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SPECTRAL SLATER INDEX OF TOURNAMENTS*

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Abstract. The Slater index $i(T)$ of a tournament T is the minimum number of arcs that must be reversed to make T transitive. In this paper, we define a parameter $\Lambda(T)$ from the spectrum of the skew-adjacency matrix of T, called the spectral Slater index. This parameter is a measure of remoteness between the spectrum of T and that of a transitive tournament. We show that $\Lambda(T) \leq 8\, i(T)$ and we characterize the tournaments with maximal spectral Slater index. As an application, an improved lower bound on the Slater index of doubly regular tournaments is given.

Key words. Tournament, Skew-spectrum, Spectral distance, Slater index.

AMS subject classifications. 05C20, 05C50.

1. Introduction. A *tournament* is a digraph in which every pair of vertices is joined by exactly one arc. If (x, y) is an arc, then we say that x dominates y and we write $x \to y$. A tournament is *transitive* if whenever u dominates v and v dominates w, then u dominates w. An n-tournament is a tournament with n vertices. Unless mentioned otherwise, all considered n-tournaments have vertex set $\{1, \dots, n\}$.

The *distance* between two n-tournaments T and T' is the number $d(T, T')$ of pairs $\{i, j\}$ from $\{1, \dots, n\}$ for which the arc between i and j has not the same direction in T and in T'. The *Slater index* $i(T)$ of a tournament T is the minimum number of arcs that must be reversed to make T transitive [21]. This index can be interpreted as the minimum distance between T and the set of transitive tournaments. Alon [3] and Charbit *et al.* [4] independently proved that the problem of finding the Slater index of a tournament is NP-hard.

Let $i(n) = \max i(T)$, where the maximum is taken over all n-tournaments. There are several works about finding bounds on $i(n)$ (see Erdős and Moon [8], Reid [18] and Jung [14]). Using probabilistic methods, Spencer [22, 23] and de la Vega [6] proved that for sufficiently large n, there exist some constant numbers c_1 and c_2 such that

$$\frac{1}{2}\binom{n}{2} - c_1 n^{\frac{3}{2}} \leq i(n) \leq \frac{1}{2}\binom{n}{2} - c_2 n^{\frac{3}{2}}. \tag{1.1}$$

However, there is no explicit construction of an n-tournament T such that $i(T) = \frac{1}{2}\binom{n}{2} - O(n^{\frac{3}{2}})$. Good candidates are doubly regular tournaments. A tournament is *doubly regular* if there is a constant $k \geq 1$ such that each unordered pair of vertices is jointly dominated by exactly k vertices. Doubly regular tournaments exist only for orders $n \equiv 3 \pmod 4$. A result due to Satake [20, Theorem 2.4] is equivalent to the following theorem.

THEOREM 1.1. *The Slater index of doubly regular n-tournaments is at least $\frac{1}{2}\binom{n}{2} - n^{\frac{3}{2}} \log_2(2n)$.*

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The distance between tournaments can also be interpreted in terms of their skew-adjacency matrices. The *skew-adjacency* matrix of a n-tournament is defined as the $n \times n$ skew-symmetric matrix $S = [s_{ij}]$, where $s_{ij} = 1$ if $i \to j$, $s_{ij} = -1$ if $j \to i$ and $s_{ii} = 0$. The *skew-spectrum* of a tournament is the spectrum of its skew-adjacency matrix. Let T and T' be two n-tournaments with skew-adjacency matrices S and S' respectively. It is easy to check that $d(T,T') = \frac{1}{8}\left\|S-S'\right\|_F^2$, where $\left\|S-S'\right\|_F$ is the Frobenius norm of $S-S'$. As the matrices S and S' are skew-symmetric, their nonzero eigenvalues are purely imaginary numbers. Let $\{i\lambda_1,\dots,i\lambda_n\}$ and $\{i\lambda'_1,\dots,i\lambda'_n\}$ be the spectra of S and S' arranged such that $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \cdots \geq \lambda'_n$. It follows from Wielandt–Hoffman theorem [10] that

$$\sum_{j=1}^{n}(\lambda_j-\lambda'_j)^2 \leq \left\|S-S'\right\|_F^2,$$

hence

$$\sum_{j=1}^{n}(\lambda_j-\lambda'_j)^2 \leq 8\,d(T,T'). \tag{1.2}$$

Motivated by this result, we define the *spectral distance* $\lambda(T,T')$ between T and T' as:

$$\lambda(T,T') = \sum_{j=1}^{n}(\lambda_j-\lambda'_j)^2.$$

The concept of spectral distance is defined for graphs in different ways with respect to various matrices [1, 12, 13].

The *spectral Slater index* of T, denoted by $\Lambda(T)$ is the spectral distance between T and a transitive tournament. By (1.2), we get

$$\Lambda(T) \leq 8\,i(T). \tag{1.3}$$

Our main result below gives an upper bound on the spectral Slater index of tournaments.

Theorem 1.2. *Let T be an n-tournament.*

(i) *If n is even, then*

$$\Lambda(T) \leq 2n\,(n-1) - 4\sqrt{n-1}\sum_{k=1}^{n/2}\cot\left(\frac{(2k-1)\,\pi}{2n}\right).$$

Equality holds iff the skew-spectrum of T is $\left\{\left[\pm i\sqrt{n-1}\right]^{\frac{n}{2}}\right\}$.

(ii) *If n is odd, then*

$$\Lambda(T) \leq 2n\,(n-1) - 4\sqrt{n}\sum_{k=1}^{(n-1)/2}\cot\left(\frac{(2k-1)\,\pi}{2n}\right).$$

Equality holds iff the skew-spectrum of T is $\left\{[0]^1, [\pm i\sqrt{n}]^{\frac{n-1}{2}}\right\}$.

The class of tournaments whose spectral Slater index attains the bounds in Theorem 1.2 will be characterized in Section 4. This class is related to doubly regular tournaments. In Section 5, we improve Satake's lower bound [20] on the Slater index of doubly regular tournaments.

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2. Preliminaries. Let T be an n-tournament and let S be its skew-adjacency matrix. McCarthy and Benjamin [16, Proposition 1] proved that the determinant of a tournament is equal to zero if and only if it has an odd number of vertices. It follows that T has n nonzero eigenvalues if n is even and $n-1$ nonzero eigenvalues if n is odd. Moreover, as S is skew-symmetric, these eigenvalues are purely imaginary and occur as conjugate pairs $\pm i\lambda_1, \ldots, \pm i\lambda_{\lfloor n/2 \rfloor}$. Throughout this paper, the nonzero eigenvalues will always be arranged such that $\lambda_1 \geq \cdots \geq \lambda_{\lfloor n/2 \rfloor} > 0$. Since the off-diagonal entries of S are from the set $\{-1, 1\}$, we have

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \lambda_k^2 = \frac{1}{2}\mathrm{trace}(-S^2) = \frac{n(n-1)}{2}. \tag{2.4}$$

Tournaments with the same skew-spectrum can be obtained by switching. The operation of *switching a tournament* with respect to a subset X of $[n] := \{1, \ldots, n\}$ consists of reversing all the arcs between X and $[n] \setminus X$. This operation defines an equivalence relation between tournaments. It is well known [17] that two tournaments are switching equivalent if and only if their skew-adjacency matrices are $\{-1, 1\}$-diagonally similar. Therefore, the skew-spectrum of a tournament is invariant under switching.

Let T_1 and T_2 be two n-tournaments and let $\{\pm i\alpha_1, \ldots, \pm i\alpha_{\lfloor n/2 \rfloor}\}$ and $\{\pm i\beta_1, \ldots, \pm i\beta_{\lfloor n/2 \rfloor}\}$ be the sets of their nonzero eigenvalues. Using (2.4), the spectral distance between T_1 and T_2 can be expressed as follows:

$$\lambda(T_1, T_2) = 2n(n-1) - 4\sum_{k=1}^{\lfloor n/2 \rfloor} \alpha_k \beta_k. \tag{2.5}$$

The nonzero eigenvalues of an n-transitive tournament are $\pm i \cot\left(\frac{(2k-1)\pi}{2n}\right)$ where $k \in \{1, \ldots, \lfloor n/2 \rfloor\}$ (see for example [26]). Then, by definition and using (2.5), the spectral Slater index of an n-tournament T with nonzero eigenvalues $\pm i\lambda_1, \ldots, \pm i\lambda_{\lfloor n/2 \rfloor}$ is

$$\Lambda(T) = 2n(n-1) - 4\sum_{k=1}^{\lfloor n/2 \rfloor} \lambda_k \cot\left(\frac{(2k-1)\,\pi}{2n}\right). \tag{2.6}$$

By definition, the spectral distance between two tournaments is equal to zero if and only if they have the same skew-spectrum. It follows that the spectral Slater index of an n-tournament T is equal to zero if and only if its nonzero eigenvalues are $\pm i \cot\left(\frac{(2k-1)\pi}{2n}\right)$ where $k \in \{1, \ldots, \lfloor n/2 \rfloor\}$, or equivalently it is switching equivalent to a transitive n-tournament [7]. Tournaments with this property can have arbitrarily large Slater index. For example, let C_n be the n-tournament defined as follows. For any unordered pair $\{i, j\}$ with $1 \leq i < j \leq n$, i dominates j if and only if $j - i \leq \left\lfloor \frac{n}{2} \right\rfloor$. By switching the tournament C_n with respect to $X := \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, we obtain a transitive tournament. Hence, the spectral Slater index of C_n is equal to 0. However, Woirgard [27, Theorem 2] proved that the Slater index of C_n is $\frac{(n^2-1)}{8}$ if n is odd and $\frac{(n^2-2n)}{8}$ if n is even.

3. Proof of Theorem 1.2. Let d be a positive integer and let c be a positive real number. We denote by $K_{d,c}$ the set of points $(x_1, \ldots, x_d)$ of $\mathbb{R}^d$ such that $x_1 \geq \cdots \geq x_d \geq 0$ and $\sum x_i = c$. Clearly, the set $K_{d,c}$ is convex and compact. Moreover, we have the following.

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LEMMA 3.1. *The extreme points of $K_{d,c}$ are $P_1, \dots, P_d$, where P_j is the d-tuple whose j first coordinates are equal to $\frac{c}{j}$ and the others are equal to 0.*

Proof. Let $j \in \{1, \dots, d\}$ and assume that $P_j = \frac{P+Q}{2}$ for some points $P := (p_1, \dots, p_d)$ and $Q := (q_1, \dots, q_d)$ of $K_{d,c}$. Clearly, we have

$$\frac{p_l + q_l}{2} = \begin{cases} \frac{c}{j} & \text{if } l \in \{1, \dots, j\}, \\ 0 & \text{if } l \in \{j+1, \dots, d\}. \end{cases}$$

Then, for $l \in \{j+1, \dots, d\}$, $p_l = q_l = 0$. Moreover, as $p_1 \geq \cdots \geq p_d \geq 0$ and $q_1 \geq \cdots \geq q_d \geq 0$, we have $p_l = p_{l+1}$ and $q_l = q_{l+1}$ for $l \in \{1, \dots, j-1\}$. We conclude that $P = Q = P_j$, because $\sum p_i = \sum q_i = c$. Hence, P_j is an extreme point of $K_{d,c}$.

Consider now an arbitrary point $P := (p_1, \dots, p_d)$ of $K_{d,c}$. Let $\gamma_j = j(p_j - p_{j+1})/c$ for $j \in \{1, \dots, d-1\}$ and $\gamma_d = dp_d/c$. It is not hard to check that $P = \sum \gamma_j P_j$. Moreover $\gamma_j \geq 0$ for $j \in \{1, \dots, d\}$ and $\sum \gamma_j = 1$. Then every point of $K_{d,c}$ is a convex combination of $P_1, \dots, P_d$, which proves the result. ∎

In addition to Lemma 3.1, the proof of Theorem 1.2 requires the following result.

PROPOSITION 3.2. *Let $n \geq 2$ be an integer and let $m = \lfloor n/2 \rfloor$. For every point $(x_1, \dots, x_m)$ of $K_{m,c}$, we have*

$$\sum_{k=1}^{m} \sqrt{x_k} \cot\left(\frac{(2k-1)\pi}{2n}\right) \geq \sqrt{\frac{c}{m}} \sum_{k=1}^{m} \cot\left(\frac{(2k-1)\pi}{2n}\right).$$

Equality holds if and only if $x_i = \dfrac{c}{m}$ for $i \in \{1, \dots, m\}$.

Proof. Consider the function f from the set $K_{m,c}$ to the field of real numbers defined by:

$$f(x_1, \dots, x_m) = \sum_{k=1}^{m} \sqrt{x_k} \cot\left(\frac{(2k-1)\pi}{2n}\right).$$

As $K_{m,c}$ is a convex compact set of $\mathbb{R}^m$ and f is a continuous concave function, by Bauer's maximum principle, f attains its minimum at some extreme point of $K_{m,c}$, namely, by Lemma 3.1, $P_1 := (c, 0, \dots, 0), \dots, P_m := \left(\frac{c}{m}, \dots, \frac{c}{m}\right)$.

We prove that $f(P_k) < f(P_{k-1})$ for every $k \in \{2, \dots, m\}$, which is equivalent after some simplifications to

$$\frac{\cot\left((2k-1)\frac{\pi}{2n}\right)}{\sum_{j=1}^{k-1} \cot\left((2j-1)\frac{\pi}{2n}\right)} < \frac{\sqrt{k}}{\sqrt{k-1}} - 1.$$

Obviously,

$$\sum_{j=1}^{k-1} \cot\left((2j-1)\frac{\pi}{2n}\right) \geq \cot\left(\frac{\pi}{2n}\right).$$

Then,

$$\frac{\cot\left((2k-1)\frac{\pi}{2n}\right)}{\sum_{j=1}^{k-1} \cot\left((2j-1)\frac{\pi}{2n}\right)} \leq \frac{\cot\left((2k-1)\frac{\pi}{2n}\right)}{\cot\left(\frac{\pi}{2n}\right)}.$$

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As the function $x \longmapsto x\cot(x)$ is decreasing on $\left(0, \frac{\pi}{2}\right]$, we have

$$\cot\left(\frac{\pi}{2n}\right) > (2k-1)\cot\left((2k-1)\frac{\pi}{2n}\right).$$

Then,

$$\frac{\cot\left((2k-1)\frac{\pi}{2n}\right)}{\sum_{j=1}^{k-1}\cot\left((2j-1)\frac{\pi}{2n}\right)} < \frac{1}{2k-1} < \frac{\sqrt{k}}{\sqrt{k-1}} - 1.$$

It follows that $f(P_k) < f(P_{k-1})$ and hence P_m is the unique point for which the function f attains its minimum. ∎

Proof of Theorem 1.2. Let T be an n-tournament and let S be its skew-adjacency matrix. Denote by $\pm i\lambda_1, \ldots, \pm i\lambda_m$ the nonzero eigenvalues of S. By (2.6), we have

$$\Lambda(T) = 2n(n-1) - 4\sum_{k=1}^{m}\lambda_k \cot\left(\frac{(2k-1)\,\pi}{2n}\right).$$

Moreover, by (2.4), we have $(\lambda_1^2, \ldots, \lambda_m^2) \in K_{m,c}$ where $c = \frac{n(n-1)}{2}$. Then by Proposition 3.2,

$$\Lambda(T) \leq 2n(n-1) - 4\sqrt{\frac{c}{m}}\sum_{k=1}^{m}\cot\left(\frac{(2k-1)\,\pi}{2n}\right),$$

with equality if and only if $\lambda_1 = \cdots = \lambda_m = \sqrt{\frac{c}{m}}$.

If n is even, then $m = \frac{n}{2}$ and hence

$$\Lambda(T) \leq 2n(n-1) - 4\sqrt{n-1}\sum_{k=1}^{n/2}\cot\left(\frac{(2k-1)\,\pi}{2n}\right),$$

equality holds if and only if $\lambda_1 = \cdots = \lambda_{n/2} = \sqrt{n-1}$ or equivalently the spectrum of S is $\left\{\left[\pm i\sqrt{n-1}\right]^{\frac{n}{2}}\right\}$. Similarly, if n is odd then $m = \frac{n-1}{2}$ and hence

$$\Lambda(T) \leq 2n\,(n-1) - 4\sqrt{n}\sum_{k=1}^{(n-1)/2}\cot\left(\frac{(2k-1)\,\pi}{2n}\right),$$

with equality if and only if the spectrum of S is $\left\{[0]^1, [\pm i\sqrt{n}]^{\frac{n-1}{2}}\right\}$. ∎

4. Tournaments with large spectral Slater index. In this section, we characterize the class of tournaments whose spectral Slater index attains the bound in Theorem 1.2.

Let T be an n-tournament and let S be its skew-adjacency matrix. The spectrum of S is $\left\{\left[\pm i\sqrt{n-1}\right]^{\frac{n}{2}}\right\}$ if and only if S is a *skew-conference matrix*, that is ,$S^2 = (1-n)I_n$. Indeed, if S is a skew-conference matrix, then the minimal polynomial of S is $X^2 + (n-1)$. Hence, its eigenvalues are $\pm i\sqrt{n-1}$, each of multiplicity $\frac{n}{2}$. Conversely, if the spectrum of S is $\left\{\left[\pm i\sqrt{n-1}\right]^{\frac{n}{2}}\right\}$, then the minimal polynomial of S is $X^2 + (n-1)$ and hence $S^2 = (1-n)I_n$.

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Skew-conference $n \times n$ matrices have maximum determinant among $\{0, \pm 1\}$-skew-symmetric matrices and exist only if $n = 2$ or n is divisible by 4. Wallis [25] conjectured that a skew-conference matrix exists for every such order. The reader is referred to [15] for infinite families of skew-conference matrices.

We characterize now, for odd integers n, the n-tournaments with skew-spectrum $\left\{[0]^1, [\pm i\sqrt{n}]^{\frac{n-1}{2}}\right\}$.

It is easy to see that a real matrix S is skew-conference if and only if $S + I$ is a skew-Hadamard matrix. Reid and Brown [19, Theorem 2] proved that for every nonnegative integer k, the existence of a skew-Hadamard matrix of order $4k + 4$ is equivalent to the existence of a doubly regular tournament of order $4k + 3$. From their proof, we have the following result, which is essential to our characterization.

Theorem 4.1. *Let S be the skew-adjacency matrix of an n-tournament T and let*

$$\widehat{S} = \begin{pmatrix} S & \mathbf{1} \\ -\mathbf{1}^t & 0 \end{pmatrix},$$

where $\mathbf{1}$ is the all-ones column vector. The tournament T is doubly regular if and only if $\widehat{S}$ is a skew-conference matrix.

In addition to Theorem 4.1, we need the following lemma.

Lemma 4.2. *Let T be an n-tournament and let S be its skew-adjacency matrix. The skew-spectrum of T is $\left\{[0]^1, [\pm i\sqrt{n}]^{\frac{n-1}{2}}\right\}$ if and only if there exists a column vector $\mathbf{x} \in \{\pm 1\}^n$ such that $\widehat{S} := \begin{pmatrix} S & \mathbf{x} \\ -\mathbf{x}^t & 0 \end{pmatrix}$ is a skew-conference matrix.*

The direct implication of Lemma 4.2 was stated in [9, Lemma 4.7] with $n \equiv 3 \pmod 4$, but the proof does not use this condition. For the converse, we apply Cauchy's interlacing theorem to $i\widehat{S}$ and iS. These matrices are Hermitian; moreover, as $\widehat{S}$ is a skew-conference matrix, the spectrum of $i\widehat{S}$ is $\left\{[\pm\sqrt{n}]^{\frac{n+1}{2}}\right\}$. By interlacing and the fact that 0 is an eigenvalue of iS, $\pm\sqrt{n}$ are eigenvalues of iS each with multiplicity $\frac{n-1}{2}$.

Theorem 4.3. *An n-tournament has skew-spectrum $\left\{[0]^1, [\pm i\sqrt{n}]^{\frac{n-1}{2}}\right\}$ if and only if it is switching equivalent to a doubly regular tournament.*

Proof. Let T be a an n-tournament and let S be its skew-adjacency matrix. Suppose that the skew-spectrum of T is $\left\{[0]^1, [\pm i\sqrt{n}]^{\frac{n-1}{2}}\right\}$. Consider the matrix $\widehat{S}$ as described in Lemma 4.2. Let D be the diagonal matrix whose diagonal entries are the coordinates of $\mathbf{x}$, and let $\widehat{D} = \begin{pmatrix} D & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix}$. It is easy to check that $\widehat{D}\widehat{S}\widehat{D} = \begin{pmatrix} DSD & \mathbf{1} \\ -\mathbf{1}^t & 0 \end{pmatrix}$.

Let T' be the tournament whose skew-adjacency matrix is DSD. This tournament is obtained from T by switching. Moreover, $\widehat{D}\widehat{S}\widehat{D}$ is a skew-conference matrix; hence, by Theorem 4.1, T' is doubly regular. This proves the direct implication.

Conversely, assume that T is switching equivalent to a doubly regular tournament T'. There is a diagonal matrix D with diagonal entries from $\{-1, 1\}$ such that the skew-adjacency matrix of T' is DSD. By Theorem 4.1, the matrix:

$$\begin{pmatrix} DSD & \mathbf{1} \\ -\mathbf{1}^t & 0 \end{pmatrix},$$

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is skew-conference. Therefore, by Lemma 4.2, the skew-spectrum of T' and, *a fortiori*, that of T is $\left\{[0]^1, [\pm i\sqrt{n}]^{\frac{n-1}{2}}\right\}$. ∎

5. Slater index of doubly regular tournaments. In this section, we improve Satake's lower bound given in Theorem 1.1.

Theorem 5.1. *Let T be a doubly regular n-tournament, then*

$$i(T) \geq \frac{1}{2}\binom{n+1}{2} - \frac{\sqrt{n}}{2\pi}\left((n+1)\log(n+1) + (n+3)\right).$$

Proof. Let $\widehat{T}$ be the tournament obtained from T by adding a vertex dominated by all its vertices. By Theorem 4.1, the skew-adjacency matrix of $\widehat{T}$ is a skew-conference matrix, and its skew-spectrum is $\left\{[\pm i\sqrt{n}]^{\frac{n+1}{2}}\right\}$. Hence, by (2.6)

$$\Lambda(\widehat{T}) = 2n\,(n+1) - 4\sqrt{n}\sum_{k=1}^{(n+1)/2}\cot\left(\frac{(2k-1)\,\pi}{2(n+1)}\right).$$

The sum in the right-hand side was considered by Cochrane and Peral in [5, Lemma 3]. They proved that

$$\sum_{k=1}^{(n+1)/2}\cot\left(\frac{(2k-1)\,\pi}{2(n+1)}\right) \leq \frac{(n+1)\log(n+1)}{\pi} + \frac{(n+1)}{\pi}(\gamma - \log(\pi/4)) + \frac{1}{\pi} + \frac{1}{6\pi(n+1)},$$

where $\gamma = 0.577...$ is Euler's constant. As $\gamma - \log(\pi/4) \leq 1$, we get

$$\sum_{k=1}^{(n+1)/2}\cot\left(\frac{(2k-1)\,\pi}{2(n+1)}\right) \leq \frac{(n+1)\log(n+1) + (n+3)}{\pi}.$$

Then,

$$\Lambda(\widehat{T}) \geq 2n\,(n+1) - \frac{4\sqrt{n}}{\pi}\left((n+1)\log(n+1) + (n+3)\right).$$

Clearly, $i(T) = i(\widehat{T})$, moreover, by (1.3), we have $i(\widehat{T}) \geq \frac{1}{8}\Lambda(\widehat{T})$, therefore

$$i(T) \geq \frac{n\,(n+1)}{4} - \frac{\sqrt{n}}{2\pi}\left((n+1)\log(n+1) + (n+3)\right).$$ ∎

6. Remarks.

1. The spectral distance for graphs, considered in [12], has a close connection with the well-studied graph energy. For tournaments, and more generally for oriented graphs, the concept of skew-energy is investigated in several works [2, 7, 11]. Following [2], the *skew-energy* of a tournament T, denoted by $\mathcal{E}_s(T)$, is the sum of the absolute value of its eigenvalues. A relationship between skew-energy and spectral distance of tournaments can be obtained from Chebyshev's sum inequality. Indeed, by applying this inequality to (2.5), we get

$$\lambda(T_1, T_2) \leq 2n(n-1) - \frac{4}{\lfloor n/2\rfloor}\left(\sum_{k=1}^{\lfloor n/2\rfloor}\alpha_k\right)\left(\sum_{k=1}^{\lfloor n/2\rfloor}\beta_k\right).$$

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This inequality can be written as follows:

$$\lambda(T_1, T_2) \leq 2n(n-1) - \frac{1}{\lfloor n/2 \rfloor}\mathcal{E}_s(T_1)\mathcal{E}_s(T_2).$$

2. Let T and R be n-tournaments. If the skew-adjacency matrix of R is a skew-conference matrix, then using (2.5), we get

$$\lambda(T, R) = 2n(n-1) - 2\sqrt{n-1}\,\mathcal{E}_s(T).$$

It is shown in [11] and [7] that an n-tournament has minimal skew-energy if and only if it is switching equivalent to a transitive tournament. The skew-energy of a transitive m-tournament is

$$2\sum_{k=1}^{\lfloor m/2 \rfloor} \cot\left(\frac{(2k-1)\,\pi}{2m}\right).$$

Hence,

$$\lambda(T, R) \leq 2n(n-1) - 4\sqrt{n-1}\sum_{k=1}^{n/2} \cot\left(\frac{(2k-1)\,\pi}{2n}\right),$$

with equality if and only if T is switching equivalent to a transitive tournament. Similarly, if R is doubly regular, then

$$\lambda(T, R) \leq 2n(n-1) - 4\sqrt{n}\sum_{k=1}^{(n-1)/2} \cot\left(\frac{(2k-1)\,\pi}{2n}\right),$$

with equality if and only if T is switching equivalent to a transitive tournament.

3. The *spectral diameter* of the set of n-tournaments, denoted by $\Delta(n)$, is the maximum possible spectral distance between any two n-tournaments. Assuming the existence of a skew-conference matrix of order n, Theorem 1.2 provides a lower bound on the spectral diameter. In other words,
   (i) If $n \equiv 0 \pmod 4$, then $\Delta(n) \geq 2n\,(n-1) - 4\sqrt{n-1}\sum_{k=1}^{n/2} \cot\left(\frac{(2k-1)\pi}{2n}\right)$.
   (ii) If $n \equiv 3 \pmod 4$, then $\Delta(n) \geq 2n\,(n-1) - 4\sqrt{n}\sum_{k=1}^{(n-1)/2} \cot\left(\frac{(2k-1)\pi}{2n}\right)$.

We suspect that these inequalities are in fact equalities. More strongly,

CONJECTURE 6.1. *If T_1 and T_2 are two n-tournaments with maximum spectral distance, then one of them is switching equivalent to a transitive tournament.*

Using SageMath [24], we have verified this conjecture up to $n = 10$.

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