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FULL RANK FACTORIZATION AND THE FLANDERS THEOREM*

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Abstract. In this paper, a method is given that obtains a full rank factorization of a rectangular matrix. It is studied when a matrix has a full rank factorization in echelon form. If this factorization exists, it is proven to be unique. Applying the full rank factorization in echelon form the Flanders theorem and its converse in a particular case are proven.

Key words. Echelon form of a matrix, LU factorization, Full rank factorization, Flanders theorem.

AMS subject classifications. 15A15, 15A23.

1. Introduction. Triangular factorizations of matrices play an important role in solving linear systems. It is known that the LDU factorization is unique for square nonsingular matrices and for full row rank rectangular matrices. In any other case, the LDU factorization is not unique and the orders of L, D and U are greater than the rank of the initial matrix.

We focus our attention on matrices $A \in \mathbb{R}^{n\times m}$ with $\operatorname{rank}(A) = r \leq \min\{n, m\}$, where the LDU factorization of A is not unique. For this kind of matrices, it is useful to consider the *full rank factorization of* A, that is, a decomposition in the form $A = FG$ with $F \in \mathbb{R}^{n\times r}$, $G \in \mathbb{R}^{r\times m}$ and $\operatorname{rank}(F) = \operatorname{rank}(G) = r$. The full rank factorization of any nonzero matrix is not unique. In addition, if $A = FG$ is a full rank factorization of A, then any other full rank factorization can be written in the form $A = (FM^{-1})(MG)$, where $M \in \mathbb{R}^{r\times r}$ is a nonsingular matrix.

If the full rank factorization of A is given by $A = LDU$, where $L \in \mathbb{R}^{n\times r}$ is in lower echelon form, $D = \operatorname{diag}(d_1, d_2, \ldots, d_r)$ is nonsingular and $U \in \mathbb{R}^{r\times m}$ is in upper echelon form, then this factorization is called a *full rank factorization in echelon form of* A.

In this paper we give a method to obtain a full rank factorization of a rectangular matrix and we study when this decomposition can be in echelon form. Moreover, if the factorization in echelon form exists, we prove that it is unique. Finally, applying

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the full rank factorization in echelon form, we give a simple proof of the Flanders theorem [4] for matrices $A \in \mathbb{R}^{n\times r}$ and $B \in \mathbb{R}^{r\times n}$ with $\operatorname{rank}(A) = \operatorname{rank}(B) = r$, as well as of the converse result.

The full rank factorization of matrices has different applications, for instance, in control theory to obtain minimal realizations of polynomial transfer matrices by using the Silverman-Ho algorithm [2, 3]; in numerical analysis to obtain a Cholesky full rank factorization or to extend the thin QR factorization to rectangular matrices without full rank [6]; in matrix analysis to obtain the nonzero eigenvalues and their associated eigenvectors or the singular values of a matrix.

Moreover, this factorization allows us to characterize some particular classes of matrices because not all matrices have a full rank factorization in echelon form. Specifically, in [1, 5] it is proven that totally positive (resp., strictly totally positive) matrices, that is matrices with all theirs minors greater than or equal to zero (resp., greater than zero), and totally nonpositive (resp. negative) matrices, that is matrices with all theirs minors less than or equal to zero (resp. less than zero), have full rank factorizations in echelon form. Totally positive (or strictly totally positive) matrices appear in numerical mathematics, economics, statistics etc., whereas totally nonpositive (or negative) matrices are a generalization of N-matrices which have applications in economic problems.

2. Quasi-Gauss elimination process. It is known that the Gauss elimination process consists of producing zeros in a column of a matrix by adding to each row an appropriate multiple of a fixed row, and the Neville elimination method obtains the zeros in a column by adding to each row an appropriate multiple of the previous one. In both processes, reordering of rows may be necessary. In this sense, the Gauss elimination method can be considered more general than the Neville elimination process, because if the Neville process with no pivoting can be applied, then the Gauss process with no pivoting can also be applied, but the converse is not true in general.

When we can apply the Gauss elimination process with no pivoting to a singular matrix, the factorization obtained is not unique and it is not a full rank factorization. Therefore, in this paper we consider a new method which allows us to obtain a full rank factorization of a singular matrix. This method, which we call *quasi-Gauss elimination process*, is based on the Gaussian and the quasi-Neville elimination [5].

Moreover, as in the Gauss and Neville processes, we can assure that the quasi-Gauss elimination process is more general than the quasi-Neville elimination process, as we will see in Remark 2.3.

We denote by $F_n^{\{j_1,j_2,\ldots,j_k\}}$ (resp., $C_n^{\{j_1,j_2,\ldots,j_k\}}$) the matrix obtained from the $n \times n$ identity matrix by deleting the columns (resp., rows) $j_1, j_2, \ldots, j_k$, and we can

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suppose, without loss of generality, that A has no zero rows or columns. This is so because, if A has the $j_1, j_2, \cdots, j_s$ zero rows and the $i_1, i_2, \cdots, i_r$ zero columns, $1 \leq s \leq n$, $1 \leq r \leq m$, using $F_n^{\{j_1,j_2,\cdots,j_s\}}$ and $C_m^{\{i_1,i_2,\cdots,i_r\}}$ we obtain

$$A = F_n^{\{j_1,j_2,\cdots,j_s\}} \tilde{A} C_m^{\{i_1,i_2,\cdots,i_r\}},$$

where $\tilde{A} \in \mathbb{R}^{(n-s)\times(m-r)}$ has no zero rows or columns. If $\tilde{L}\tilde{D}\tilde{U}$ is a full rank factorization of $\tilde{A}$ then

$$LDU = \left(F_n^{\{j_1,j_2,\cdots,j_s\}} \tilde{L}\right) \tilde{D} \left(\tilde{U} C_m^{\{i_1,i_2,\cdots,i_r\}}\right) \tag{2.1}$$

is a full rank factorization of A. Note that if $\tilde{L}\tilde{D}\tilde{U}$ is a full rank factorization in echelon form of $\tilde{A}$, then (2.1) is a full rank factorization in echelon form of A.

From now on, we denote by $E_{i,j}(m_{ij})$ the *elementary matrix* which differs from the identity matrix only in its (i,j) entry m_{ij}.

Algorithm 2.1 (*Quasi-Gauss elimination process*).

- Consider $A \in \mathbb{R}^{n\times m}$ with $\operatorname{rank}(A) = r \leq \min\{n, m\}$. If A has no zero rows let $\bar{A} = A$. Otherwise, $\bar{A}$ is obtained from A by deleting its zero rows, that is, $A = F_n^{\{i_1,i_2,\ldots,i_s\}} \bar{A}$, where $i_1, i_2, \ldots, i_s$ are the indices of the zero rows of A.
- Apply the first iteration of the Gauss elimination process to $\bar{A}$ to obtain $A_{(1)} = E_{(1)}\bar{A}$, where $E_{(1)}$ is the product of the corresponding elementary matrices in the first iteration of the Gauss algorithm, i.e.,
$$E_{(1)} = E^{(1)}_{n,1}(m_{n1}) E^{(1)}_{n-1,1}(m_{n-1,1}) \ldots E^{(1)}_{2,1}(m_{21}).$$
- If $A_{(1)}$ has no zero rows, then $\bar{A}_{(1)} = A_{(1)}$. Otherwise, obtain $\bar{A}_{(1)}$ from $A_{(1)}$ by deleting the zero rows.
- Continue in this way until an $r \times m$ matrix DU is obtained, where $D \in \mathbb{R}^{r\times r}$ is a nonsingular diagonal matrix and $U \in \mathbb{R}^{r\times m}$ is an upper echelon matrix with $\operatorname{rank}(U) = r$.

Note that, since the Gauss elimination process has been applied, it follows that $A = LDU$, where $L \in \mathbb{R}^{n\times r}$ with $\operatorname{rank}(L) = r$. Moreover, when pivoting is not necessary, L is a lower echelon matrix and the full rank LDU factorization obtained is in echelon form, as explained in the following example.

Example 2.2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ -1 & -1 & 2 & -2 & -5 \\ 2 & 3 & -1 & 0 & -2 \\ 2 & 3 & 5 & -1 & 2 \\ 0 & 1 & 4 & 0 & -1 \end{bmatrix}.$$

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Since A has no zero rows, we have $\bar{A} = A$. Then by applying the first iteration of the Gauss elimination process

$$A_{(1)} = E_{4,1}^{(1)}(-2)E_{3,1}^{(1)}(-2)E_{2,1}^{(1)}(1)\bar{A} = \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -4 & -12 \\ 0 & 1 & 9 & -5 & -8 \\ 0 & 1 & 4 & 0 & -1 \end{bmatrix}.$$

By deleting the zero row we obtain

$$A_{(1)} = F_5^{\{2\}} \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & -4 & -12 \\ 0 & 1 & 9 & -5 & -8 \\ 0 & 1 & 4 & 0 & -1 \end{bmatrix} = F_5^{\{2\}} \bar{A}_{(1)}.$$

Now the second iteration of the Gauss process is applied to $\bar{A}_{(1)}$ giving

$$A_{(2)} = E_{4,2}^{(2)}(-1)E_{3,2}^{(2)}(-1)\bar{A}_{(1)} = \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & -4 & -12 \\ 0 & 0 & 6 & -1 & 4 \\ 0 & 0 & 1 & 4 & 11 \end{bmatrix}.$$

This matrix has no zero row, so $\bar{A}_{(2)} = A_{(2)}$ and following with the third iteration of the Gauss elimination process we obtain

$$A_{(3)} = E_{4,3}^{(3)}(-1/6)\bar{A}_{(2)} = \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & -4 & -12 \\ 0 & 0 & 6 & -1 & 4 \\ 0 & 0 & 0 & 25/6 & 31/3 \end{bmatrix}$$

which can be written as

$$A_{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 25/6 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & -4 & -12 \\ 0 & 0 & 1 & -1/6 & 2/3 \\ 0 & 0 & 0 & 1 & 62/25 \end{bmatrix} = DU,$$

where $D \in \mathbb{R}^{4\times 4}$ is a nonsingular diagonal matrix and $U \in \mathbb{R}^{4\times 5}$ is an upper echelon matrix with $\text{rank}(U) = 4$. Finally, we have that the full rank factorization in echelon form of A is

$$A = \left(E_{2,1}^{(1)}(-1)E_{3,1}^{(1)}(2)E_{4,1}^{(1)}(2)F_5^{\{2\}}E_{3,2}^{(2)}(1)E_{4,2}^{(2)}(1)E_{4,3}^{(3)}(1/6)\right) DU = LDU,$$

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where $L \in \mathbb{R}^{5\times 4}$ is the following lower echelon matrix with $\text{rank}(L) = 4$,

$$L = E_{2,1}^{(1)}(-1)E_{3,1}^{(1)}(2)E_{4,1}^{(1)}(2)F_5^{\{2\}}E_{3,2}^{(2)}(1)E_{4,2}^{(2)}(1)E_{4,3}^{(3)}(1/6) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1/6 & 1 \end{bmatrix}.$$

Remark 2.3. Note that it is not possible to apply the quasi-Neville elimination process to A without pivoting. Therefore, the full rank factorization of A obtained applying this method is not a full rank factorization in echelon form. Specifically, if we apply the quasi-Neville elimination process to A we have that

$$A_{(1)} = E_2^{(1)}(1)E_3^{(1)}(-2)E_4^{(1)}(-1)\bar{A} = \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -4 & -12 \\ 0 & 0 & 6 & -1 & 4 \\ 0 & 1 & 4 & 0 & -1 \end{bmatrix} = F_5^{\{2\}}\bar{A}_{(1)},$$

where

$$\bar{A}_{(1)} = \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & -4 & -12 \\ 0 & 0 & 6 & -1 & 4 \\ 0 & 1 & 4 & 0 & -1 \end{bmatrix}.$$

Now, it is not possible to apply the quasi-Neville elimination process to $\bar{A}_{(1)}$ without interchange of rows. Finally, the full rank factorization of A that we obtain applying this method is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 6 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 & 2 & 5 \\ 0 & 1 & 3 & -4 & -12 \\ 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & 0 & -25 & -62 \end{bmatrix} = FG$$

which is not in echelon form. Therefore, as we comment at the beginning of this section, we can conclude that the quasi-Gauss elimination process is more general that the quasi-Neville elimination process.

3. Full rank factorization in echelon form. In this section we derive a necessary and sufficient condition for a matrix to have full rank decomposition in echelon form. Moreover, we prove that if the full rank factorization in echelon form exists then it is unique.

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THEOREM 3.1. *Let $A \in \mathbb{R}^{n\times m}$ be a matrix with* $\operatorname{rank}(A) = r \le \min\{n, m\}$. *Then A admits a full rank factorization in echelon form if and only if the upper echelon form of the first r linearly independent rows of A can be obtained with no pivoting.*

Proof. Let $A_1 \in \mathbb{R}^{r\times m}$ be the matrix formed by the r first linearly independent rows of A. Then, there exists a unique reduced lower echelon matrix $F_1 \in \mathbb{R}^{n\times r}$ such that $A = F_1A_1$.

Suppose A_1 is transformed to an upper echelon form with no pivoting, so there exists a *unique factorization* $L_1D_1U_1$, where $L_1 \in \mathbb{R}^{r\times r}$ is a unit lower triangular matrix, $D_1 \in \mathbb{R}^{r\times r}$ is a nonsingular diagonal matrix and $U_1 \in \mathbb{R}^{r\times m}$ is an upper echelon matrix with $\operatorname{rank}(U) = r$. Therefore,

$$A = F_1(L_1D_1U_1) = (F_1L_1)D_1U_1 = LDU,$$

where $L = F_1L_1 \in \mathbb{R}^{n\times r}$ is a lower echelon matrix, $D = D_1 \in \mathbb{R}^{r\times r}$ is a nonsingular diagonal matrix, $U = U_1 \in \mathbb{R}^{r\times m}$ is an upper echelon matrix and $\operatorname{rank}(U) = \operatorname{rank}(L) = r$.

Now, suppose that A has the full rank factorization in echelon form $A = LDU$, where $L \in \mathbb{R}^{n\times r}$ is a lower echelon matrix, $U \in \mathbb{R}^{r\times m}$ is an upper echelon matrix, $\operatorname{rank}(L) = \operatorname{rank}(U) = r$ and $D \in \mathbb{R}^{r\times r}$ is a nonsingular matrix. The lower echelon matrix L has the following structure

$$L = \begin{array}{cc} \begin{bmatrix} L_{11} & O & \cdots & O \\ L_{21} & L_{22} & \cdots & O \\ \vdots & \vdots & & \vdots \\ L_{r1} & L_{r2} & \cdots & L_{rr} \end{bmatrix} & \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_r \end{array} \end{array} \qquad s_1 + s_2 + \cdots + s_r = n,$$

where

$$L_{ij} = \begin{bmatrix} l_{s_1+s_2+\cdots+s_{i-1}+1,j} \\ l_{s_1+s_2+\cdots+s_{i-1}+2,j} \\ \vdots \\ l_{s_1+s_2+\cdots+s_{i-1}+s_i,j} \end{bmatrix} \in \mathbb{R}^{s_i\times 1},$$

for $i, j = 1, 2, \ldots, r$ with $j < i$, and

$$L_{ii} = \begin{bmatrix} 1 \\ l_{s_1+s_2+\cdots+s_{i-1}+2,i} \\ l_{s_1+s_2+\cdots+s_{i-1}+3,i} \\ \vdots \\ l_{s_1+s_2+\cdots+s_{i-1}+s_i,i} \end{bmatrix} \in \mathbb{R}^{s_i\times 1}$$

for $i = j = 1, 2, \ldots, r$.

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From this structure we have that rows $1, s_1+1, s_1+s_2+1, \ldots, s_1+s_2+\ldots+s_{r-1}+1$ of A are linearly independent, whereas rows $s_1+s_2+\cdots+s_{i-1}+2$, $s_1+s_2+\cdots+s_{i-1}+3, \ldots, s_1+s_2+\cdots+s_{i-1}+s_i$ are linear combinations of the rows $1, s_1+1$, $s_1+s_2+1, \ldots, s_1+s_2+\ldots+s_{i-1}+1$, for $i=1,2,\ldots,r$, where we put $s_0=0$ if $i=1$.

Moreover, L can be written as $L=F\bar{L}$, where F is a matrix in reduced lower echelon form, with the leading 1's in the corresponding first linearly independent rows of A, i.e., rows $1, s_1+1, s_1+s_2+1, \ldots, s_1+s_2+\ldots+s_{r-1}+1$, and $\bar{L}$ is equal to

$$\bar{L}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ l_{s_1+1,1} & 1 & \cdots & 0 & 0 \\ l_{s_1+s_2+1,1} & l_{s_1+s_2+1,2} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ l_{s_1+s_2+\ldots+s_{r-2}+1,1} & l_{s_1+s_2+\ldots+s_{r-2}+1,2} & \cdots & 1 & 0 \\ l_{s_1+s_2+\ldots+s_{r-1}+1,1} & l_{s_1+s_2+\ldots+s_{r-1}+1,2} & \cdots & l_{s_1+s_2+\ldots+s_{r-1}+1,r-1} & 1 \end{bmatrix}.$$

Consequently it follows that

$$A[\{1, s_1+1, s_1+s_2+1, \ldots, s_1+s_2+\ldots s_{r-1}+1\}|\{1,2,\ldots,m\}]=\bar{L}DU.$$

In other words, the upper echelon form of the first r linearly independent rows of A can be obtained without interchange of rows. $\square$

Remark 3.2. If the leading principal submatrices of the first r linearly independent rows are nonsingular, then the matrix formed by these rows is reducible with no pivoting. So this is a sufficient condition to guarantee that the upper echelon form of the matrix formed by the first r linearly independent rows of A can be obtained with no pivoting, but not, in general, a necessary condition.

If $A_1 \in \mathbb{R}^{r\times m}$ is the submatrix formed by the first r linearly independent rows, it is not difficult to prove that the necessary and sufficient conditions for this matrix to be reducible to the echelon form with no pivoting are:

1. If $r=m$: $\det A_1[1,2,\ldots,k] \neq 0, \quad \forall k=1,2,\ldots,r$.
2. If $r<m$, suppose that $A_1[1,2,\ldots,k+1]$ is the first leading principal submatrix such that $\det A_1[1,2,\ldots,k+1]=0$. Since A_1 has full row rank there exists, at least, an index j, $k+1<j\leq m$ such that

$$\det A_1[1,2,\ldots,k,k+1|1,2,\ldots,k,j] \neq 0.$$

Let j_0 be the first index for which inequality holds, then we need that

$$\det A_1[1,2,\ldots,k,s|1,2,\ldots,k,t]=0,$$

for all $s=k+2,k+3,\ldots,r$ and $t=k+1,k+2,\ldots,j_0-1$.

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REMARK 3.3. The full rank factorization in echelon form allows us to know, from the rows of L with a leading 1, the linear independent rows beginning from the top of A. Therefore, if we have two different full rank factorizations in echelon form, then one row can be independent or dependent of the rows above it at the same time, which is absurd. Taking into account this comment, we can give the following result.

THEOREM 3.4. *Let $A \in \mathbb{R}^{n\times m}$ be a matrix with* $\operatorname{rank}(A) = r \leq \min\{n, m\}$. *If A has a full rank factorization in echelon form, then this factorization is unique.*

Proof. Suppose that there exist two full rank factorizations in echelon form of A

$$A = L_1D_1U_1 = L_2D_2U_2,$$

where $L_1,\ L_2 \in \mathbb{R}^{n\times r}$ are lower echelon matrices with $\operatorname{rank}(L_1) = \operatorname{rank}(L_2) = r$, $D_1 = \operatorname{diag}(d_{11}, d_{12}, \ldots, d_{1r})$ and $D_2 = \operatorname{diag}(d_{21}, d_{22}, \ldots, d_{2r})$ nonsingular matrices and $U_1,\ U_2 \in \mathbb{R}^{r\times m}$ are upper echelon matrices with $\operatorname{rank}(U_1) = \operatorname{rank}(U_2) = r$.

From Remark 3.3 we have that necessarily $L_1 = L_2 = L$. Then,

$$A = LD_1U_1 = LD_2U_2.$$

Since L can be written, in a unique way, as $L = FL_{11}$, where F is a reduced lower echelon matrix and L_{11} is a unit lower triangular matrix, then

$$A = FL_{11}D_1U_1 = FL_{11}D_2U_2.$$

From this equality $L_{11}D_1U_1$ and $L_{11}D_2U_2$ are two different factorizations with no pivoting of the submatrix $A_1 \in \mathbb{R}^{r\times m}$ formed by the first r linearly independent rows of A, which it is not possible. Therefore, $D_1 = D_2$ and $U_1 = U_2$. ☐

REMARK 3.5. We have proven that the full rank factorization in echelon form of A exists if the upper echelon form of the first r linearly independent rows can be obtained with no pivoting, and in this case we have obtained the factorization. We want to point that if $\tilde{A}_1$ is the submatrix formed by any r linear independent rows of A and the full rank factorization in echelon form of A exists, then it can be obtained from $\tilde{A}_1$ if the following conditions hold:

(i) The matrix F_1, such that $A = F_1\tilde{A}_1$, is in lower echelon form.
(ii) The echelon form of $\tilde{A}_1$ can be obtained without interchange of rows.

4. The Flanders theorem. The full rank factorization in echelon form allows us to give a simple prove of the Flanders theorem in the case that $A \in \mathbb{R}^{n\times r}$, $B \in \mathbb{R}^{r\times n}$ and $\operatorname{rank}(A) = \operatorname{rank}(B) = r$. Flanders [4] proved that the difference between the Jordan blocks sizes associated with the eigenvalue zero of matrices AB and BA is -1, 0 or 1 for all blocks. We prove that this difference is always equal to 1 in this particular case.

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THEOREM 4.1. *Let $C = AB \in \mathbb{R}^{n\times n}$ and $D = BA \in \mathbb{R}^{r\times r}$ be matrices with $A \in \mathbb{R}^{n\times r}$, $B \in \mathbb{R}^{r\times n}$ and* $\mathrm{rank}(A) = \mathrm{rank}(B) = r$*. Then C and D have the same elementary divisors with nonzero roots. Moreover, if $k_1 \geq k_2 \geq \cdots \geq k_p$ (resp. $k'_1 \geq k'_2 \geq \cdots \geq k'_p$) are the Jordan blocks sizes associated with the eigenvalue zero in AB (resp. BA), then $k_i - k'_i = 1$ for all i.*

Proof. Since $\mathrm{rank}(C) = r$, the product AB is a full rank factorization of C. Suppose that its Jordan form is

$$J_C = \begin{bmatrix} J_{C_0} & O \\ O & J_t \end{bmatrix},$$

where $J_t \in \mathbb{R}^{n_t\times n_t}$ is the block containing the Jordan blocks associated with the eigenvalues $\lambda_i \neq 0$, J_{C_0} is the Jordan block associated with the eigenvalue $\lambda = 0$, with $k_1 \geq k_2 \geq \cdots \geq k_p \geq 1$ are the sizes of the corresponding Jordan blocks, that is,

$$J_{C_0} = \begin{bmatrix} J_0^{(k_1)} & O & \dots & O \\ O & J_0^{(k_2)} & \dots & O \\ \vdots & \vdots & & \vdots \\ O & O & \dots & J_0^{(k_p)} \end{bmatrix}.$$

Suppose that $k_1 \geq k_2 \geq \cdots \geq k_q > k_{q+1} = \cdots = k_p = 1$. The full rank factorization in echelon form of J_C is $L_{J_C}U_{J_C}$, where $L_{J_C} \in \mathbb{R}^{(k_1+\cdots+k_q+c+n_t)\times(k_1-1+\cdots+k_q-1+n_t)}$ and $U_{J_C} \in \mathbb{R}^{(k_1-1+\cdots+k_q-1+n_t)\times(k_1+\cdots+k_q+c+n_t)}$, with $c = k_{q+1} + \cdots + k_p$, are the following matrices:

$$L_{J_C} = \left[\begin{array}{c|c|c|c|c} I_{k_1-1} & O & \dots & O & O \\ 0\dots0 & 0\dots0 & \dots & 0\dots0 & 0\dots0 \\ \hline O & I_{k_2-1} & \dots & O & O \\ 0\dots0 & 0\dots0 & \dots & 0\dots0 & 0\dots0 \\ \hline \vdots & \vdots & & \vdots & \vdots \\ \hline O & O & \dots & I_{k_q-1} & O \\ 0\dots0 & 0\dots0 & \dots & 0\dots0 & 0\dots0 \\ \hline O & O & \dots & O & O \\ \hline O & O & \dots & O & J_t \end{array}\right],$$

$$U_{J_C} = \left[\begin{array}{cc|cc|c|cc|c|c} 0 & I_{k_1-1} & 0 & O & \cdots & 0 & O & O & O \\ \hline 0 & O & 0 & I_{k_2-1} & \cdots & 0 & O & O & O \\ \hline \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & O & 0 & O & \cdots & 0 & I_{k_q-1} & O & O \\ \hline 0 & O & 0 & O & \cdots & 0 & O & O & I_{n_t} \end{array}\right].$$

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Let S be the nonsingular matrix such that

$$S^{-1}CS = J_C = L_{J_C}U_{J_C}.$$

From this expression we have

$$C = (SL_{J_C})(U_{J_C}S^{-1}) = F_C G_C,$$

that is, $F_C G_C$ is a full rank factorization of C. Since AB is also a full rank factorization of C there exists a nonsingular matrix M such that

$$C = F_C M^{-1} M G_C = (SL_{J_C})M^{-1}M(U_{J_C}S^{-1}) = (SL_{J_C}M^{-1})(MU_{J_C}S^{-1}) = AB.$$

Then

$$\begin{aligned} D = BA = (MU_{J_C}S^{-1})(SL_{J_C}M^{-1}) = (MU_{J_C})(L_{J_C}M^{-1}) = \\ = M(U_{J_C}L_{J_C})M^{-1} = M \begin{bmatrix} J_{D_0} & O \\ O & J_t \end{bmatrix} M^{-1}, \end{aligned}$$

where

$$J_{D_0} = \begin{bmatrix} J_0^{(k_1-1)} & O & \dots & O \\ O & J_0^{(k_2-1)} & \dots & O \\ \vdots & \vdots & & \vdots \\ O & O & \dots & J_0^{(k_q-1)} \end{bmatrix}.$$

Therefore, the result holds. □

Observe that

1. If $k_1 = k_2 = \cdots = k_p = 1$ then $J_{C_0} = O$ and D is similar to J_t.
2. $\operatorname{rank}(D) = \operatorname{rank}(C^2)$.

Remark 4.2. Consider $A \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(A) = r < n$. Let $A = FU$ be a full rank factorization of A, with $F \in \mathbb{R}^{n\times r}$, $U \in \mathbb{R}^{r\times n}$ and $\operatorname{rank}(F) = \operatorname{rank}(U) = r$. If $A_2 = UF$, then by Theorem 4.1 we have $\operatorname{rank}(A^2) = \operatorname{rank}(A_2)$. In the case where A_2 is singular, we apply Theorem 4.1 again and we obtain a new matrix A_3 such that $\operatorname{rank}(A^3) = \operatorname{rank}(A_2^2) = \operatorname{rank}(A_3)$. Proceeding in this way, we construct a sequence of matrices $A_2, A_3, \ldots, A_w$ such that

$$\operatorname{rank}(A_i) = \operatorname{rank}(A^i), \qquad i = 2, 3, \ldots, w$$

with A_w nonsingular. If we know the Jordan structure of A_w, and taking into account the rank of the matrices A_i, $i = 2, 3, \ldots, w-1, w$, we obtain the Jordan structure of A.

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Now we obtain the converse result.

Theorem 4.3. *Let C and D be matrices with the same elementary divisors with nonzero root and let $k_1 \geq k_2 \geq \cdots \geq k_p$, $k'_1 \geq k'_2 \geq \cdots \geq k'_p$ be the Jordan blocks sizes associated with the eigenvalue zero in C and D, respectively, such that $k_i - k'_i = 1$ for all i. Then there exist two matrices A and B with full column and row rank, respectively, such that both $AB = C$ and $BA = D$ exist.*

Proof. By the hypothesis the Jordan form of C is

$$J_C = \begin{bmatrix} J_{C_0} & O \\ O & J_t \end{bmatrix} = \left[\begin{array}{cccc|c} J_0^{(k_1)} & O & \dots & O & O \\ O & J_0^{(k_2)} & \dots & O & O \\ \vdots & \vdots & & \vdots & \vdots \\ O & O & \dots & J_0^{(k_p)} & O \\ \hline O & O & \dots & O & J_t \end{array}\right].$$

If we suppose that $k_1 \geq k_2 \geq \cdots \geq k_q > k_{q+1} = \cdots = k_p = 1$, then $\operatorname{rank}(C) = (k_1 - 1) + (k_2 - 1) + \cdots + (k_q - 1) + n_t = r$. By theorem 4.1, J_C admits the following full rank factorization in echelon form

$$J_C = L_{J_C} U_{J_C},$$

where $L_{J_C} \in \mathbb{R}^{n \times r}$ and $U_{J_C} \in \mathbb{R}^{r \times n}$, with n the order of C.

Let $S_c \in \mathbb{R}^{n \times n}$ be the nonsingular matrix such that

$$S_c^{-1} C S_c = J_C = L_{J_C} U_{J_C} \quad \Longrightarrow \quad C = (S_c L_{J_C}) \left(U_{J_C} S_c^{-1}\right) = F_C G_C,$$

where $F_C \in \mathbb{R}^{n \times r}$ has full column rank and $G_C \in \mathbb{R}^{r \times n}$ has full row rank.

On the other hand, since the Jordan form of D is

$$J_D = \begin{bmatrix} J_{D_0} & O \\ O & J_t \end{bmatrix} = \left[\begin{array}{cccc|c} J_0^{(k_1-1)} & O & \dots & O & O \\ O & J_0^{(k_2-1)} & \dots & O & O \\ \vdots & \vdots & & \vdots & \vdots \\ O & O & \dots & J_0^{(k_q-1)} & O \\ \hline O & O & \dots & O & J_t \end{array}\right] = U_{J_C} L_{J_C},$$

we have that $D \in \mathbb{R}^{r \times r}$ and $\operatorname{rank}(D) = (k_1 - 2) + (k_2 - 2) + \cdots + (k_q - 2) + n_t = r - q = \operatorname{rank}(C^2)$.

Let $S_d \in \mathbb{R}^{r \times r}$ be the nonsingular matrix such that

$$D = S_d J_D S_d^{-1} = S_d U_{J_C} L_{J_C} S_d^{-1} = S_d (U_{J_C} S_c^{-1})(S_c L_{J_C}) S_d^{-1} = (S_d G_C)(F_C S_d^{-1}).$$

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If we write $A = F_C S_d^{-1} \in \mathbb{R}^{n\times r}$ and $B = S_d G_C \in \mathbb{R}^{r\times n}$, then $\operatorname{rank}(A) = \operatorname{rank}(B) = r$ and

$$C = F_C G_C = (F_C S_d^{-1})(S_d G_C) = AB, \quad D = (S_d G_C)(F_C S_d^{-1}) = BA. \square$$

REFERENCES

[1] R. Cantó, B. Ricarte, and A. M. Urbano. Full rank factorization in echelon form of totally nonpositive (negative) rectangular matrices. *Linear Algebra and its Applications*, to appear.
[2] R. Cantó, B. Ricarte, and A. M. Urbano. Computation of realizations of complete singular systems. Preprint.
[3] L. Dai. *Singular control systems.* Lecture Notes in Control and Information Sciences, 118, Springer-Verlag, 1989.
[4] H. Flanders. Elementary divisors of AB and BA. *Proceedings of the American Mathematical Society*, 2(6):871–874, 1951.
[5] M. Gassó and J.R. Torregrosa. A totally positive factorization of rectangular matrices by the Neville elimination. *SIAM Journal on Matrix Analysis and Applications*, 25:986–994, 2004.
[6] J. H. Golub and C. F. Van Loan. *Matrix Computations.* The John Hopkins Univ. Press, 1966.