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$\mathrm{M}_\vee$- MATRICES : A GENERALIZATION OF M-MATRICES BASED ON EVENTUALLY NONNEGATIVE MATRICES*

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Abstract. An $\mathrm{M}_\vee$- matrix has the form $A = sI - B$, where $s \geq \rho(B) \geq 0$ and B is eventually nonnegative; i.e., B^k is entrywise nonnegative for all sufficiently large integers k. A theory of $\mathrm{M}_\vee$- matrices is developed here that parallels the theory of M-matrices, in particular as it regards exponential nonnegativity, spectral properties, semipositivity, monotonicity, inverse nonnegativity and diagonal dominance.

Key words. M-matrix, Eventually nonnegative matrix, Exponentially nonnegative matrix, Perron-Frobenius.

AMS subject classifications. 15A48.

1. Introduction.

An M-matrix has the form $A = sI - B$, where $B \in \mathbb{R}^{n \times n}$ is (entrywise) nonnegative with $s \geq \rho(B)$, the spectral radius of B. M-matrices have been studied extensively and their properties documented in several books; e.g., [2, 7]. Evidently, M-matrices and in particular their spectral properties and eigenstructure are inextricably related to the Perron-Frobenius theory of nonnegative matrices. Additionally, M-matrices occur in numerous applications. An extensive theory of M-matrices has been developed relative to their role in numerical analysis (e.g., splittings in iterative methods and discretization of differential equations), in modeling of the economy, optimization and Markov chains; see e.g., [1, 2].

Several generalizations of the notion of an M-matrix have been given. In one case, entrywise nonnegativity of B is replaced by cone nonnegativity. The cone can be either finitely or infinitely generated, resulting in different approaches and results; see e.g., [2, 13] and references therein. Another interesting generalization is in terms of block partitions of B; see [10].

More recently, some generalizations of M-matrices have been introduced that are

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based on replacing nonnegativity of the matrix B with conditions related to eventual nonnegativity (positivity) of B; i.e., the existence of a positive integer $k_0 \geq 1$ such that B^k is entrywise nonnegative (positive) for all $k \geq k_0$. We briefly describe these generalizations next.

In [8], pseudo M-matrices are introduced. These are matrices of the form $A = sI - B$, where $s > \rho(B) > 0$, $\rho(B)$ is a simple eigenvalue that is strictly greater than the modulus of any other eigenvalue of B, and the left and right eigenvectors associated with $\rho(B)$ are strictly positive. It turns out that these assumptions on B are equivalent to eventual positivity of B; see [8, Theorem 1]. In [8, Theorem 8] it is shown that the inverse of a pseudo M-matrix is eventually positive; see also [5, Theorem 3.2].

In [9], matrices of the form $A = sI - B$ are also considered, where $s > \rho(B)$ with B irreducible and eventually nonnegative. In [9, Theorem 4.2] it is shown that if an eventually nonnegative matrix B is irreducible and the index of the eigenvalue 0 of B is at most 1, then there exists $\beta > \rho(B)$ such that $A = sI - B$ has a positive inverse for all $s \in (\rho(B), \beta)$.

In [5], the authors consider the above classes of matrices, as well as the class of EM-matrices, which are of the form $A = sI - B$, where $s \geq \rho(B) > 0$ and B is eventually nonnegative. The classes of M-matrices and pseudo M-matrices are both proper subsets of the class of EM-matrices; see [5, Fig. 2.1]. Properties of the eigenvalues, inverses, and splittings of a proper superset of EM-matrices are developed in [5].

In this paper we define and examine the following class of matrices, which is slightly different from those considered in [5, 8, 9]. An $M_\vee$- matrix has the form $A = sI - B$, where $s \geq \rho(B) \geq 0$ with B eventually nonnegative. The class of $M_\vee$- matrices is a proper superset of the class of EM-matrices. In Sections 3 and 4, we show that $M_\vee$- matrices satisfy many analogous properties of M-matrices. This is accomplished by borrowing theory and techniques used in the cone-theoretic generalizations of M-matrices, and by drawing on recent advancements in the study of eventually nonnegative matrices. The latter have been the focus of study in recent years by many researchers, partly because they generalize and shed new light on the class of nonnegative matrices (see [3, 4, 8, 11, 18, 19]) and partly because of their applications in systems theory (see [12, 15, 16, 17]).

2. Notation and definitions.

Given $X \in \mathbb{R}^{n\times n}$, the spectrum of X is denoted by $\sigma(X)$ and its spectral radius by $\rho(X) = \max\{|\lambda| \mid \lambda \in \sigma(X)\}$. The degree of 0 as a root of the minimal polynomial of A is denoted by $\mathrm{index}_0(A)$, and if A is invertible, then $\mathrm{index}_0(A) = 0$. Recall that $\mathrm{index}_0(A)$ coincides with the size of the largest nilpotent Jordan block in the Jordan canonical form of A.

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DEFINITION 2.1. A matrix $X \in \mathbb{R}^{n\times n}$ has

- the *Perron-Frobenius property* if $\rho(X) > 0$, $\rho(X) \in \sigma(X)$ and there exists a nonnegative eigenvector corresponding to $\rho(X)$;
- the *strong Perron-Frobenius property* if, in addition to having the Perron-Frobenius property, $\rho(X)$ is a simple eigenvalue such that

$$\rho(X) > |\lambda| \quad \text{for all} \quad \lambda \in \sigma(X), \ \ \lambda \neq \rho(X)$$

and the corresponding eigenvector is strictly positive.

Note that this definition of the Perron-Frobenius property is taken from Noutsos [11] and excludes $\rho(X) = 0$ (whereas, for example in [6, 14], $\rho(X) = 0$ is allowed).

The *nonnegative orthant* in $\mathbb{R}^n$, that is, the set of all nonnegative vectors in $\mathbb{R}^n$, is denoted by $\mathbb{R}^n_+$. For $x \in \mathbb{R}^n$, we use the notation $x \geq 0$ interchangeably with $x \in \mathbb{R}^n_+$.

DEFINITION 2.2. An $n \times n$ matrix $B = [b_{ij}]$ is called:

- *nonnegative* (*positive*), denoted by $B \geq 0$ ($B > 0$), if $b_{ij} \geq 0$ ($b_{ij} > 0$) for all i and j;
- *eventually nonnegative* (*positive*), denoted by $B \overset{\text{v}}{\geq} 0$ ($B \overset{\text{v}}{>} 0$) , if there exists a nonnegative integer k_0 such that $B^k \geq 0$ ($B^k > 0$) for all $k \geq k_0$. In each case, we denote the smallest such nonnegative integer by $k_0 = k_0(B)$ and refer to it as the *power index* of B with respect to eventual nonnegativity (positivity);
- *exponentially nonnegative* (*positive*) if $\forall t \geq 0$, $e^{tB} = \displaystyle\sum_{k=0}^{\infty} \frac{t^k B^k}{k!} \geq 0$ ($e^{tB} > 0$);
- *eventually exponentially nonnegative* (*positive*) if $\exists t_0 \in [0, \infty)$ such that $\forall t \geq t_0$, $e^{tB} \geq 0$ ($e^{tB} > 0$). In each case, we denote the smallest such nonnegative number by $t_0 = t_0(B)$ and refer to it as the *exponential index* of B with respect to eventual exponential nonnegativity (positivity).

Note that if $B \overset{\text{v}}{\geq} 0$ with power index k_0 such that B^{k_0} is nonnegative but not positive, and if in addition $B \overset{\text{v}}{>} 0$, then the power index of B with respect to eventual positivity is greater than k_0. When the context is clear, we omit writing with respect to eventual nonnegativity (positivity). Similar remarks apply to the exponential index.

Next is a simple but useful observation regarding the power index of an eventually nonnegative matrix.

LEMMA 2.3. *Let $B \in \mathbb{R}^{n\times n}$ and suppose that for some integer $m \geq 2$,*

$$B^{m-1} \ngeq 0 \quad \textit{and} \quad B^k \geq 0, \ \ k = m, m+1, \ldots, 2m-1.$$

Then $B \overset{\text{v}}{\geq} 0$ with power index $k_0(B) = m$.

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Proof. Let B and m be as prescribed and notice that for every $k \geq 2m$, there exist integers $p \geq 2$ and $r \in \{0, 1, \ldots, m-1\}$ such that $k = p\,m + r$. Thus

$$B^k \;=\; (B^m)^{p-1}\, B^{m+r} \;\geq\; 0.$$

It follows that $B \overset{\mathrm{v}}{\geq} 0$ and that its power index is m. $\square$

Definition 2.4. An $n \times n$ matrix $A = [a_{ij}]$ is called:

- a *Z-matrix* if $a_{ij} \leq 0$ for all $i \neq j$. In this case, $-A$ is called *essentially nonnegative*.
- an *M-matrix* if $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B) \geq 0$;
- an M_{v}*-matrix* if $A = sI - B$, where $B \overset{\mathrm{v}}{\geq} 0$ and $s \geq \rho(B) \geq 0$.

3. Basic properties of $\mathbf{M}_{\vee}$- matrices. By the Perron-Frobenius Theorem, for every $B \geq 0$, $\rho(B) \in \sigma(B)$. As a consequence, the minimum of the real parts of the eigenvalues of an M-matrix A belongs to $\sigma(A)$. Analogous results hold for eventually nonnegative and $\mathrm{M}_{\vee}$- matrices. To prove this, we first quote some results from [8, 11]. Note that in the second theorem below from [11], the assumption that B is not nilpotent is added; this assumption is needed as observed in [6]. Furthermore, the converse of Theorem 3.2 is not necessarily true.

Theorem 3.1. ([8, Theorem 1] and [11, Theorem 2.2]) *For a matrix $B \in \mathbb{R}^{n\times n}$, the following are equivalent:*

(i) *Both matrices B and B^T have the strong Perron-Frobenius property.*

(ii) *B is eventually positive.*

(iii) *B^T is eventually positive.*

Theorem 3.2. ([11, Theorem 2.3]) *Let $B \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix that is not nilpotent. Then both B and B^T have the Perron-Frobenius property.*

Example 3.3. (See [12, Example 3.11].) A non-nilpotent eventually nonnegative matrix need not have the strong Perron-Frobenius property. To see this, consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

By Lemma 2.3, B is eventually nonnegative with $k_0(B) = 2$ and spectral radius $\rho(B) = 2$ of algebraic multiplicity two. Note also that B is not eventually positive since B^k is reducible for all $k \geq 2$.

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The following basic properties of an $\mathrm{M}_\vee$ - matrix are immediate consequences of its definition, as well as Theorems 3.1 and 3.2.

THEOREM 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be an $M_\vee$ - matrix, i.e., $A = sI - B$ with $B \overset{\mathrm{v}}{\geq} 0$ and $s \geq \rho(B) \geq 0$. Then*

(i) $s - \rho(B) \in \sigma(A)$*;*

(ii) $\mathrm{Re}\,\lambda \geq 0$ *for all* $\lambda \in \sigma(A)$*;*

(iii) $\det A \geq 0$ *and* $\det A = 0$ *if and only if* $s = \rho(B)$*;*

(iv) *if, in particular,* $\rho(B) > 0$*, then there exists an eigenvector* $x \geq 0$ *of* A *and an eigenvector* $y \geq 0$ *of* A^T *corresponding to* $\lambda(A) = s - \rho(B)$*;*

(v) *if, in particular,* $B \overset{\mathrm{v}}{>} 0$ *and* $s > \rho(B)$*, then in* (iv) $x > 0$, $y > 0$ *and in* (ii) $\mathrm{Re}\,\lambda > 0$ *for all* $\lambda \in \sigma(A)$*.*

In the following result, different representations of an $\mathrm{M}_\vee$ - matrix are considered (analogous to different representations of an M-matrix).

THEOREM 3.5. *Let $A \in \mathbb{R}^{n\times n}$ be an $M_\vee$ - matrix. Then in any representation $A = tI - \hat{B}$ with $\hat{B} \overset{\mathrm{v}}{\geq} 0$, it follows that $t \geq \rho(\hat{B})$. If, in addition, A is nonsingular, then $t > \rho(\hat{B})$.*

Proof. Since A is an $\mathrm{M}_\vee$ - matrix, $A = sI - B \in \mathbb{R}^{n\times n}$ for some $B \overset{\mathrm{v}}{\geq} 0$ and some $s \geq \rho(B) \geq 0$. Let $A = tI - \hat{B}$, where $\hat{B} \overset{\mathrm{v}}{\geq} 0$. If B is nilpotent, then $0 = \rho(B) \in \sigma(B)$, and by Theorem 3.2, this containment also holds if B is not nilpotent. A similar containment holds for $\hat{B}$. If $t \geq s$, then $\rho(\hat{B}) = \rho((t-s)I + B) = \rho(B) + t - s$. Hence, $t = s - \rho(B) + \rho(\hat{B}) \geq \rho(\hat{B})$. Similarly, if $t \leq s$, then $\rho(B) = \rho(\hat{B}) + s - t$. Hence, $t = s - \rho(B) + \rho(\hat{B}) \geq \rho(\hat{B})$. If A is nonsingular, then by Theorem 3.4 (iii) it follows that $s > \rho(B)$ and so $t > \rho(\hat{B})$. ◻

As with M-matrices (see [2, Chapter 6, Lemma (4.1)]), we can now show that the class of $\mathrm{M}_\vee$ - matrices is the closure of the class of nonsingular $\mathrm{M}_\vee$ - matrices.

PROPOSITION 3.6. *Let $A = sI - B \in \mathbb{R}^{n\times n}$, where $B \overset{\mathrm{v}}{\geq} 0$. Then A is an $M_\vee$ - matrix if and only if $A + \epsilon I$ is a nonsingular $M_\vee$ - matrix for each $\epsilon > 0$.*

Proof. If $A + \epsilon I = (s+\epsilon)I - B$ is a nonsingular $\mathrm{M}_\vee$ - matrix for each $\epsilon > 0$, then by Theorem 3.5, $s + \epsilon > \rho(B)$ for each $\epsilon > 0$. Letting $\epsilon \to 0+$ gives $s \geq \rho(B)$, i.e., A is an $\mathrm{M}_\vee$ - matrix. Conversely, let $A = sI - B$ be an $\mathrm{M}_\vee$ - matrix, where $B \overset{\mathrm{v}}{\geq} 0$ and $s \geq \rho(B) \geq 0$. Thus, for every $\epsilon > 0$, $A + \epsilon I = (s+\epsilon)I - B$ with $B \overset{\mathrm{v}}{\geq} 0$ and $s + \epsilon > s \geq \rho(B)$. That is, $A + \epsilon I$ is a nonsingular $\mathrm{M}_\vee$ - matrix. ◻

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Given an M-matrix A, clearly $-A$ is essentially nonnegative and so $-A + \alpha I \geq 0$ for sufficiently large $\alpha \geq 0$. Thus $e^{-tA} = e^{-t\alpha}\, e^{-t(A-\alpha I)} \geq 0$ for all $t \geq 0$; that is $-A$ is exponentially nonnegative. In the following theorem, this property is extended to a special class of $\mathrm{M}_\vee$- matrices with exponential nonnegativity replaced by eventual exponential positivity.

THEOREM 3.7. *Let $A = sI - B \in \mathbb{R}^{n\times n}$ be an $M_\vee$- matrix with $B \overset{\mathrm{v}}{>} 0$ (and thus $s \geq \rho(B) > 0$). Then $-A$ is eventually exponentially positive. That is, there exists $t_0 \geq 0$ such that $e^{-tA} > 0$ for all $t \geq t_0$.*

Proof. Let $A = sI - B$, where $B = sI - A \overset{\mathrm{v}}{>} 0$ with power index k_0. As $B^m > 0$ for all $m \geq k_0$, there exists sufficiently large $t_0 > 0$ so that for all $t \geq t_0$, the sum of the first $k_0 - 1$ terms of the series $e^{tB} = \displaystyle\sum_{m=0}^{\infty} \frac{t^m B^m}{m!}$ is dominated by the term $\dfrac{t^{k_0} B^{k_0}}{k_0!}$, rendering e^{tB} positive for all $t \geq t_0$. It follows that $e^{-tA} = e^{-ts} e^{tB}$ is positive for all $t \geq t_0$. That is, $-A$ is eventually exponentially positive as claimed. □

If B is only eventually nonnegative, then a further condition is imposed to yield a result similar to the above theorem. To prove this, we first need the following theorem.

THEOREM 3.8. ([12, Theorem 3.7]) *Let $B \in \mathbb{R}^{n\times n}$ such that $B \overset{\mathrm{v}}{\geq} 0$ and* $\mathrm{index}_0(B) \leq 1$. *Then B is eventually exponentially nonnegative.*

THEOREM 3.9. *Let $A = sI - B \in \mathbb{R}^{n\times n}$ be an $M_\vee$- matrix with $B \overset{\mathrm{v}}{\geq} 0$ (and thus $s \geq \rho(B) \geq 0$). Suppose that* $\mathrm{index}_0(B) \leq 1$. *Then $-A$ is an eventually exponentially nonnegative matrix.*

Proof. The result follows readily from Theorem 3.8 and the fact that $e^{-tA} = e^{-ts} e^{tB}$. □

COROLLARY 3.10. *Let $A = sI - B \in \mathbb{R}^{n\times n}$ be an $M_\vee$- matrix such that $B + \alpha I \overset{\mathrm{v}}{\geq} 0$ for some $\alpha \in \mathbb{R}$ with $-\alpha \notin \sigma(B)$. Then $-A$ is an eventually exponentially nonnegative matrix.*

Proof. Since $A = (s+\alpha)I - (B+\alpha I)$ is an $\mathrm{M}_\vee$- matrix and $B + \alpha I \overset{\mathrm{v}}{\geq} 0$, it follows from Theorem 3.5 that $s + \alpha \geq \rho(B + \alpha I)$. As $\mathrm{index}_0(B + \alpha I) = 0 \leq 1$, the corollary follows by applying Theorem 3.9 to $A = (s+\alpha)I - (B+\alpha I)$. □

EXAMPLE 3.11. (see [12, Example 3.9]) Consider $A = 3I - B$, where

$$B = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

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It can be easily verified that $B^k \geq 0$ for $k = 2, 3$ and thus, by Lemma 2.3, for all $k \geq 2$. Hence $B \overset{\text{v}}{\geq} 0$. Since $\rho(B) = 2$, A is an M$_\vee$- matrix. As $\text{index}_0(B) = 1$, by Theorem 3.9 it follows that $-A$ is eventually exponentially nonnegative. For illustration, we compute e^{-tA} for $t = 1, 2$ to respectively be

$$e^{-3}\begin{bmatrix} 1.5431 & 1.1752 & 2.3404 & -0.0100 \\ 1.1752 & 1.5431 & 4.0487 & 2.9625 \\ 0 & 0 & 4.1945 & 3.1945 \\ 0 & 0 & 3.1945 & 4.1945 \end{bmatrix}, e^{-6}\begin{bmatrix} 3.7622 & 3.6269 & 18.1543 & 10.9006 \\ 3.6269 & 3.7622 & 35.4439 & 29.9195 \\ 0 & 0 & 27.7991 & 26.7991 \\ 0 & 0 & 26.7991 & 27.7991 \end{bmatrix}.$$

Hence $-A$ is an eventually exponentially nonnegative matrix with exponential index t_0 such that $t_0 > 1$.

Example 3.12. (See [12, Example 3.11].) When $\text{index}_0(B) > 1$, the conclusion of Theorem 3.9 is not in general true. To see this, consider the matrix B in Example 3.3, which is eventually nonnegative with $k_0(B) = 2$ and $\text{index}_0(B) = 2$, and let $A = sI - B$ with $s \geq \rho(B)$. As

$$B^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix} \quad (k = 2, 3, \ldots),$$

it follows that the $(3, 1)$ and $(4, 2)$ entries of e^{tB} (and thus e^{-tA}) are negative for all $t > 0$. That is, $-A$ is not eventually exponentially nonnegative.

4. Monotonicity, semipositivity and inverse nonnegativity. There are several properties of a Z-matrix A that are equivalent to A being an M-matrix. These properties are documented in the often cited Theorems 2.3 and 4.6 in [2]: positive stability, semipositivity, inverse nonnegativity and monotonicity among others. In the cone-theoretic generalizations of M-matrices (see [13]), these properties are generalized and shown to play an analogous characterizing role. In the following theorems we examine the form and role these properties take in the context of M$_\vee$- matrices.

Theorem 4.1. *Let $A = sI - B \in \mathbb{R}^{n\times n}$, where $B \overset{\text{v}}{\geq} 0$ has power index $k_0 \geq 0$. Let K be the cone defined as $K = B^{k_0}\mathbb{R}^n_+$. Consider the following conditions:*

(i) *A is an invertible M$_\vee$- matrix.*

(ii) $s > \rho(B)$ (positive stability of A).

(iii) A^{-1} *exists and* $A^{-1}K \subseteq \mathbb{R}^n_+$ (inverse nonnegativity).

(iv) $Ax \in K \implies x \geq 0$ (monotonicity).

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Then (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii)$\Longleftrightarrow$(iv)*. If, in addition, B is not nilpotent, then all conditions* (i)-(iv) *are equivalent.*

Proof.

(i)$\Longrightarrow$(ii). This implication follows by Theorem 3.5 and invertibility of A.

(ii)$\Longrightarrow$(i). It follows from the definition of an $\mathrm{M}_\vee$- matrix and Theorem 3.4 (i).

(iii)$\Longrightarrow$(iv). Assume (iii) holds and consider $y = Ax \in K$. As A^{-1} exists, $x = A^{-1}y \geq 0$.

(iv)$\Longrightarrow$(iii). Assume (iv) holds. First notice that A must be invertible because if $Au = 0 \in K$ then $u \geq 0$. Also $A(-u) = 0 \in K$ and so $u \leq 0$; that is, $u = 0$. Consider now $y = A^{-1}B^{k_0}x$, where $x \geq 0$. Then $Ay = B^{k_0}x \in K$ and so $y \geq 0$.

(ii)$\Longrightarrow$(iii). If (ii) holds, then $\rho(B/s) < 1$ and so

$$A^{-1} = \frac{1}{s}(I - B/s)^{-1} = \frac{1}{s}\sum_{q=0}^{\infty}\frac{B^q}{s^q}.$$

Consequently, for all $x \geq 0$,

$$A^{-1}B^{k_0}\, x \;=\; \frac{1}{s}\sum_{q=0}^{\infty}\frac{B^{q+k_0}}{s^q}\, x \;\geq\; 0.$$

Now suppose that B is not nilpotent, that is, $\rho(B) > 0$. To prove that (i)-(iv) are equivalent it is sufficient to show that (iii)$\Longrightarrow$(ii).

(iii)$\Longrightarrow$(ii). By Theorem 3.2, B has the Perron-Frobenius property, i.e., there exists nonzero $x \geq 0$ so that $Bx = \rho(B)x$. Assume (iii) holds and consider $\mu = s - \rho(B) \in \sigma(A) \cap \mathbb{R}$. As $B^{k_0}x = \rho(B)^{k_0}x$ and since $\rho(B) > 0$, it follows that $x \in K$ and thus $A^{-1}x \geq 0$. But $Ax = \mu x$ and so $x = \mu A^{-1}x$. It follows that $\mu > 0$. ◻

Remark 4.2.

(a) The implication (iii)$\Longrightarrow$(ii) or (i) in Theorem 4.1 is not in general true if B is nilpotent. For example, consider $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \overset{\mathrm{v}}{\geq} 0$, which has power index $k_0 = 2$. Thus $K = B^2\mathbb{R}^2_+ = \{0\}$. For any $s < 0$, $A = sI - B$ is invertible and $A^{-1}K = \{0\} \subset \mathbb{R}^2_+$; however, A is not an $\mathrm{M}_\vee$- matrix because its eigenvalues are negative.

(b) It is well known that when an M-matrix is invertible, its inverse is nonnegative. As mentioned earlier, in [8, Theorem 8] it is shown that the inverse of a pseudo M-matrix is eventually positive. In [9, Theorem 4.2] it is shown that if B is an irreducible eventually nonnegative matrix with $index_0(B) \leq 1$, then there exists $t > \rho(B)$

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such that for all $s \in (\rho(B),\, t)$, $(sI - B)^{-1} > 0$. The situation with the inverse of an $\mathrm{M}_\vee$ - matrix A is different. Notice that condition (iii) of Theorem 4.1 is equivalent to $A^{-1}B^{k_0} \geq 0$. In general, if A is an invertible $\mathrm{M}_\vee$ - matrix, A^{-1} is neither nonnegative nor eventually nonnegative; for example, if $A = I - B$ with B as in Remark 4.2 (a), then the $(1, 2)$ entry of $(A^{-1})^k$ is negative for all $k \geq 1$.

More equivalent conditions for a matrix to be an $\mathrm{M}_\vee$ - matrix can be added if some *a priori* assumptions are made. A result about pseudo M-matrices similar to the equivalence of (i) and (iii) below can be found in [5, Theorem 3.11].

THEOREM 4.3. *Let $A = sI - B \in \mathbb{R}^{n \times n}$, where $B \overset{\mathrm{v}}{\geq} 0$ has a positive eigenvector (corresponding to $\rho(B)$). Consider the following conditions:*

(i) *A is an $M_\vee$ - matrix.*

(ii) *There exists an invertible diagonal matrix $D \geq 0$ such that the row sums of AD are nonnegative.*

(iii) *There exists $x > 0$ such that $Ax \geq 0$* (semipositivity).

Then (i)$\Longrightarrow$(ii)$\Longleftrightarrow$(iii). *If, in addition, B is not nilpotent, then all conditions* (i)-(iii) *are equivalent.*

Proof. Let A be as prescribed and let $w > 0$ so that $Bw = \rho(B)w$.

(i)$\Longrightarrow$(ii). Consider the diagonal matrix $D = \mathrm{diag}(w)$ whose diagonal entries are the entries of w and let e denote the vector of all ones. Then $s \geq \rho(B)$ and so $ADe = Aw = sw - Bw = (s - \rho(B))w \geq 0$.

(ii)$\Longleftrightarrow$(iii). This equivalence follows by setting $x = De > 0$.

Now suppose that B is not nilpotent and assume (ii) holds, i.e., $Ay \geq 0$, where $y = De > 0$. Then

$$sy - By \geq 0. \tag{4.1}$$

Since B is not nilpotent, by Theorem 3.2, B^T has the Perron-Frobenius property. Let $z \geq 0$ be an eigenvector of B^T corresponding to $\rho(B^T)$, i.e., $z^T B = \rho(B) z^T$. Then, by (4.1),

$$0 \leq s\, z^T y - z^T B y = s\, z^T y - \rho(B)\, z^T y = (s - \rho(B))\, z^T y.$$

Since $z^T y \geq 0$ and $z^T y \neq 0$, it follows that $s \geq \rho(B)$ and thus A is an $\mathrm{M}_\vee$ - matrix, so (ii) implies (i). $\square$

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Remark 4.4.

(a) Let $A = -\frac{1}{4}I + B$, with B as in Remark 4.2 (a), and $x = [2,1]^T$. Then Theorem 4.3 (iii) holds, but A is not an $\mathrm{M}_\vee$- matrix. Thus the implication (iii)$\Longrightarrow$(i) is not in general true if B is nilpotent.

(b) The existence of a positive eigenvector in Theorem 4.3 is necessary. To see this, consider $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \overset{\mathrm{v}}{\geq} 0$, which is nilpotent and has no positive eigenvector. Let

$$A = \frac{1}{4}I - B = \begin{bmatrix} -\frac{3}{4} & -1 \\ 1 & \frac{5}{4} \end{bmatrix}.$$

Notice that there is no $x > 0$ such that $Ax \geq 0$, but A is an $\mathrm{M}_\vee$- matrix. That is, condition (i) of Theorem 4.3 holds but not (iii).

(c) Let $A = sI - B \in \mathbb{R}^{n\times n}$ be invertible, where $B \overset{\mathrm{v}}{\geq} 0$ is not nilpotent and has a positive eigenvector (corresponding to $\rho(B)$). Then condition (ii) with row sums of AD positive and condition (iii) with $Ax > 0$ of Theorem 4.3 are equivalent to A being an (invertible) $\mathrm{M}_\vee$- matrix.

In the context of invertible M-matrices (i.e., when $B \geq 0$), condition (ii) of Theorem 4.3 is associated with *diagonal dominance* of AD because the diagonal entries are positive and the off-diagonal entries are nonpositive. As a consequence, when A is an invertible M-matrix, the inequality $ADe > 0$ can be interpreted as saying that the columns of A can be scaled by positive numbers so that the diagonal entries dominate (in modulus) the row sums of the (moduli of the) off-diagonal entries. This is not the case with $\mathrm{M}_\vee$- matrices because off-diagonal entries of $B \overset{\mathrm{v}}{\geq} 0$ can very well be negative. For example, the matrix A below is an invertible $\mathrm{M}_\vee$- matrix for which (ii) of Theorem 4.3 holds for $D = \mathrm{diag}(w)$ as in the proof; however, AD is not diagonally dominant. Let

$$A \;=\; sI - B \;=\; 9.5\,I - \begin{bmatrix} -0.1 & 20 & 47 \\ -0.2 & 1 & 1 \\ 0.3 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 9.6 & -20 & -47 \\ 0.2 & 8.5 & -1 \\ -0.3 & -5 & 1.5 \end{bmatrix}.$$

The matrix A is an invertible $\mathrm{M}_\vee$- matrix because $B \overset{\mathrm{v}}{>} 0$ with $\rho(B) = 9.4834$ (to 4 decimal places). Letting $D = \mathrm{diag}(w)$, where $w = [0.9799, 0.0004, 0.1996]^T$ is an eigenvector of B corresponding to $\rho(B)$, gives

$$AD = \begin{bmatrix} 9.4068 & -0.0086 & -9.3819 \\ 0.1960 & 0.0036 & -0.1996 \\ -0.2940 & -0.0021 & 0.2994 \end{bmatrix}.$$

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Note that $ADe \geq 0$, however, AD is not diagonally dominant.

In the next theorem we present some properties of singular M$_\vee$- matrices analogous to the properties of singular, irreducible M-matrices found in [2, Chapter 6, Theorem (4.16)].

THEOREM 4.5. *Let $A = sI - B \in \mathbb{R}^{n\times n}$ be a singular $M_\vee$- matrix, where $B \overset{\text{v}}{>} 0$. Then the following hold.*

(i) *A has rank $n-1$.*

(ii) *There exists a vector $x > 0$ such that $Ax = 0$.*

(iii) *If for some vector u, $Au \geq 0$, then $u = 0$* (almost monotonicity).

Proof. As A is singular, by Theorem 3.4 (iii) it follows that $s = \rho(B)$.

(i) By Theorem 3.1, B has the strong Perron-Frobenius property and so $\rho(B)$ is a simple eigenvalue of B. Thus, $0 = s - \rho(B)$ is a simple eigenvalue of A.

(ii) As B has the strong Perron-Frobenius property, there exists $x > 0$ such that $Bx = \rho(B)x$, i.e., $Ax = \rho(B)x - Bx = 0$.

(iii) By Theorem 3.1, B^T also has the strong Perron-Frobenius property and so there exists $z > 0$ such that $z^T B = \rho(B) z^T$. Let u be such that $Au \geq 0$. If $Au \neq 0$, then $z^T Au > 0$. However,

$$z^T Au = \rho(B) z^T u - z^T Bu = \rho(B) z^T u - \rho(B) z^T u = 0,$$

a contradiction, showing that $Au = 0$. ∎

The following is a comparison condition for M$_\vee$- matrices analogous to a known result for M-matrices (see e.g., [7, Theorem 2.5.4]).

THEOREM 4.6. *Let $A = sI - B \in \mathbb{R}^{n\times n}$ and $E = sI - F \in \mathbb{R}^{n\times n}$, where $B, F \overset{\text{v}}{\geq} 0$ are not nilpotent. Suppose that at least one of B, B^T, F or F^T has a positive eigenvector* (*corresponding to the spectral radius*). *If A is an $M_\vee$- matrix and $A \leq E$, then E is an $M_\vee$- matrix.*

Proof. Suppose that there is a vector $x > 0$ such that $Bx = \rho(B)x$. Since $F \overset{\text{v}}{\geq} 0$, by Theorem 3.2, there exists nonzero vector $y \geq 0$ such that $y^T F = \rho(F) y^T$. Also $A \leq E$ implies that $B \geq F$ and so

$$\rho(B)\, y^T x \,=\, y^T Bx \,\geq\, y^T Fx \,=\, \rho(F)\, y^T x.$$

As $y^T x > 0$, it follows that $\rho(B) \geq \rho(F)$. If A is an M$_\vee$- matrix, then by Theorem 3.5,

$$s \geq \rho(B) \geq \rho(F)$$

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and thus E is an $\mathrm{M}_\vee$- matrix. The proof is similar if one of B^T, F or F^T has a positive eigenvector corresponding to its spectral radius. □

Note that a weakening of the assumptions regarding B and F in Theorem 4.6 is possible. Its proof remains valid if it is assumed that either (i) B or B^T has a positive eigenvector and F is not nilpotent, or (ii) F or F^T has a positive eigenvector and B is not nilpotent.

An important aspect of M-matrix theory is principal submatrix inheritance: every principal submatrix of an M-matrix is also an M-matrix. This fact can be viewed as a consequence of the monotonicity of the spectral radius of a nonnegative matrix as a function of its entries. Another consequence is that all principal minors of an M-matrix are nonnegative (i.e., every M-matrix is a P_0-matrix). These facts do not carry over to $\mathrm{M}_\vee$- matrices as seen in the next example.

Example 4.7. Not all principal submatrices of an $\mathrm{M}_\vee$- matrix are $\mathrm{M}_\vee$- matrices. Also not all $\mathrm{M}_\vee$- matrices are P_0-matrices. To see this, consider

$$B = \begin{bmatrix} 9.5 & 1 & 1.5 \\ -14.5 & 16 & 10.5 \\ 10.5 & -3 & 4.5 \end{bmatrix}$$

for which $\rho(B) = 12$ is a simple dominant eigenvalue having positive left and right eigenvectors. That is, B and B^T satisfy the strong Perron-Frobenius property and so by Theorem 3.1, $B \overset{\mathrm{v}}{>} 0$. As a consequence,

$$A = 12.5\, I - B = \begin{bmatrix} 3 & -1 & -1.5 \\ 14.5 & -3.5 & -10.5 \\ -10.5 & 3 & 8 \end{bmatrix}$$

is an invertible $\mathrm{M}_\vee$- matrix. Clearly, A is not a P_0-matrix since the $(2,2)$ entry is negative. Also the $(2,2)$ entry is a principal submatrix of A that is not an $\mathrm{M}_\vee$- matrix. The principal submatrix of A lying in rows and columns 1 and 2 is also not an $\mathrm{M}_\vee$- matrix because it does not have a real eigenvalue, violating Theorem 3.4 (i). Furthermore, using Lemma 2.3, it can be verified that the power index of B is $k_0 = 19$. Note that A^{-1} is not positive, but $A^{-1}B^{19} > 0$, illustrating Theorem 4.1 (iii). Referring to Remark 4.2 (b), note that $A^{-1} \overset{\mathrm{v}}{>} 0$ with $k_0 = 3$. Also, by [9, Theorem 4.2], since B is irreducible and invertible, there exists $\beta > \rho(B)$ so that $sI - B$ has a positive inverse for all $s \in (\rho(B), \beta)$.

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