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EXPLICIT SOLUTIONS OF REGULAR LINEAR DISCRETE-TIME DESCRIPTOR SYSTEMS WITH CONSTANT COEFFICIENTS*

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Abstract. Explicit solution formulas are presented for systems of the form $Ex^{k+1} = Ax^k + f^k$ with $k \in \mathbb{K}$, where $\mathbb{K} \subset \mathbb{Z}$ is a discrete interval and the pencil $\lambda E - A$ is regular. Different results are obtained when one starts with an initial condition at the point $k = 0$ and calculates into the future (i.e., $Ex^{k+1} = Ax^k + f^k$ with $k \in \mathbb{N}$) and when one wants to get a complete solution (i.e., $Ex^{k+1} = Ax^k + f^k$ with $k \in \mathbb{Z}$).

Key words. Descriptor system, Strangeness index, Linear discrete descriptor system, Explicit solution, Backward Leslie model.

AMS subject classifications. 39A05, 15A06.

1. Introduction.

We denote sequences of vectors by $\{x^k\}_{k\in\mathbb{D}}$ for arbitrary discrete intervals $\mathbb{D} \subset \mathbb{Z}$. The k-th vector of such a sequence x^k is also called the k-th element of $\{x^k\}_{k\in\mathbb{D}}$ and further x_i^k denotes the i-th (block-)row of the vector x^k. To introduce the notion of a discrete-time descriptor system let us first define two discrete intervals in the following way.

$$\mathbb{K} := \{k \in \mathbb{Z} : k_b \leq k \leq k_f\}, k_b \in \mathbb{Z} \cup \{-\infty\}, k_f \in \mathbb{Z} \cup \{\infty\},$$
$$\mathbb{K}^+ := \begin{cases} \mathbb{K} & \text{if } k_f = \infty, \\ \mathbb{K} \cup \{k_f + 1\} & \text{if } k_f < \infty. \end{cases}$$

With this definition we call

$$Ex^{k+1} = Ax^k + f^k, \quad x^{k_0} = x_0, \quad k \in \mathbb{K} \tag{1.1}$$

a *linear discrete-time descriptor system with constant coefficients*, where $E, A \in \mathbb{C}^{n,n}$, $x^k \in \mathbb{C}^n$ for $k \in \mathbb{K}^+$ are the state vectors, $f^k \in \mathbb{C}^{n,n}$ for $k \in \mathbb{K}$ are the inhomogeneities and $x_0 \in \mathbb{C}^n$ is an initial condition given at the point $k_0 \in \mathbb{K}^+$. Other names for systems of the form (1.1) include *linear time-invariant discrete-time descriptor system*, *linear singular system* (e.g., [12]), *linear semi-state system*, and *linear generalized state-space system*. The sequence $\{x^k\}_{k\in\mathbb{K}^+}$ is called a solution of (1.1) if its elements fulfill all the equations. The continuous-time counterpart to (1.1) is called *linear continuous-time descriptor system with constant coefficients* and given by

$$E\dot{x}(t) = Ax(t) + f(t), \quad x(t_0) = x_0, \quad t \in \mathbb{R}, \tag{1.2}$$

where $E, A \in \mathbb{C}^{n,n}$, $x(t) \in \mathbb{C}^n$ is the state vector, $f(t) \in \mathbb{C}^n$ is the inhomogeneity, $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t, and $x_0 \in \mathbb{R}^n$ is an initial condition given at

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the point $t_0 \in \mathbb{R}$. Assuming that the pencil $\lambda E - A$ is *regular*, i.e., $\det \lambda E - A \neq 0$ for some $\lambda \in \mathbb{C}$, one can explicitly write down the unique solution of (1.2) with the help of the Drazin inverse, as shown in [8]. The purpose of this paper is to obtain corresponding results for the discrete-time case (1.1).
Equations of the form (1.1) arise naturally by approximating $\dot{x}(t)$ in (1.2) via an explicit finite difference method. Equation (1.1) is also a special form of a singular Leontief model in economics, see [5, 9]. Another application of (1.1) is the backward Leslie model [3]. The Leslie model is used to describe the evolution of the age distribution of a population at discrete time points. Therefore the population is divided into n distinct equidistant age classes, e.g., 0-1 years, 1-2 years, ... Then the vector $x^k \in \mathbb{R}^n$ in the Leslie model describes the number of individuals in each of the age classes at the discrete time point k. It is further assumed that all successive discrete time points k and $k+1$ correspond to two time points in real time that are as far apart as the extent of one age classes is. With these assumptions, the Leslie model is given by

$$x^{k+1} = \underbrace{\begin{bmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & \cdots & \cdots & 0 \\ 0 & s_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \end{bmatrix}}_{=:L} x^k,$$

with $L \in \mathbb{R}^{n,n}$ where $b_i \geq 0$ for $i = 1, \ldots, n$ are the birth rates of the i-th age class in one period of time and $s_i \in [0,1]$ for $i = 1, \ldots, n-1$ are the survival rates of the i-th age class in one period of time. Since in most cases elderly individuals are not likely to produce offsprings we can assume that $b_n = 0$ and thus L is singular. Given an age distribution we can use the Leslie model to estimate the age distribution in the future. If, however, we want to determine an age distribution in the past, given a present age distribution $\hat{x}$, we have to solve the Leslie Model backwards, i.e., we have to solve a system of the form

$$Lx^{l+1} = x^l, \text{ with } x^0 = \hat{x}, \tag{1.3}$$

with L singular. System (1.3) is a special case of system (1.1).
Throughout the paper we will assume that $\lambda E - A$ is a regular pencil. As shown in [6], any regular pencil can be reduced to the *Kronecker/Weierstraß canonical form*

$$P(\lambda E - A) Q = \lambda \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix}, \tag{1.4}$$

where $P, Q \in \mathbb{C}^{n,n}$ are invertible, $I_k \in \mathbb{C}^{k,k}$ is the identity matrix of dimension k, $J \in \mathbb{C}^{n_f, n_f}$ is in Jordan canonical form, $N \in \mathbb{C}^{n_\infty, n_\infty}$ is a nilpotent matrix in Jordan canonical form and $n_f, n_\infty \in \mathbb{N}$ with $n_f + n_\infty = n$. We see that n_f is the number of finite eigenvalues and n_∞ is the number of infinite eigenvalues. The form (1.4) allows

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to determine the solution of (1.1) in changed coordinates, i.e., by transforming system (1.1) through

$$P^{-1}EQ^{-1}Qx^{k+1} = P^{-1}AQ^{-1}Qx^k + P^{-1}f^k, \tag{1.5}$$

we do not change the set of solutions. Adapting the state and the inhomogeneity to the new coordinates we define

$$\begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix} := Qx^k \text{ and } \begin{bmatrix} f_1^k \\ f_2^k \end{bmatrix} := P^{-1}f^k,$$

with partitioning analog to (1.5), i.e., $x_1^k, f_1^k \in \mathbb{C}^{n_f}$ and $x_2^k, f_2^k \in \mathbb{C}^{n_\infty}$. Then from (1.5) we see that in the new coordinates (1.1) can be decomposed into the two subproblems

$$x_1^{k+1} = Jx_1^k + f_1^k, \tag{1.6a}$$
$$Nx_2^{k+1} = x_2^k + f_2^k, \tag{1.6b}$$

of which we can compute the solutions separately, see [2]. In this paper, however, we will determine representations of the solution in the original coordinates.

2. The Drazin inverse. The Drazin inverse is a generalization of the inverse of a matrix to potentially singular square matrices. The properties of the Drazin inverse make it very useful for finding solutions of systems of the form (1.1).

DEFINITION 2.1. [8] Let $E, A \in \mathbb{C}^{n,n}$, let the matrix pencil $\lambda E - A$ be regular and let the Kronecker canonical form of $\lambda E - A$ be given by (1.4). Then the quantity ν defined by $N^\nu = 0$, $N^{\nu-1} \neq 0$, i.e., by the index of nilpotency of N in (1.4), if the nilpotent block in (1.4) is present and by $\nu = 0$ if it is absent, is called the *index* of the matrix pencil $\lambda E - A$, and denoted by $\operatorname{ind}(\lambda E - A) = \nu$.

DEFINITION 2.2. Let $E \in \mathbb{C}^{n,n}$. Further, let ν be the index of the matrix pencil $\lambda E - I_n$. Then ν is also called the *index* of E and denoted by $\operatorname{ind}(E) = \nu$.

DEFINITION 2.3. Let $E \in \mathbb{C}^{n,n}$ have the index ν. A matrix $X \in \mathbb{C}^{n,n}$ satisfying

$$\begin{aligned} EX &= XE, \\ XEX &= X, \\ XE^{\nu+1} &= E^\nu, \end{aligned} \tag{2.1}$$

is called a *Drazin inverse* of E and denoted by E^D.

As shown in [8] the Drazin inverse of a matrix is uniquely determined. Several properties of the Drazin inverse will be used frequently in section 3, which is why we review them here.

LEMMA 2.4. *Consider matrices $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$. Then*

$$\begin{aligned} EA^D &= A^DE, \\ E^DA &= AE^D, \\ E^DA^D &= A^DE^D. \end{aligned} \tag{2.2}$$

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Proof. See [8, Lemma 2.21]. □

Also, the following Theorem will be necessary in the next section. It represents a decomposition of a general square matrix into a part belonging to the non-zero eigenvalues and a part belonging to the zero eigenvalues.

THEOREM 2.5. *Let $E \in \mathbb{C}^{n,n}$ with $\nu = \mathrm{ind}(E)$. Then there is one and only one decomposition*

$$E = \tilde{C} + \tilde{N} \tag{2.3}$$

with the properties

$$\tilde{C}\tilde{N} = \tilde{N}\tilde{C} = 0, \quad \tilde{N}^{\nu} = 0, \quad \tilde{N}^{\nu-1} \neq 0, \quad \mathrm{ind}(\tilde{C}) \leq 1. \tag{2.4}$$

For this decomposition the following statements hold:

$$\begin{aligned} &\tilde{C}^D\tilde{N} = \tilde{N}\tilde{C}^D = 0, \\ &E^D = \tilde{C}^D, \\ &\tilde{C}\tilde{C}^D\tilde{C} = \tilde{C}, \\ &\tilde{C}^D\tilde{C} = E^D E, \\ &\tilde{C} = EE^D E, \quad \tilde{N} = E\left(I - E^D E\right). \end{aligned} \tag{2.5}$$

Proof. See [8, Theorem 2.22]. □

Note, that the Drazin inverse and the decomposition (2.3) can easily be computed via the Jordan canonical form of E. To see this, assume that $E \in \mathbb{C}^{n,n}$ has the Jordan canonical form

$$E = S \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} S^{-1}, \tag{2.6}$$

where $S \in \mathbb{C}^{n,n}$ is invertible, J is invertible, and N only has zero as an eigenvalue. Then, the Drazin inverse of E is given by

$$E^D = S \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} S^{-1},$$

which can be shown by proving the properties (2.1) through basic computations. The matrices of decomposition (2.3) in this case can be written as

$$\tilde{C} = S \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} S^{-1} \text{ and } \tilde{N} = S \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} S^{-1}, \tag{2.7}$$

which shows that $\tilde{N}$ only has zero as an eigenvalue and $\tilde{C}$ has all the non-zero eigenvalues of E. However, $\tilde{C}$ may also have the eigenvalue zero although in this case the eigenvalue zero is non-defective. We also remark that

$$P_E := E^D E \tag{2.8}$$

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is a projector which follows from the properties of the Drazin inverse (2.1). Again, using the Jordan canonical form (2.6) we can write this projector as

$$P_E = E^D E = S \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S^{-1},$$

which shows that P_E is, in functional analytic terms, the Riesz projection corresponding to the non-zero eigenvalues of E. Analogously, $(I - P_E) = (I - E^D E)$ is the Riesz projection corresponding to the zero eigenvalues of E.

3. Basic theorems. Assume that $\lambda E - A$ is a regular pencil and that E and A do not commute. Using a nice trick, which is due to Campbell (see [4]), we can in this case rewrite system (1.1) as a system with commuting coefficient matrices.

LEMMA 3.1. *[4] Let $E, A \in \mathbb{C}^{n,n}$ with $\lambda E - A$ regular. Let $\tilde{\lambda} \in \mathbb{C}$ be chosen such that the matrix $\tilde{\lambda} E - A$ is nonsingular. Then the matrices*

$$\tilde{E} := \left(\tilde{\lambda} E - A\right)^{-1} E, \qquad \tilde{A} := \left(\tilde{\lambda} E - A\right)^{-1} A$$

commute.

Thus, by multiplying (1.1) from the left with the invertible matrix $\left(\tilde{\lambda} E - A\right)^{-1}$ and using the notation of Lemma 3.1 as well as $\tilde{f}^k := (\tilde{\lambda} E - A)^{-1} f^k$ we see that we obtain the equivalent system

$$\tilde{E} x^{k+1} = \tilde{A} x^k + \tilde{f}^k, \tag{3.1}$$

with commuting coefficient matrices $\tilde{E}$ and $\tilde{A}$. Note, that this transformation does not change the state space/the coordinates of the system (1.1), since the multiplication is only executed from the left. Thus, the set of solutions is not changed. Because of Lemma 3.1 we will assume in the following that E and A already commute. Using (2.2) together with the definition (2.8) we see that in the case that E and A commute we also have

$$P_E P_A = P_A P_E,$$

which means that $P_A P_E$ is again a projector. Also $P_A(I - P_E)$, $(I - P_A)P_E$, and $(I - P_A)(I - P_E)$ are projectors.

REMARK 3.2. In the following we are going to decompose the complete problem (1.1) into four subproblems by projecting (1.1) through the following projectors:

$P_A P_E$	corresponds to the non-zero, finite eigenvalues of $\lambda E - A$,
$P_A(I - P_E)$	corresponds to the infinite eigenvalues of $\lambda E - A$,
$(I - P_A)P_E$	corresponds to the zero eigenvalues of $\lambda E - A$,
$(I - P_A)(I - P_E)$	corresponds to the remaining (singular) part of $\lambda E - A$.

Assuming that $\lambda E - A$ is regular we are going to see that the singular part belonging to the projector $(I - P_A)(I - P_E)$ vanishes.

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Like [8, Lemma 2.24] we start by splitting system (1.1) into the two subsystems which occur by projecting (1.1) through P_E and $(I - P_E)$. The subsystem projected through P_E corresponds to (1.6a) and thus to the matrix $\tilde{C}$ as in 2.3 whereas the subsystem projected through $(I - P_E)$ corresponds to (1.6b) and thus the matrix $\tilde{N}$ as in 2.3.

Lemma 3.3. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ and $E = \tilde{C} + \tilde{N}$ be the decomposition (2.3). Then system (1.1) is equivalent (in the sense that there is a one-to-one correspondence of solutions) to the system*

$$\tilde{C}x_1^{k+1} = Ax_1^k + f_1^k, \tag{3.2a}$$
$$\tilde{N}x_2^{k+1} = Ax_2^k + f_2^k, \tag{3.2b}$$

for $k \in \mathbb{K}$, where

$$\begin{aligned} x_1^k &:= E^D E x^k, \qquad & x_2^k &:= \left(I - E^D E\right) x^k, \\ f_1^k &:= E^D E f^k, \qquad & f_2^k &:= \left(I - E^D E\right) f^k, \end{aligned} \tag{3.3}$$

for all $k \in \mathbb{K}^+$. With (3.3) subsystem (3.2a) is equivalent to the standard difference equation

$$x_1^{k+1} = E^D A x_1^k + E^D f_1^k, \text{ for } k \in \mathbb{K}, \tag{3.4}$$

Proof. Since $x^k = E^D E x^k + (I - E^D E)x^k = x_1^k + x_2^k$ we find that (1.1) is equivalent to

$$\left(\tilde{C} + \tilde{N}\right)\left(x_1^{k+1} + x_2^{k+1}\right) = A\left(x_1^k + x_2^k\right) + f^k. \tag{3.5}$$

Using (2.5), (2.2) we see that $\tilde{N}x_1^{k+1} = 0$, $\tilde{C}x_2^{k+1} = 0$, $\tilde{N}f_1^{k+1} = 0$, and $\tilde{C}f_2^{k+1} = 0$. Projecting (3.5) with $P_E = E^D E = \tilde{C}^D\tilde{C}$, i.e., multiplying (3.5) with $\tilde{C}^D\tilde{C}$ from the left first leads to (3.2a) and then to (3.2b). Multiplying (3.2a) by $\tilde{C}^D = E^D$ and adding $(I - \tilde{C}^D\tilde{C})x_1^{k+1} = 0$ finally gives the equivalence of (3.2a) and (3.4). A more detailed proof of Lemma 3.3 can be found in [1, Lemma 6]. ∎

System (3.2a) corresponds to the system projected by P_E and thus to the finite eigenvalues of $\lambda E - A$ whereas system (3.2b) corresponds to the system projected by $(I - P_E)$ and thus to the infinite eigenvalues of $\lambda E - A$. Because of the linearity of (1.1) we first consider the homogeneous case.

Analogous to [8, Lemma 2.25] we obtain the following Lemma.

Lemma 3.4. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$, $k_0 \in \mathbb{Z}$ and $v \in \mathbb{C}^n$. Then the following statements hold.*

1. *Let $\hat{v} = E^D E v$. Then*

$$x^k := (E^D A)^{k-k_0}\hat{v}, \quad k = k_0, k_0 + 1, \ldots \tag{3.6}$$

solves the homogeneous linear discrete-time descriptor system

$$Ex^{k+1} = Ax^k, \quad k = k_0, k_0 + 1, \ldots \tag{3.7}$$

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2. *Let* $\hat{v} = A^D A v$. *Then*

$$x^k := (A^D E)^{k_0 - k} \hat{v}, \quad k = k_0, k_0 - 1, \ldots \tag{3.8}$$

solves the homogeneous linear discrete-time descriptor system

$$E x^{k+1} = A x^k, \quad k = k_0 - 1, k_0 - 2, \ldots \tag{3.9}$$

3. *Let* $\hat{v} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$. *Then*

$$x^k := \begin{cases} (E^D A)^{k - k_0} \hat{v}, & k = k_0, k_0 - 1, \ldots \\ (A^D E)^{k_0 - k} \hat{v}, & k = k_0 - 1, k_0 - 2, \ldots \end{cases} \tag{3.10}$$

solves the homogeneous linear discrete-time descriptor system

$$E x^{k+1} = A x^k, \quad k \in \mathbb{Z}. \tag{3.11}$$

Proof.

1. With (2.1) and (2.2) we get

$$\begin{aligned} E x^{k+1} &= E(E^D A)(E^D A)^{k - k_0} E^D E v \\ &= A(E^D A)^{k - k_0} E^D E E^D E v \\ &= A(E^D A)^{k - k_0} E^D E v \\ &= A x^k, \end{aligned}$$

   for all $k = k_0, k_0 + 1, \ldots$

2. In this case we obtain

$$\begin{aligned} A x^k &= A(A^D E)^{k_0 - k} A^D A v \\ &= A(A^D E)(A^D E)^{k_0 - k - 1} A^D A v \\ &= E(A^D E)^{k_0 - k - 1} A^D A A^D A v \\ &= E(A^D E)^{k_0 - k - 1} A^D A v \\ &= E x^{k+1}, \end{aligned}$$

   for all $k = k_0 - 1, k_0 - 2, \ldots$

3. This follows from 1. and 2., since the definitions of x^{k_0} from 1. and 2. coincide. ∎

Since by (2.2)

$$(E^D A)^{k - k_0} E^D E v = E^D E (E^D A)^{k - k_0} v,$$

it is clear, that the solution x^k stays in the subspace $\operatorname{range}\left(E^D E\right)$ for all $k \geq k_0$. An analogous conclusion is possible for the case 2. in Lemma 3.4. In case 3. of Lemma 3.4 the solution even stays in $\operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$ all the time. To verify that all solutions of the homogeneous discrete-time descriptor system are in the form given by Lemma 3.4 we make the following observation.

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THEOREM 3.5. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ and suppose that the pencil $\lambda E - A$ is regular. Then,*

$$(I - E^D E)A^D A = (I - E^D E). \tag{3.12}$$

Proof. See [8, Lemma 2.26] as well as [1, Lemma 20]. ∎

REMARK 3.6. Using (3.12) and (2.8) we see that under the assumption that E and A commute we have

$$\begin{aligned}(I - P_A)(I - P_E) &= (I - E^D E) - (I - E^D E)A^D A = 0,\\ P_A(I - P_E) &= (I - P_E)P_A = (I - P_E), \text{ and}\\ (I - P_A)P_E &= (I - P_A).\end{aligned}$$

Looking back at Remark 3.2 this proves that under the assumptions of Theorem 3.5 the part belonging to the projector $(I - P_A)(I - P_E)$ indeed vanishes and thus the complete problem (1.1) really splits up into only three subproblems: one on $P_A P_E$, one on $(I - P_E)$, and one on $(I - P_A)$.

According to [8, Theorem 2.27] we obtain the following Theorem.

THEOREM 3.7. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ be such that $\lambda E - A$ is regular. Also, let $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *Let $\{x^k\}_{k \geq k_0}$ be any solution of (3.7). Then $\{x^k\}_{k \geq k_0}$ has the form (3.6) for some $\hat{v} \in \operatorname{range}\left(E^D E\right)$.*
2. *Let $\{x^k\}_{k \leq k_0}$ be any solution of (3.9). Then $\{x^k\}_{k \leq k_0}$ has the form (3.8) for some $\hat{v} \in \operatorname{range}\left(A^D A\right)$.*
3. *Let $\{x^k\}_{k \in \mathbb{Z}}$ be any solution of (3.11). Then $\{x^k\}_{k \in \mathbb{Z}}$ has the form (3.10) for some $\hat{v} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.*

Proof. Using the decomposition (2.3), (2.5), and (2.2) we have

$$A\tilde{N} = AE(I - E^D E) = E(I - E^D E)A = \tilde{N}A. \tag{3.13}$$

Furthermore, we see that for any $x \in \mathbb{C}^n$ with $A\tilde{N}x = 0$ we also have

$$(I - E^D E)A^D A\tilde{N}x = 0.$$

Using (3.12) this implies

$$(I - E^D E)\tilde{N}x = 0.$$

Thus, using (2.5) we have shown that for any $x \in \mathbb{C}^n$ with $A\tilde{N}x = 0$ we have

$$\tilde{N}x = 0. \tag{3.14}$$

Let $\{x^k\}_{k \in \mathbb{Z}}$ be any solution of (3.7). From Lemma 3.3 we get $\{x_1^k\}_{k \geq k_0}$, $\{x_2^k\}_{k \geq k_0}$ with $x^k = x_1^k + x_2^k$ which solve (3.2), respectively. With $\nu = \operatorname{ind}(E)$, using (2.4), (3.2), and (3.13) one then obtains

$$0 = \tilde{N}^\nu x_2^{k+1} = \tilde{N}^{\nu-1}Ax_2^k = A\tilde{N}^{\nu-1}x_2^k,$$

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for all $k \geq k_0$. From this and from (3.14) we see that we also have $\tilde{N}^{\nu-1}x_2^k = 0$ for all $k \geq k_0$. Discarding the identity for $k = k_0$ then yields

$$\tilde{N}^{\nu-1}x_2^k = 0, \quad k \geq k_0 + 1.$$

Shifting the index k, i.e., replacing k by $k+1$ shows that

$$\tilde{N}^{\nu-1}x_2^{k+1} = 0, \quad k+1 \geq k_0 + 1,$$

which is the same as

$$\tilde{N}^{\nu-1}x_2^{k+1} = 0, \quad k \geq k_0.$$

By repeating this procedure $\nu - 2$ times we finally get

$$\tilde{N}x_2^k = 0, \quad k \geq k_0.$$

Using (3.2) once again, this implies

$$Ax_2^k = 0,$$

and thus with (3.3) and (3.12) we have

$$x_2^k = (I - E^D E)x_2^k = (I - E^D E)A^D Ax_2^k = 0,$$

which means that $x^k = x_1^k$ for all $k \geq k_0$. Therefore, from Lemma 3.3 we know that $\{x_1^k\}$ solves

$$x_1^{k+1} = (E^D A)x_1^k,$$

for all $k \geq k_0$. Applying this formula recursively shows that

$$x_1^k = (E^D A)^{k-k_0}x_1^{k_0},$$

for every $k \geq k_0$. Summing up those implications we have that for all $k \geq k_0$

$$x^k = x_1^k = (E^D A)^{k-k_0}x_1^{k_0} = (E^D A)^{k-k_0}E^D E x^{k_0}, \tag{3.15}$$

which shows part 1. To prove part 2., let $\{x^k\}_{k \leq k_0}$ be any solution of (3.9). Set $l_0 := -k_0$ and $y^l := x^{-l}$ for $l \geq l_0$. By replacing $k = -l$ in (3.9) one obtains

$$Ex^{-l+1} = Ax^{-l}, \quad -l = -l_0 - 1, -l_0 - 2, \ldots,$$

which is equivalent to

$$Ex^{-(l-1)} = Ax^{-l}, \quad l = l_0 + 1, l_0 + 2, \ldots$$

By definition we can see that $\{y^l\}_{l \geq l_0}$ is a solution of

$$Ey^{l-1} = Ay^l, \quad l \geq l_0 + 1,$$

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and also a solution of

$$Ay^{l+1} = Ey^l, \quad l \geq l_0.$$

Using identity (3.15) for this reversed system means that

$$y^l = (A^D E)^{l-l_0} A^D A y^{l_0},$$

for all $l \geq l_0$. Undoing the replacements then yields

$$x^{-l} = (A^D E)^{l-l_0} A^D A x^{-l_0},$$

for all $l \geq l_0$ and thus

$$x^k = (A^D E)^{-k+k_0} A^D A x^{k_0},$$

for all $-k \geq -k_0$. Again, summing up these results shows that

$$x^k = (A^D E)^{k_0-k} A^D A x^{k_0}, \tag{3.16}$$

for all $k \leq k_0$. Finally, to prove part 3., let $\{x^k\}_{k\in\mathbb{Z}}$ be any solution of (3.11). Then from (3.15) we have

$$x^k = (E^D A)^{k-k_0} E^D E x^{k_0},$$

for all $k \geq k_0$ and especially for $k = k_0$ we see that

$$x^{k_0} = E^D E x^{k_0} \in \operatorname{range}\left(E^D E\right).$$

Also we know from (3.16) that

$$x^k = (A^D E)^{k_0-k} A^D A x^{k_0},$$

for all $k \leq k_0$ and especially for $k = k_0$ we see that

$$x^{k_0} = A^D A x^{k_0} \in \operatorname{range}\left(A^D A\right).$$

Thus, the claim of the Theorem follows with $\hat{v} = x^{k_0}$. □

One may think that it is not meaningful to look at case 3. of Theorem 3.7, since in most cases one starts at some time point and then calculates into the future. But as shown by the following Lemma 3.8, also those solutions (where one starts at $k_0 \in \mathbb{Z}$ and calculates a solution for $k \geq k_0$) are almost completely in the subspace $\operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.

Lemma 3.8. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ be such that $\lambda E - A$ is regular. Also, let $k_0 \in \mathbb{Z}$ and let $\nu_E = \operatorname{ind}(E), \nu_A = \operatorname{ind}(A)$. Then the following statements hold.*

1. *Let $\{x^k\}_{k\geq k_0}$ be any solution of (3.7). Then for all $k \geq k_0 + \nu_A$ it holds that $x^k \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.*

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2. Let $\{x^k\}_{k\le k_0}$ be any solution of (3.9). Then for all $k \le k_0 - \nu_E$ it holds that $x^k \in \operatorname{range}\left(E^D E\right) \cap \operatorname{range}\left(A^D A\right)$.

Proof. Since $k \ge k_0+\nu_A$ it follows that there exists $\hat{k} \ge 0$ such that $k = \hat{k}+k_0+\nu_A$. From Theorem 3.7 using (3.6) and (2.1) we then know that for some $v \in \mathbb{C}^n$ we have

$$\begin{aligned}
A^D A x^k &= A^D A (E^D A)^{k-k_0} E^D E v \\
&= A^D A (E^D)^{k-k_0} A^{k-k_0} E^D E v \\
&= A^D A (E^D)^{k-k_0} A^{\nu_A} A^{\hat{k}} E^D E v \\
&= A^D A (E^D)^{k-k_0} A^D A^{\nu_A+1} A^{\hat{k}} E^D E v \\
&= (E^D)^{k-k_0} A^D A A^D A^{\nu_A+1} A^{\hat{k}} E^D E v \\
&= (E^D)^{k-k_0} A^D A^{\nu_A+1} A^{\hat{k}} E^D E v \\
&= (E^D A)^{k-k_0} E^D E v \\
&= x^k.
\end{aligned}$$

Also, we naturally get

$$\begin{aligned}
E^D E x^k &= E^D E (E^D A)^{k-k_0} E^D E v \\
&= (E^D A)^{k-k_0} E^D E v = x^k,
\end{aligned} \tag{3.17}$$

and thus the assertion of part 1. follows. As in (3.17) one gets that $A^D A x^k = x^k$. Let $k = -\hat{k} + k_0 - \nu_E$ with $\hat{k} \ge 0$. Then again for some $v \in \mathbb{C}^n$ it follows that

$$\begin{aligned}
E^D E x^k &= E^D E (A^D E)^{k_0-k} A^D A v \\
&= E^D E (A^D)^{k_0-k} E^{k_0-k} A^D A v \\
&= E^D E (A^D)^{k_0-k} E^{\nu_E} E^{\hat{k}} A^D A v \\
&= E^D E (A^D)^{k_0-k} E^D E^{\nu_E+1} E^{\hat{k}} A^D A v \\
&= (A^D)^{k_0-k} E^D E E^D E^{\nu_E+1} E^{\hat{k}} A^D A v \\
&= (A^D)^{k_0-k} E^D E^{\nu_E+1} E^{\hat{k}} A^D A v \\
&= (A^D E)^{k_0-k} A^D A v \\
&= x^k,
\end{aligned}$$

which proves part 2. □

To understand the relevance of Lemma 3.8, consider a homogeneous forward system, i.e., a system of the type (3.7), and assume that a consistent initial condition $x^{k_0} \in \operatorname{range}\left(E^D E\right)$ is given. Assuming that $EA = AE$ we can apply the projector $A^D A$ to x^{k_0} obtaining a new consistent initial condition

$$\tilde{x}^{k_0} = A^D A x^{k_0} \in \operatorname{range}\left(E^D E\right) \cap \operatorname{range}\left(A^D A\right).$$

Lemma 3.8 and the proof of Lemma 3.8 then show that for all $k \ge k_0 + \operatorname{ind}(A)$ we have

$$\tilde{x}^k = x^k.$$

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Thus, it seems reasonable to demand

$$x^{k_0} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right) \tag{3.18}$$

in the first place.
Also, only in case that (3.18) holds, we get something like an invertibility of the operator that calculates x^{k+1} from x^k. To understand this, imagine that a fixed x^{k_0} is given. From this we calculate a finite number of steps κ into the future. Thus, we have $x^{k_0+\kappa}$. From this state we then calculate κ steps back into the past to obtain $\tilde{x}^{k_0}$. We then have $x^{k_0} = \tilde{x}^{k_0}$ if condition (3.18) holds. Otherwise we cannot be sure that $x^{k_0} = \tilde{x}^{k_0}$ holds, as shown in the following example.

Example 3.9. Consider the homogeneous linear discrete-time descriptor system defined by

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=E} x^{k+1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{:=A} x^k,\ k \geq 0,\ x^0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{3.19}$$

Clearly, we have $EA = AE$, $E^D = E$, $A^D = A$ and $\lambda E - A$ is regular. Thus, the pencil (E, A), corresponding to system (3.19), satisfies all assumptions of Lemma 3.8. Since the index of the matrix A is $\operatorname{ind}(A) = 1$ this means that the iterate x^1 has to be in $\operatorname{range}\left(A^D A\right)$. Indeed,

$$Ax^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \operatorname{range}\left(A^D A\right). \tag{3.20}$$

Now let us calculate back one step from (3.20), i.e., let us consider the reversed system

$$A\tilde{x}^{l+1} = E\tilde{x}^l,\ l \leq 0,\ \tilde{x}^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We see that

$$E\tilde{x}^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{x}^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and thus

$$\tilde{x}^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq x^0.$$

So far, we have characterized all the solutions of the homogeneous descriptor system. Thus, what we still need to characterize all solutions of (1.1) is one particular

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solution of the inhomogeneous system. Using [8, Theorem 2.28] we can prove the following.

THEOREM 3.10. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ be such that $\lambda E - A$ is regular. Also, let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}}$ with $f^k \in \mathbb{C}^n$ and $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *The linear discrete-time descriptor system*

$$Ex^{k+1} = Ax^k + f^k, \quad k \geq k_0,$$

has the particular solution $\{x_1^k + x_2^k\}_{k\geq k_0}$, where

$$\begin{aligned} x_1^k &:= \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j, \\ x_2^k &:= -(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i}, \end{aligned}$$

for $k \geq k_0$. For the construction of the iterate x^k only the f_k with $k \geq k_0$ have to be employed.

2. *The linear discrete-time descriptor system*

$$Ex^{k+1} = Ax^k + f^k, \quad k \leq k_0 - 1, \tag{3.21}$$

has the particular solution $\{x_1^k + x_2^k\}_{k\leq k_0}$, where

$$\begin{aligned} x_1^k &:= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1}, \\ x_2^k &:= -\sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}, \end{aligned}$$

for $k \leq k_0$. For the construction of the iterate x^k only the f_k with $k \leq k_0 - 1$ have to be employed.

Proof. Let $E = \tilde{C} + \tilde{N}$ be the decomposition (2.3). Then, using (2.2) we have the identities

$$E^D E x_1^k = \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D E E^D f^j = x_1^k,$$

$$(I - E^D E)x_2^k = -(I - E^D E)(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i} = x_2^k.$$

Using (2.5) and (2.1) one can also conclude, that for all $k \geq k_0$ it follows that

$$\tilde{C} x_1^{k+1} = \tilde{C} \sum_{j=k_0}^{k} (E^D A)^{k+1-j-1} E^D f^j$$

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$$
\begin{aligned}
&= \tilde{C}\left(\sum_{j=k_0}^{k-1}(E^DA)^{k-j}E^Df^j + E^Df^k\right)\\
&= \tilde{C}\left((E^DA)\sum_{j=k_0}^{k-1}(E^DA)^{k-j-1}E^Df^j + E^Df^k\right)\\
&= AE^DEx_1^k + E^DEf^k\\
&= Ax_1^k + E^DEf^k,
\end{aligned}
$$

and with

$$
(I-E^DE)E^{\nu_E} = \begin{cases} (I-E^DE)E^DE^{\nu_E-1} = 0, & \text{if } \nu_E \geq 1,\\ (I-E^DE) = (I-I) = 0, & \text{if } \nu_E = 0, \end{cases} \tag{3.22}
$$

we obtain

$$
\begin{aligned}
\tilde{N}x_2^{k+1} &= E(I-E^DE)x_2^{k+1}\\
&= -(I-E^DE)\sum_{i=0}^{\nu_E-1}(A^DE)^{i+1}f^{k+i+1}\\
&= -(I-E^DE)\sum_{i=0}^{\nu_E-2}(A^DE)^{i+1}f^{k+i+1}\\
&= -(I-E^DE)A^DA\sum_{i=1}^{\nu_E-1}(A^DE)^{i}f^{k+i}\\
&= -A(I-E^DE)\sum_{i=0}^{\nu_E-1}(A^DE)^{i}A^Df^{k+i} + (I-E^DE)f^k\\
&= Ax_2^k + (I-E^DE)f^k,
\end{aligned}
$$

by using (3.12). With these results and Lemma 3.3 one immediately gets that $\{x^k\}_{k\geq k_0}$ with

$$
x^k = E^DEx_1^k + (I-E^DE)x_2^k = x_1^k + x_2^k
$$

is a solution and thus part 1. of the assertion follows. To prove part 2. we perform a variable substitution. By replacing $l := -k$ and $l_0 := -k_0$ in (3.21) one gets the system

$$
Ex^{-l+1} = Ax^{-l} + f^{-l}, \quad -l \leq -l_0 - 1,
$$

which is equivalent to the system

$$
Ex^{-(l-1)} = Ax^{-l} + f^{-l}, \quad l \geq l_0 + 1.
$$

By further replacing $y^l := x^{-l}$ for $l \geq l_0$ one gets

$$
Ey^{l-1} = Ay^l + f^{-l}, \quad l \geq l_0 + 1.
$$

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Shifting the index l, i.e., replacing l by $l+1$ shows that

$$Ey^l = Ay^{l+1} + f^{-l-1}, \quad l+1 \geq l_0 + 1,$$

which in turn is equivalent to

$$Ay^{l+1} = Ey^l - f^{-l-1}, \quad l \geq l_0.$$

Setting $g^l := -f^{-l-1}$ we can finally write this equation as

$$Ay^{l+1} = Ey^l + g^l, \quad l \geq l_0.$$

From the results of the first part we then get a solution of this last system as

$$y^l = \sum_{j=l_0}^{l-1} (A^D E)^{l-j-1} A^D g^j - (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D g^{l+i}.$$

Undoing the replacement $y^l = x^{-l}$ in this equations then leads to

$$x^{-l} = -\sum_{j=l_0}^{l-1} (A^D E)^{l-j-1} A^D f^{-j-1} + (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{-(l+i)-1},$$

and undoing the replacement $k = -l$ finally gives us

$$\begin{aligned} x^k &= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} - \sum_{j=-k_0}^{-k-1} (A^D E)^{-k-j-1} A^D f^{-j-1} \\ &= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} - \sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}. \ \square \end{aligned}$$

The parts x_1^k and x_2^k of the solution in Theorem 3.10 part 1. and Lemma 3.3 correspond to each other. With (2.8) we see that $x_1^k = P_E x_1^k$ corresponds to the finite eigenvalues of $\lambda E - A$ and $x_2^k = (I - P_E) x_2^k$ corresponds to the infinite eigenvalues of $\lambda E - A$. Theorem 3.10 shows that the problem (1.1) only decomposes into two subproblems if we only want to compute the solution into one direction, i.e., $k \geq 0$. With the notation of Remark 3.2 we can say that in case 1. of Theorem 3.10 the projectors $P_E P_A$ and $P_E(I - P_A)$ can be treated as one. If, however, we move to the case where we want to get a solution for all $k \in \mathbb{Z}$, then we need all three projectors introduced in Remark 3.6. Similar to Lemma 3.3 we obtain the following result.

Lemma 3.11. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ be such that $\lambda E - A$ is regular. Further, let $E = \tilde{C} + \tilde{N}$ and analogously $A = \tilde{D} + \tilde{M}$ be decompositions as in (2.3). Let $\{x_1^k\}_{k\in\mathbb{Z}}$, $\{x_2^k\}_{k\in\mathbb{Z}}$, $\{x_3^k\}_{k\in\mathbb{Z}}$ be solutions of*

$$\begin{aligned} \tilde{C} x_1^{k+1} &= \tilde{M} x_1^k + (I - A^D A) f^k, \\ \tilde{C} x_2^{k+1} &= \tilde{D} x_2^k + A^D A E^D E f^k, \\ \tilde{N} x_3^{k+1} &= \tilde{D} x_3^k + (I - E^D E) f^k, \end{aligned}$$

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respectively. Then $\{x^k\}_{k\in\mathbb{Z}}$ *with*

$$x^k := (I - A^D A)x_1^k + A^D A E^D E x_2^k + (I - E^D E)x_3^k,$$

is a solution of

$$Ex^{k+1} = A^k + f^k.$$

Proof. First of all, by (3.12) we see that

$$\begin{aligned} & (I - A^D A) + (I - E^D E) + A^D A E^D E \\ = \;& I - A^D A + I - (I - A^D A)E^D E \\ = \;& I - A^D A + I - (I - A^D A) = I. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} \tilde{D}(I - A^D A) &= A A^D A(I - A^D A) = 0, \\ \tilde{M}(A^D A E^D E) &= A(I - A^D A)(A^D A E^D E) = 0, \\ \tilde{M}(I - E^D E) &= A(I - A^D A)(I - E^D E) \\ &= A\left((I - E^D E) - (I - E^D E)A^D A\right) = 0. \end{aligned}$$

Using all those identities together we finally obtain the following equation:

$$\begin{aligned} Ex^{k+1} &= E(I - A^D A)x_1^{k+1} + EA^D A E^D E x_2^{k+1} + E(I - E^D E)x_3^{k+1} \\ &= (I - A^D A)\tilde{C}x_1^{k+1} + A^D A E^D E \tilde{C} x_2^{k+1} + (I - E^D E)\tilde{N}x_3^{k+1} \\ &= (I - A^D A)\tilde{M}x_1^k + (I - A^D A)f^k + \\ &\quad A^D A E^D E \tilde{D} x_2^k + A^D A E^D E f^k + \\ &\quad (I - E^D E)\tilde{D}x_3^k + (I - E^D E)f^k \\ &= (I - A^D A)\tilde{M}x_1^k + A^D A E^D E \tilde{D} x_2^k + (I - E^D E)\tilde{D}x_3^k + f^k \\ &= \tilde{M}(I - A^D A)x_1^k + \tilde{D}(I - A^D A)x_1^k + \\ &\quad \tilde{M}A^D A E^D E x_2^k + \tilde{D}A^D A E^D E x_2^k + \\ &\quad \tilde{M}(I - E^D E)x_3^k + \tilde{D}(I - E^D E)x_3^k + f^k \\ &= Ax^k + f^k. \end{aligned}$$

Here we have used that $\tilde{D} = AA^D A$, and thus $\tilde{D}$ commutes with the matrices E and A. ◻

Using Lemma 3.11 we can construct a particular solution for the case $\mathbb{K} = \mathbb{Z}$, as we did in Theorem 3.10 for the case that $\mathbb{K} = \{k \in \mathbb{Z} : k_b \leq k\}$.

THEOREM 3.12. *Let* $E, A \in \mathbb{C}^{n,n}$ *with* $EA = AE$ *be such that* $\lambda E - A$ *is regular. Also, let* $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}} \subset \mathbb{C}^n$ *and* $k_0 \in \mathbb{Z}$. *Then a solution* $\{x^k\}_{k\in\mathbb{Z}}$ *of the system*

$$Ex^{k+1} = Ax^k + f^k, \quad k \in \mathbb{Z},$$

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is given by $x^k := x_1^k + x_2^k + x_3^k$, *where*

$$x_1^k := (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1},$$

$$x_2^k := \begin{cases} A^D A \displaystyle\sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j, & k \geq k_0, \\ -E^D E \displaystyle\sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}, & k \leq k_0, \end{cases}$$

$$x_3^k := -(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i},$$

for $k \in \mathbb{Z}$.

Proof. Considering the decompositions $E = \tilde{C} + \tilde{N}$ and $A = \tilde{D} + \tilde{M}$ as in (2.3), using (2.2), (3.12), and (3.22) we have

$$\begin{aligned}
\tilde{M} x_1^k &= A(I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} \\
&= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^{i+1} f^{k-i-1} \\
&= (I - A^D A) \sum_{i=0}^{\nu_A - 2} (E^D A)^{i+1} f^{k-i-1} \\
&= (I - A^D A) \sum_{i=1}^{\nu_A - 1} (E^D A)^{i} f^{k-i} \\
&= (I - A^D A) \left(\sum_{i=0}^{\nu_A - 1} (E^D A)^{i} f^{k-i} - f^k \right) \\
&= -(I - A^D A) f^k + (I - A^D A) E^D E \sum_{i=0}^{\nu_A - 1} (E^D A)^{i} f^{k-i} \\
&= -(I - A^D A) f^k + E(I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^{i} E^D f^{k-i} \\
&= -(I - A^D A) f^k + E x_1^{k+1} \\
&= -(I - A^D A) f^k + (\tilde{C} + \tilde{N}) x_1^{k+1} \\
&= -(I - A^D A) f^k + \tilde{C} x_1^{k+1},
\end{aligned}$$

where the last identity holds, since x_1^k has the form $x_1^k = (I - A^D A) y_1^k$ for some y_1^k and

$$\begin{aligned}
\tilde{N} x_1^k &= E(I - E^D E)(I - A^D A) y_1^k \\
&= E \left(I - A^D A - E^D E (I - A^D A) \right) y_1^k = 0,
\end{aligned} \tag{3.23}$$

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because of (3.12). As in Theorem 3.10, part 1. one obtains

$$\tilde{N}x_3^{k+1} = Ax_3^k + (I - E^D E)f^k = (\tilde{D} + \tilde{M})x_3^k + (I - E^D E)f^k.$$

Again as in (3.23) it follows that

$$\tilde{M}x_3^k = 0,$$

and thus

$$\tilde{N}x_3^{k+1} = \tilde{D}x_3^k + (I - E^D E)f^k.$$

Finally, for $k \geq k_0$ one has

$$\begin{aligned}
\tilde{C}x_2^{k+1} &= \tilde{C}A^D A \sum_{j=k_0}^{k} (E^D A)^{k-j} E^D f^j \\
&= \tilde{C}E^D A^D A \left(\sum_{j=k_0}^{k-1} (E^D A)^{k-j} f^j + f^k \right) \\
&= EE^D EE^D A^D A \left(\sum_{j=k_0}^{k-1} (E^D A)^{k-j} f^j + f^k \right) \\
&= E^D EA^D A \left(\sum_{j=k_0}^{k-1} (E^D A)^{k-j} f^j + f^k \right) \\
&= E^D EA^D A \left(E^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} f^j + f^k \right) \\
&= AA^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j + A^D AE^D Ef^k \\
&= AA^D AA^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j + A^D AE^D Ef^k \\
&= \tilde{D}A^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j + A^D AE^D Ef^k \\
&= \tilde{D}x_2^k + A^D AE^D Ef^k,
\end{aligned}$$

and for $k < k_0$ analogously,

$$\tilde{D}x_2^k = -\tilde{D}E^D E \sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}$$

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$$
\begin{aligned}
&= -AA^DAE^DEA^D \sum_{j=k+1}^{k_0} (A^DE)^{j-k-1} f^{j-1} \\
&= -AA^DE^DE \sum_{j=k+1}^{k_0} (A^DE)^{j-k-1} f^{j-1} \\
&= -AA^DE^DE \left(\sum_{j=k+2}^{k_0} (A^DE)^{j-k-1} f^{j-1} + f^k \right) \\
&= -AA^DE^DEf^k - A^DAA^DEE^DE \sum_{j=k+2}^{k_0} (A^DE)^{j-k-2} f^{j-1} \\
&= -AA^DE^DEf^k - A^DEE^DEE^DE \sum_{j=k+2}^{k_0} (A^DE)^{j-k-2} f^{j-1} \\
&= -AA^DE^DEf^k + EE^DE \left(-E^DE \sum_{j=k+2}^{k_0} (A^DE)^{j-k-2} A^D f^{j-1} \right) \\
&= -AA^DE^DEf^k + \tilde{C}x_2^{k+1}.
\end{aligned}
$$

Lemma 3.11 then implies the assertion. $\square$

x_1^k, x_2^k, and x_3^k from Lemma 3.11 and Theorem 3.12 again correspond to each other. As stated in Remark 3.6 the problem (1.1) is decomposed into three projected subsystems by Theorem 3.12 with x_1^k corresponding to the projector $(I - P_A)$, x_2^k corresponding to the projector P_AP_E, and x_3^k corresponding to the projector $(I - P_E)$.

4. Main results. In the previous section we have constructed a particular solution of the inhomogeneous problem and we have explicitly characterized all solutions of the homogeneous problem. This enables us to specify all solutions of the inhomogeneous problem.

THEOREM 4.1. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ be such that $\lambda E - A$ is regular. Also, let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}}$ with $f^k \in \mathbb{C}^n$ and $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *Every solution $\{x^k\}_{k\geq k_0}$ of*

$$
Ex^{k+1} = Ax^k + f^k, \quad k \geq k_0 \tag{4.1}
$$

satisfies

$$
\begin{aligned}
x^k &= \left(E^DA\right)^{k-k_0} E^DEv + \sum_{j=k_0}^{k-1} \left(E^DA\right)^{k-j-1} E^D f^j \\
&\quad - \left(I - E^DE\right) \sum_{i=0}^{\nu_E-1} \left(A^DE\right)^i A^D f^{k+i},
\end{aligned}
$$

for $k \geq k_0$ and for some $v \in \mathbb{C}^n$.

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2. *Every solution* $\{x^k\}_{k\le k_0}$ *of*

$$Ex^{k+1} = Ax^k + f^k, \quad k \le k_0 - 1 \tag{4.2}$$

satisfies

$$\begin{aligned} x^k &= \left(A^D E\right)^{k_0-k} A^D A v + (I - A^D A)\sum_{i=0}^{\nu_A-1}(E^D A)^i E^D f^{k-i-1} \\ &\quad - \sum_{j=k+1}^{k_0}(A^D E)^{j-k-1}A^D f^{j-1}, \end{aligned}$$

for $k \le k_0$ *and for some* $v \in \mathbb{C}^n$.

3. *Every solution* $\{x^k\}_{k\in\mathbb{Z}}$ *of*

$$Ex^{k+1} = Ax^k + f^k, \quad k \in \mathbb{Z} \tag{4.3}$$

satisfies

$$\begin{aligned} x^k &= (I - A^D A)\sum_{i=0}^{\nu_A-1}(E^D A)^i E^D f^{k-i-1} \\ &\quad + \begin{cases} (E^D A)^{k-k_0}\hat{v} + A^D A \displaystyle\sum_{j=k_0}^{k-1}(E^D A)^{k-j-1}E^D f^j, \ k \ge k_0 \\ (A^D E)^{k_0-k}\hat{v} - E^D E \displaystyle\sum_{j=k+1}^{k_0}(A^D E)^{j-k-1}A^D f^{j-1}, \ k \le k_0 \end{cases} \\ &\quad - (I - E^D E)\sum_{i=0}^{\nu_E-1}(A^D E)^i A^D f^{k+i}, \end{aligned} \tag{4.4}$$

for $k \in \mathbb{Z}$ *and for some* $\hat{v}$ *which has the form* $\hat{v} = A^D A E^D E v$*, where* $v \in \mathbb{C}^n$ *is arbitrary.*

Proof. Since the problem is linear any solution may be written as a particular solution of the inhomogeneous problem plus a solution of the homogeneous problem. Thus, we can derive the result from Theorems 3.7, 3.10, and 3.12 as well as Lemma 3.4. ∎

Since Theorem 4.1 explicitly gives all solutions of (1.1) we can also easily specify all consistent initial conditions. We only have to look at the values of all possible solutions at k_0.

Corollary 4.2. *Let the assumptions of Theorem 4.1 hold. Consider the initial condition*

$$x^{k_0} = x_0. \tag{4.5}$$

Then the following statements hold.

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1. *The initial value problem consisting of (4.1) and (4.5) possesses a solution if and only if there exists a* $v \in \mathbb{C}^n$ *with*

$$x_0 = E^D E v - \left(I - E^D E\right) \sum_{i=0}^{\nu_E - 1} \left(A^D E\right)^i A^D f^{k_0+i}.$$

If this is the case, then the solution is unique.

2. *The initial value problem consisting of (4.2) and (4.5) possesses a solution if and only if there exists a* $v \in \mathbb{C}^n$ *with*

$$x_0 = A^D A v + (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k_0-i-1}.$$

If this is the case, then the solution is unique.

3. *The problem consisting of (4.3) and (4.5) possesses a solution if and only if there exists a* $v \in \mathbb{C}^n$ *with*

$$\begin{aligned} x_0 &= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k_0-i-1} \\ &\quad + A^D A E^D E v \\ &\quad - (I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k_0+i}. \end{aligned}$$

If this is the case, then the solution is unique.

Finally, recall that the assumption $EA = AE$ in Theorem 4.1 and Corollary 4.2 is not a restriction, since due to Lemma 3.1 we can transform any system of the form (1.1) to a system of the form (3.1) with $\tilde{E}\tilde{A} = \tilde{A}\tilde{E}$ by premultiplying the original equations (1.1) by a matrix of the form $(\tilde{\lambda}E - A)^{-1}$, which exists since the pencil $\lambda E - A$ is assumed to be regular. Thus, the assumptions of Theorem 4.1 and Corollary 4.2 can essentially be reduced to the regularity of the matrix pencil $\lambda E - A$ by performing the following replacements in Theorem 4.1 and Corollary 4.2:

$$E \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} E, \qquad A \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} A, \qquad f \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} f.$$

5. Conclusion. In this text we concentrated on regular systems, i.e., on systems of the form (1.1) where the matrix pencil $\lambda E - A$ is regular. For such systems we have presented the explicit solution with the help of the Drazin inverse. In contrast to the continuous-time case, one has to distinguish between four different cases for such systems. The first case is where one has an initial condition given at point $k_0 \in \mathbb{Z}$ and only wants to get a solution for indices $k \geq k_0$. The second case is where one has an initial condition given at point $k_0 \in \mathbb{Z}$ and only wants to get a solution for indices $k \leq k_0$. These first two cases are closely related, since the first case can be transformed into the second one by a variable substitution, as shown in the proof of Theorem 3.10.

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The third case is really different from the first two cases. Here, also an initial condition is given at some point $k_0 \in \mathbb{Z}$ but one is looking for a solution for indices $k \geq k_0$, as well as for indices $k \leq k_0$. This puts stronger restrictions on the initial condition, i.e., the set of consistent initial conditions in the third case is smaller than in the first or second case, as we saw in Corollary 4.2.
The fourth case has not been examined in this paper. It is the case where one only wants to get a solution on a finite interval, i.e., a solution for all $k \in \mathbb{Z}$ with $k_b \leq k \leq k_f$ where $k_b \in \mathbb{Z}$ and $k_f \in \mathbb{Z}$. This case is more complicated, as boundary value conditions have to be introduced on both ends of the interval to fix a unique solution. By first introducing only an initial condition for x^{k_b}, we can see from the solution formulas in Theorem 4.1 that all but the last $\operatorname{ind}(E)+1$ elements are already uniquely determined. The additional boundary condition for x^{k_f} then only fixes these last $\operatorname{ind}(E)+1$ elements

$$x^{k_f-\operatorname{ind}(E)}, x^{k_f-\operatorname{ind}(E)+1}, \ldots, x^{k_f}$$

of the solution and thus can be considered irrelevant. This fourth case has been studied in [10, 11].
The results in this paper could in principle be used to actually compute solutions of systems of the form (1.1) on a computer but another method employing the singular value decomposition is better suited for this purpose; see [2, Chapter 5].

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