced by $(0, 1, -1)$*-matrices.*

Proof. It is known that cut-vertex reduction is valid over any field [10]; the statement about field independence is an immediate consequence.

The existence of a universally optimal matrix is established by methods similar to those in Theorems 4.9 and 4.12 in [13]. If $\mathrm{mr}(G) - \mathrm{mr}(G - v) < 2$, then $\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G[V_i])$ [3]. Let $\breve{A}_i$ be the $n \times n$ matrix obtained from A_i by embedding it in the appropriate place (setting all other entries 0). The matrix $A = \breve{A}_1 + \cdots + \breve{A}_h$ is optimal for G.

If $\mathrm{mr}(G) - \mathrm{mr}(G - v) = 2$, for $i = 1, \ldots, h$, let $\breve{A}_i$ be the $n \times n$ matrix obtained from $\widetilde{A}_i$ by embedding it in the appropriate place (setting all other entries 0). The matrix A constructed from $\breve{A}_1 + \cdots + \breve{A}_h$ by setting entries in row and column v to 1 as needed to obtain $\mathcal{G}(A) = G$ is optimal for G. □

THEOREM 2.20. *A block-cycle-clique graph* G *has a universally optimal matrix of the form* $\mathcal{A}(G) + D$*, where* D *is an integer diagonal matrix, and the minimum rank of a block-cycle-clique matrix is field independent.*

Proof. Note that the result of deleting a vertex from a block-cycle-clique graph is one or more smaller block-cycle-clique graphs. The proof is by induction. Assume true for all block-cycle-clique graphs of order less than n and let G be a block-cycle-clique graph of order n. If G is a clique or a cycle then G has a universally optimal matrix of the form $\mathcal{A}(G) + D$ (cf. Observation 2.4 and Proposition 2.5) and the minimum rank of a such graph is field independent. If not, then G has a cut-vertex. Label the components as in Theorem 2.19. By the induction hypothesis, for all $i = 1, \ldots, h$, $G[V_i]$ and $G[W_i]$ have universally optimal matrices of the form $A_i = \mathcal{A}(G[V_i]) + D_i$ and $\widetilde{A_i} = \mathcal{A}(G[W_i]) + \widetilde{D_i}$, respectively, where $D_i, \widetilde{D_i}$ are integer diagonal matrices. So by Theorem 2.19, G has a universally optimal matrix of the form $\mathcal{A}(G) + D$ and the minimum rank of G is field independent. □

COROLLARY 2.21. *A unicyclic graph has field independent minimum rank and has a universally optimal matrix of the form* $\mathcal{A}(G) + D$*, where* D *is an integer diagonal matrix.*

COROLLARY 2.22. *A corona* G *of the form* $K_t \circ K_s$ *or* $C_t \circ K_s$ *has field independent minimum rank and has a universally optimal matrix of the form* $\mathcal{A}(G) + D$*, where* D *is an integer diagonal matrix.*

In [1], it was shown that $\mathrm{mr}(K_t \circ K_s) = t + 1$. In [4], it was shown that $\mathrm{mr}(C_t \circ K_1) = 2t - \lfloor \frac{t}{2} \rfloor$ (for $t \geq 4$).

PROPOSITION 2.23. *For* $s \geq 2$, $\mathrm{mr}(C_t \circ K_s) = 2t - 2$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 403-419, July 2009

Proof. We can cover $C_t \circ K_s$ with the cycle and t copies of K_{s+1}, so $\mathrm{mr}(C_t \circ K_s) \leq 2t-2$. All s vertices in two consecutive copies of K_s and all but one vertex of each of the remaining K_s are a zero forcing set, so $\mathrm{Z}(C_t \circ K_s) \leq ts - t + 2$ and $2t - 2 = (ts+t) - (ts - t + 2) \leq \mathrm{mr}((C_t \circ K_s)$. □

The (m,k)-*pineapple* (with $m \geq 3, k \geq 2$) is $P_{m,k} = K_m \cup K_{1,k}$ such that $K_m \cap K_{1,k}$ is the vertex of $K_{1,k}$ of degree k; $P_{5,3}$ is shown in Figure 2.2.

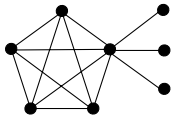


FIG. 2.2. *The pineapple* $P_{5,3}$.

COROLLARY 2.24. *Every pineapple* $P_{m,k}$ $(m \geq 3, k \geq 2)$ *has field independent minimum rank, has a universally optimal matrix of the form* $\mathcal{A}(P_{m,k}) + D$ *where* D *is an integer diagonal matrix, and* $\mathrm{mr}^F(P_{m,k}) = 3$ *for every field* F.

Proof. The first two statements follow from Theorem 2.20. By construction, $\mathrm{mr}(P_{m,k}) \leq \mathrm{mr}(K_m) + \mathrm{mr}(K_{1,k}) = 1 + 2$. Since $P_{m,k}$ contains $\ltimes$ as an induced subgraph, $\mathrm{mr}(P_{m,k}) \geq 3$ [5]. □

In all previous examples, whenever a graph had field independent minimum rank, it also had a universally optimal matrix. However, this need not always be the case. The next example exhibits a (disconnected) graph that has field independent minimum rank, but that does not have a universally optimal matrix.

EXAMPLE 2.25. Let G be the disjoint union of $K_{3,3,3}$ and $\overline{P_3 \cup 2K_3}$. We show that G has field independent minimum rank but G does not have a universally optimal matrix. Note first that $\mathrm{mr}^F(G) = \mathrm{mr}^F(K_{3,3,3}) + \mathrm{mr}^F(\overline{P_3 \cup 2K_3})$ and if $\mathcal{G}^F(A) = G$ then $A = A_1 \oplus A_2$, where $\mathcal{G}^F(A_1) = K_{3,3,3}$ and $\mathcal{G}^F(A_2) = \overline{P_3 \cup 2K_3}$.

In [5, 6], it is shown that for $\mathrm{char}\, F \neq 2$, $\mathrm{mr}^F(K_{3,3,3}) = 3$ and $\mathrm{mr}^F(\overline{P_3 \cup 2K_3}) = 2$. For $\mathrm{char}\, F = 2$, $\mathrm{mr}^F(K_{3,3,3}) = 2$ and $\mathrm{mr}^F(\overline{P_3 \cup 2K_3}) > 2$ [5, 6]. It is easy to construct a matrix A such that $\mathcal{G}^{\mathbb{Z}_2}(A) = \overline{P_3 \cup 2K_3}$ and $\mathrm{rank}^{\mathbb{Z}_2}(A) = 3$, so for $\mathrm{char}\, F = 2$, $\mathrm{mr}^F(\overline{P_3 \cup 2K_3}) = 3$. Thus,

$$\mathrm{mr}^F(G) = \mathrm{mr}^F(K_{3,3,3}) + \mathrm{mr}^F(\overline{P_3 \cup 2K_3}) = \begin{cases} 3+2=5 & \text{if } \mathrm{char}\, F \neq 2; \\ 2+3=5 & \text{if } \mathrm{char}\, F = 2. \end{cases}$$

Thus, G has field independent minimum rank.

Now suppose $A = A_1 \oplus A_2$ is an optimal integer matrix for G over $\mathbb{R}$. Necessarily, $\mathrm{rank}(A_2) = 2$. Then by Corollary 2.3, $\mathrm{rank}^{\mathbb{Z}_2}(A_2) \leq 2$. Since $\mathrm{mr}^{\mathbb{Z}_2}(\overline{P_3 \cup 2K_3}) = 3$, $\mathcal{G}^{\mathbb{Z}_2}(A_2) \neq \overline{P_3 \cup 2K_3}$. Thus, $\mathcal{G}^{\mathbb{Z}_2}(A) \neq G$ and A is not a universally optimal matrix.

Electronic Journal of Linear Algebra ISSN 1081-3810
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3. Additional Field Dependence Results. With the exception of Q_n and $P_s \Box P_t$ for $s \neq t$, the field independence or lack thereof has been established for all the families of graphs in the AIM Minimum Rank Graph Catalog: Families of Graphs [15]. These results are summarized in Table 1.1; many were established in Section 2 by exhibiting universally optimal matrices. This section contains the remaining justifications, which involve lack of field independence of minimum rank. In this section, when we state the minimum rank over $\mathbb{Z}_2$ of a particular graph, this minimum rank was exhaustively computed using all possible diagonals; the computations are available in a worksheet [9] using the open-source mathematics software *Sage* [20].

EXAMPLE 3.1. For the 6th wheel, $\mathrm{mr}^{\mathbb{Z}_2}(W_6) = 4 > 3 = \mathrm{mr}(W_6)$ (it is well known that for any infinite field F, $\mathrm{mr}^F(W_n) = n - 3$ because W_n can be constructed from C_{n-1} by adding one vertex that is adjacent to all the other vertices).

EXAMPLE 3.2. For the 5th Möbius ladder shown in Figure 3.1, $\mathrm{mr}^{\mathbb{Z}_2}(M_5) = 8 > 6 = \mathrm{mr}(M_5)$ [1].

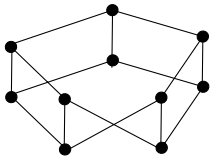


FIG. 3.1. *The 5th Möbius ladder M_5.*

The sth *half-graph*, denoted H_s, is the graph constructed from (disjoint) graphs K_s and $\overline{K_s}$, having vertices $u_1, \ldots, u_s$ and $v_1, \ldots, v_s$, respectively, by adding all edges $\{u_i, v_j\}$ such that $i + j \leq s + 1$. Figure 3.2 shows H_3, with the vertices of the K_3 being colored black and the vertices of the $\overline{K_3}$ colored white.

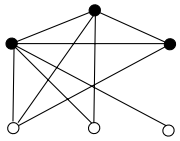


FIG. 3.2. *The 3rd half-graph H_3.*

PROPOSITION 3.3. *For every half-graph,* $\mathrm{mr}(H_s) = s$ *and* $\mathrm{Z}(H_s) = M(H_s) = s$.

Proof. Let $u_1, \ldots, u_s, v_1, \ldots, v_s$ be the vertices of H_s as described above, taken in that order. Let $L_s = [\ell_{ij}]$ be the $(0,1)$-matrix defined by $\ell_{ij} = 1$ if and only if $i + j \leq s + 1$. Let $A_s = \begin{bmatrix} L_s^2 & L_s \\ L_s & I_s \end{bmatrix}$. Then $\mathcal{G}(A_s) = H_s$ and $\mathrm{rank}(A_s) = s$. It is easy to

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 403-419, July 2009

see that the vertices $u_1, \ldots, u_s$ form a zero forcing set, so

$$s = |H_s| - s \leq |H_s| - \mathrm{Z}(H_s) \leq \mathrm{mr}(H_s) \leq \mathrm{rank}(A_s) = s.$$

Thus, $\mathrm{mr}(H_s) = s$ and $\mathrm{Z}(H_s) = M(H_s) = s$. □

EXAMPLE 3.4. For the 3rd half-graph shown in Figure 3.2, $\mathrm{mr}^{\mathbb{Z}_2}(H_3) = 4 > 3 = \mathrm{mr}(H_3)$.

EXAMPLE 3.5. The graphs $\overline{C_6}$, $K_3 \square K_2$, $K_3 \square P_2$, $C_3 \square P_2$ are isomorphic and $\mathrm{mr}^{\mathbb{Z}_2}(C_3 \square P_2) = 4 > 3 = \mathrm{mr}(C_3 \square P_2)$ [1].

EXAMPLE 3.6. For the tree T shown in Figure 3.3, $\mathrm{mr}^{\mathbb{Z}_2}(\overline{T}) = 4 > 3 = \mathrm{mr}(\overline{T})$ [1].

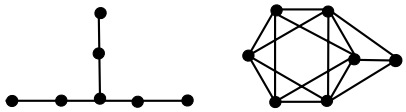


FIG. 3.3. *A tree T and its complement $\overline{T}$.*

EXAMPLE 3.7. For the 2-tree H shown in Figure 3.4, $\mathrm{mr}^{\mathbb{Z}_2}(\overline{H}) = 5 > 4 = \mathrm{mr}(\overline{H})$ [14].

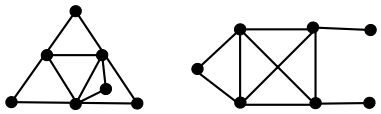


FIG. 3.4. *A 2-tree H and its complement $\overline{H}$.*

EXAMPLE 3.8. The line graph of K_7 has $\mathrm{mr}^{\mathbb{Z}_2}(L(K_7)) = 6 > 5 = \mathrm{mr}(L(K_7))$ [1]. Note that $L(K_7)$ is a strongly regular graph with parameters $(21, 10, 5, 4)$.

EXAMPLE 3.9. The graphs $C_5 \square C_3, C_5 \square K_3$ are isomorphic and $\mathrm{mr}^{\mathbb{Z}_2}(C_5 \square K_3) = 10 > 9 = \mathrm{mr}(C_5 \square K_3)$ [1].

EXAMPLE 3.10. For the complete multipartite graph $K_{2,2,2,2}$, $\mathrm{mr}^{\mathbb{Z}_2}(K_{2,2,2,2}) = 4 > 2 = \mathrm{mr}(K_{2,2,2,2})$ [5]. Therefore, $K_{2,2,2,2}$ does not have a universally optimal matrix. (Note that $K_{3,3,3}$ has already established that complete multipartite graphs need not be field independent.)

Let $\mathcal{C} = (G_1, \ldots, G_h)$ be an ordered covering of the graph G and for $k = 2, \ldots, h$, let $R_k = \{e : \exists j < k \text{ such that } e \in (E_{G_j} \cap E_{G_k})\}$. The *overlap* of $\mathcal{C}$ is $\mathrm{over}_{\mathcal{C}}(G) = \max_{k=2,\ldots,h} |R_k|$.

PROPOSITION 3.11. *Let F be a field and let G be a graph. If $\mathcal{C} = (G_1, \ldots, G_h)$*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 403-419, July 2009

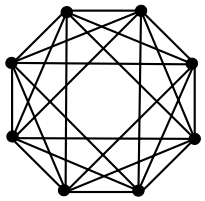


FIG. 3.5. *The complete multipartite graph* $K_{2,2,2,2}$.

is an ordered covering of G, *and* $\operatorname{over}_{\mathcal{C}}(G) < |F| - 1$ *or* F *is infinite, then*

$$\operatorname{mr}^F(G) \leq \sum_{i=1}^{h} \operatorname{mr}^F(G_i).$$

Proof. Let A_i be constructed by embedding an optimal matrix for G_i in the appropriate place in a $|G| \times |G|$ matrix. For $e \in E_G$, we will denote by $a_e^{(i)}$ the entry of A_i corresponding to e. We show how to select c_i such that for $A = \sum_{i=1}^{h} c_i A_i$, $\mathcal{G}^F(A) = G$. Let $c_1 = 1$ and assume $c_1, \ldots, c_{k-1}$ have been chosen such no cancelation has occurred for any off-diagonal entry. Choose a nonzero c_k such that for each $e \in R_k$,

$$c_k a_e^{(k)} \neq -\left(\sum_{j=1}^{k-1} c_j a_e^{(j)}\right). \quad \square$$

PROPOSITION 3.12. *For* $P_s \boxtimes P_t$, *the minimum rank is the same over all fields of order greater than* 2, *and* $\operatorname{mr}^{\mathbb{Z}_2}(P_3 \boxtimes P_3) = 6 > 4 = \operatorname{mr}(P_3 \boxtimes P_3)$.

Proof. The graph $P_s \boxtimes P_t$ has an ordered covering $\mathcal{C} = (G_1, \ldots, G_{(s-1)(t-1)})$ with $G_i = K_4$, proceeding row by row. With the covering just described, $\operatorname{over}_{\mathcal{C}}(P_s \boxtimes P_t) = 2$ and $c_{\mathcal{C}}(e) = 2$ or 1 for all edges $e \in E_{P_s \boxtimes P_t}$. If $|F| > 3$, $\operatorname{over}_{\mathcal{C}}(P_s \boxtimes P_t) < |F| - 1$; for $F = \mathbb{Z}_3$, note that $c_{\mathcal{C}}(e) \not\equiv 0 \mod 3$ for all $e \in E_{P_s \boxtimes P_t}$. Thus, by Propositions 3.11 and 2.9, $\operatorname{mr}^F(P_s \boxtimes P_t) \leq \sum_{i=1}^{h} \operatorname{mr}^F(G_i) = \operatorname{mr}(P_s \boxtimes P_t)$. By [1], $M(P_s \boxtimes P_t) = \mathrm{Z}(P_s \boxtimes P_t)$, so $\operatorname{mr}(P_s \boxtimes P_t) \leq \operatorname{mr}^F(P_s \boxtimes P_t)$. $\square$

Many of our examples show a difference in minimum rank only over $\mathbb{Z}_2$, but this need not be the case. In [6], it was shown it is possible for a graph to have minimum rank that differs only over small fields, as in the next example.

EXAMPLE 3.13. Consider the graph $\overline{3K_2 \cup K_1}$ shown in Figure 3.6 and let F be a field. It is shown in [5] that for an infinite field, $\operatorname{mr}^F(\overline{3K_2 \cup K_1}) = 2$. For finite fields, it is shown in [6] that $\operatorname{mr}^F(\overline{3K_2 \cup K_1}) = 2$ if and only if ($\operatorname{char} F \neq 2$ and $|F| > 3$) or ($\operatorname{char} F = 2$ and $|F| > 2$), or equivalently, $|F| \geq 4$. In particular, $3 = \operatorname{rank}^{\mathbb{Z}_3}(\mathcal{A}(\overline{3K_2 \cup K_1})) \geq \operatorname{mr}^{\mathbb{Z}_3}(\overline{3K_2 \cup K_1}) \geq 3$ and $\operatorname{mr}^{\mathbb{Z}_2}(\overline{3K_2 \cup K_1}) = 4$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 403-419, July 2009

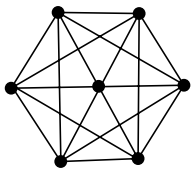


FIG. 3.6. *The graph* $\overline{3K_2 \cup K_1}$.

THEOREM 3.14. *Let F be a field such that the characteristic of F is* 0 *or* 2, *or* $|F| \geq 6$. *Then* $\mathrm{mr}^F(Q_n) = 2^{n-1}$.

Proof. In [1], it is shown that $\mathrm{mr}^F(Q_n) \geq 2^{n-1}$, with equality for $F = \mathbb{R}$ or $\mathrm{char}\, F = 2$. We extend the technique used for $\mathbb{R}$ (which requires $\sqrt{2} \in F$) to other fields.

Suppose that there exist nonzero $\alpha, \beta \in F$ such that $\alpha^2 + \beta^2 = 1$. We recursively define two sequences of matrices L_n, H_n such that $L_n^2 = I$, $\mathcal{G}^F(H_n) = Q_n$, and $\mathrm{rank}^F(H_n) = 2^{n-1}$. Let

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Given H_{n-1} and L_{n-1}, define

$$H_n = \begin{bmatrix} L_{n-1} & I \\ I & L_{n-1} \end{bmatrix}, \quad L_n = \begin{bmatrix} \alpha L_{n-1} & \beta I \\ \beta I & -\alpha L_{n-1} \end{bmatrix}.$$

By induction, $L_n^2 = I$. Clearly, $\mathcal{G}^F(H_n) = Q_n$. Since

$$\begin{bmatrix} I & 0 \\ -L_{n-1} & I \end{bmatrix} \begin{bmatrix} L_{n-1} & I \\ I & L_{n-1} \end{bmatrix} = \begin{bmatrix} L_{n-1} & I \\ 0 & 0 \end{bmatrix},$$

it holds that $\mathrm{rank}^F(H_n) = 2^{n-1}$.

Note that for $\mathrm{char}\, F = 0$, $(\frac{3}{5})^2 + (\frac{4}{5})^2 = 1$. Now consider Z_p. The sums $\alpha^2 + \beta^2$ and $\gamma^2 + \delta^2$ are considered *essentially different* if $\{\alpha^2, \beta^2\} \neq \{\gamma^2, \delta^2\}$ (as unordered sets). If there are two essentially different expressions for 1 as the sum of two squares, then one of them must have both elements nonzero, and we can construct L_n, H_n as above. In [18], it is shown that the number of essentially different ways to express 1 as a sum of squares in a finite field F is $\lfloor \frac{|F|+10}{8} \rfloor$, and note that if $|F| \geq 6$ then $\lfloor \frac{|F|+10}{8} \rfloor \geq 2$. □

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