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A NEW SOLVABLE CONDITION FOR A PAIR OF GENERALIZED SYLVESTER EQUATIONS*

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Abstract. A necessary and sufficient condition is given for the quaternion matrix equations $A_iX + YB_i = C_i$ $(i = 1, 2)$ to have a pair of common solutions X and Y. As a consequence, the results partially answer a question posed by Y.H. Liu (Y.H. Liu, Ranks of solutions of the linear matrix equation $AX + YB = C$, Comput. Math. Appl., 52 (2006), pp. 861-872).

Key words. Quaternion matrix equation, Generalized Sylvester equation, Generalized inverse, Minimal rank, Maximal rank.

AMS subject classifications. 15A03, 15A09, 15A24, 15A33.

1. Introduction. Throughout this paper, we denote the real number field by $\mathbb{R}$, the complex number field by $\mathbb{C}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid \ i^2 = j^2 = k^2 = ijk = -1, \, a_0, a_1, a_2, a_3 \ \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, the identity matrix with the appropriate size by I, the transpose of a matrix A by A^T, the column right space, the row left space of a matrix A over $\mathbb{H}$ by $\mathcal{R}(A)$, $\mathcal{N}(A)$, respectively, a reflexive inverse of a matrix A by A^+ which satisfies simultaneously $AA^+A = A$ and $A^+AA^+ = A^+$. Moreover, R_A and L_A stand for the two projectors $L_A = I - A^+A$, $R_A = I - AA^+$ induced by A. By [1], for a quaternion matrix A, $\dim \mathcal{R}(A) = \dim \mathcal{N}(A)$, which is called the rank of A and denoted by $r(A)$.

Many problems in systems and control theory require the solution of the generalized Sylvester matrix equation $AX+YB = C$. Roth [2] gave a necessary and sufficient condition for the consistency of this matrix equation, which was called Roth's theorem on the equivalence of block diagonal matrices. Since Roth's paper appeared

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in 1952, Roth's theorem has been widely extended (see, e.g., [2]-[16]). Perturbation analysis of generalized Sylvester eigenspaces of matrix quadruples [17] leads to a pair of generalized Sylvester equations of the form

$$A_1X + YB_1 = C_1, A_2X + YB_2 = C_2. \tag{1.1}$$

In 1994, Wimmer [12] gave a necessary and sufficient condition for the consistency of (1.1) over $\mathbb{C}$ by matrix pencils. In 2002, Wang, Sun and Li [14] established a necessary and sufficient condition for the existence of constant solutions with bi(skew)symmetric constrains to (1.1) over a finite central algebra. Liu [16] in 2006 presented a necessary and sufficient condition for the pair of equations in (1.1) to have a common solution X or Y over $\mathbb{C}$, respectively, and proposed an open problem: find a necessary and sufficient condition for system (1.1) to have a pair of solutions X and Y by ranks.

Motivated by the work mentioned above and keeping applications and interests of quaternion matrices in view (e.g., [18]-[34]), in this paper we investigate the above open problem over $\mathbb{H}$. In Section 2, we establish a necessary and sufficient condition for (1.1) to have a pair of solutions X and Y over $\mathbb{H}$. In section 3, we present a counterexample to illustrate the errors in Liu's paper [16]. A conclusion and a further research topic related to (1.1) are also given.

2. Main results. The following lemma is due to Marsaglia and Styan [35], which can also be generalized to $\mathbb{H}$.

LEMMA 2.1. *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$ and $C \in \mathbb{H}^{l\times n}$. Then they satisfy the following:*
(*a*) $r\begin{bmatrix} A & B \end{bmatrix} = r(A) + r(R_AB) = r(B) + r(R_BA)$.
(*b*) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CL_A) = r(C) + r(AL_C)$.
(*c*) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(R_BAL_C)$.

From Lemma 2.1 we can easily get the following.

LEMMA 2.2. *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$ and $E \in \mathbb{H}^{l\times i}$. Then*
(*a*) $r(CL_A) = r\begin{bmatrix} A \\ C \end{bmatrix} - r(A)$.
(*b*) $r\begin{bmatrix} B & AL_C \end{bmatrix} = r\begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C)$.
(*c*) $r\begin{bmatrix} C \\ R_BA \end{bmatrix} = r\begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B)$.

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(d) $r\begin{bmatrix} A & BL_D \\ R_EC & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{bmatrix} - r(D) - r(E).$

The following three lemmas are due to Baksalary and Kala [6], Tian [36],[37], respectively, which can be generalized to $\mathbb{H}$.

LEMMA 2.3. *Let $A \in \mathbb{H}^{m\times p}, B \in \mathbb{H}^{q\times n}$ and $C \in \mathbb{H}^{m\times n}$ be known and $X \in \mathbb{H}^{p\times q}$ unknown. Then the matrix equation $AX + YB = C$ is solvable if and only if*

$$r\begin{bmatrix} B & A \\ 0 & C \end{bmatrix} = r(A) + r(B).$$

In this case, the general solution to the matrix equation is given by

$$X = A^{+}C + UB + L_AV,$$
$$Y = R_AC - AU + L_AWR_B,$$

where $U \in \mathbb{H}^{p\times q}, V \in \mathbb{H}^{p\times n}$ and $W \in \mathbb{H}^{m\times q}$ are arbitrary.

LEMMA 2.4. *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times p}$, $C \in \mathbb{H}^{q\times n}$ be given, $Y \in \mathbb{H}^{p\times n}$, $Z \in \mathbb{H}^{m\times q}$ be two variant matrices. Then*

$$\max_{Y,Z} r(A - BY - ZC) = \min\left\{m,\ n,\ r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right\}; \tag{2.1}$$

$$\min_{Y,Z} r(A - BY - ZC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C). \tag{2.2}$$

LEMMA 2.5. *The matrix equation $A_1X_1B_1 + A_2X_2B_2 + A_3Y + ZB_3 = C$ is solvable if and only if the following four rank equalities are all satisfied:*

$$r\begin{bmatrix} C & A_1 & A_2 & A_3 \\ B_3 & 0 & 0 & 0 \end{bmatrix} = r\,[A_1, A_2, A_3] + r(B_3),$$

$$r\begin{bmatrix} C & A_3 \\ B_1 & 0 \\ B_2 & 0 \\ B_3 & 0 \end{bmatrix} = r(A_3) + r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix},$$

$$r\begin{bmatrix} C & A_1 & A_3 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} + r\,[A_1, A_3],$$

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$$r\begin{bmatrix} C & A_2 & A_3 \\ B_1 & 0 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} + r\left[A_2, A_3\right].$$

LEMMA 2.6. *(Lemma 2.3 in [38]) Let A, B be matrices over $\mathbb{H}$ and*

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, B = \left[B_1, B_2\right], S = A_2 L_{A_1}, T = R_{B_1} B_2.$$

Then

$$A^{+} = \left[A_1^{+} - L_{A_1} S^{+} A_2 A_1^{+}, L_{A_1} S^{+}\right], B^{+} = \begin{bmatrix} B_1^{+} - B_1^{+} B_2 T^{+} R_{B_1} \\ T^{+} R_{B_1} \end{bmatrix}$$

are reflexive inverses of A and B, respectively.

LEMMA 2.7. *Suppose $A_1, A_2 \in \mathbb{H}^{m\times p}, B_1, B_2 \in \mathbb{H}^{q\times n}$ and $\widehat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ are given, $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ and $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ are any matrices with compatible dimensions. Then*
(a) $[I_p, 0]\, L_{[A_1,A_2]} V$ and $[0, I_p]\, L_{[A_1,A_2]} V$ are independent, that is, for any $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$, $[I_p, 0]\, L_{[A_1,A_2]} V$ only relates to V_2 and the change of $[0, I_p]\, L_{[A_1,A_2]} V$ only relates to V_1, if and only if

$$r\left[A_1, A_2\right] = r\left(A_1\right) + r\left(A_2\right).$$

(b) $W R_{\widehat{B}} \begin{bmatrix} I_q \\ 0 \end{bmatrix}$ and $W R_{\widehat{B}} \begin{bmatrix} 0 \\ I_q \end{bmatrix}$ are independent, that is, for any $W = \left[W_1, W_2\right]$, $W R_{\widehat{B}} \begin{bmatrix} I_q \\ 0 \end{bmatrix}$ only relates to W_1 and $W R_{\widehat{B}} \begin{bmatrix} 0 \\ I_q \end{bmatrix}$ only relates to W_2, if and only if

$$r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = r\left(B_1\right) + r\left(B_2\right).$$

Proof. From Lemma 2.6, we have

$$\begin{aligned}
&[I_p, 0]\, L_{[A_1,A_2]} V \\
&= [I_p, 0]\left(I - \begin{bmatrix} A_1^{+} - A_1^{+} A_2[(I - A_1 A_1^{+})A_2]^{+}(I - A_1 A_1^{+}) \\ [(I - A_1 A_1^{+})A_2]^{+}(I - A_1 A_1^{+}) \end{bmatrix} [A_1, A_2]\right) V \\
&= [I_p, 0]\left(I - \begin{bmatrix} A_1 A_1^{+} & A_1^{+} A_2 - A_1^{+} A_2[(I - A_1 A_1^{+})A_2]^{+}(I - A_1 A_1^{+})A_2 \\ 0 & [(I - A_1 A_1^{+})A_2]^{+}(I - A_1 A_1^{+})A_2 \end{bmatrix}\right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\
&= V_1 - \left[A_1 A_1^{+}, A_1^{+} A_2 - A_1^{+} A_2[(I - A_1 A_1^{+})A_2]^{+}(I - A_1 A_1^{+})A_2\right] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.
\end{aligned}$$

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Similarly, we have

$$\begin{aligned}&[0, I_p]\, L_{[A_1,A_2]}V \\ &= V_2 - \left[0, [(I - A_1A_1^{+})A_2]^{+}(I - A_1A_1^{+})A_2\right] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.\end{aligned}$$

Thus, $[I_p, 0]\, L_{[A_1,A_2]}V$ and $[0, I_p]\, L_{[A_1,A_2]}V$ are independent if and only if

$$A_1^{+}A_2 - A_1^{+}A_2[(I - A_1A_1^{+})A_2]^{+}(I - A_1A_1^{+})A_2 = 0.$$

According to Lemma 2.2, we have

$$\begin{aligned}& r\left(A_1^{+}A_2 - A_1^{+}A_2[(I - A_1A_1^{+})A_2]^{+}(I - A_1A_1^{+})A_2\right) \\ &= r\begin{bmatrix} (I - A_1A_1^{+})A_2 \\ A_1^{+}A_2 \end{bmatrix} - r\left((I - A_1A_1^{+})A_2\right) \\ &= r\begin{bmatrix} A_2 & A_1 \\ A_1^{+}A_2 & 0 \end{bmatrix} - r\,[A_2, A_1] \\ &= r\begin{bmatrix} A_2 & 0 \\ 0 & A_1 \end{bmatrix} - r\,[A_2, A_1].\end{aligned}$$

That is $r\,[A_1, A_2] = r\,(A_1) + r\,(A_2)$.

Similarly, we can prove (b). □

Now we give the main result of this article.

Theorem 2.8. *Suppose that every matrix equation in system (1.1) is consistent and*

$$r\,[A_1, A_2] = r\,(A_1) + r\,(A_2)\,, r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = r\,(B_1) + r\,(B_2)\,. \tag{2.3}$$

Then system (1.1) has a pair of solutions X and Y if and only if

$$r\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ -C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{2.4}$$

$$r\begin{bmatrix} A_1 & A_2 & -C_1 & C_2 \\ 0 & 0 & B_1 & B_2 \end{bmatrix} = r\,[A_1, A_2] + r\,[B_1, B_2]\,, \tag{2.5}$$

$$r\begin{bmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ A_2 & 0 & F \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r\,[B_1, B_2]\,, \tag{2.6}$$

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$$r\begin{bmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ A_2 & 0 & \widehat{F} \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r\left[B_1, B_2\right], \tag{2.7}$$

where

$$F = A_1\left(A_2^{+}C_2 - A_1^{+}C_1\right)\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^{+}\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + \Omega B_1 \tag{2.8}$$

and

$$\widehat{F} = A_2\left(A_2^{+}C_2 - A_1^{+}C_1\right)\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^{+}\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + \Omega B_2 \tag{2.9}$$

with $\Omega = \left[-A_1, A_2\right]\left[-A_1, A_2\right]^{+}\left(R_{A_2}C_2B_2^{+} - R_{A_1}C_1B_1^{+}\right).$

Proof. Clearly, system (1.1) has a pair of solutions X and Y if and only if

$$A_1X_1 + Y_1B_1 = C_1 \tag{2.10}$$

$$A_2X_2 + Y_2B_2 = C_2 \tag{2.11}$$

are consistent and $X_1 = X_2$ and $Y_1 = Y_2$. It follows from Lemma 2.3 that $A_iX_i + Y_iB_i = C_i, i = 1, 2$, are consistent if and only if

$$C_i - A_iA_i^{+}C_i - C_iB_i^{+}B_i + A_iA_i^{+}C_iB_i^{+}B_i = 0, i = 1, 2.$$

In that case, the general solutions can be written as

$$X_i = A_i^{+}C_i + U_iB_i + L_{A_i}V_i, \tag{2.12}$$

$$Y_i = R_{A_i}C_i - A_iU_i + L_{A_i}W_iR_{B_i}, \tag{2.13}$$

where $U_i \in \mathbb{H}^{p\times q}, V_i \in \mathbb{H}^{p\times n}, W_i \in \mathbb{H}^{m\times q}$, $i = 1, 2$, are arbitrary. Hence,

$$\begin{aligned} &X_1 - X_2 \\ &= A_1^{+}C_1 - A_2^{+}C_2 + \left[U_1, U_2\right]\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + \left[L_{A_1}, -L_{A_2}\right]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} &Y_1 - Y_2 \\ &= R_{A_1}C_1B_1^{+} - R_{A_2}C_2B_2^{+} + \left[-A_1, A_2\right]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \left[W_1, W_2\right]\begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix}. \end{aligned} \tag{2.15}$$

Obviously, the equations (2.10) and (2.11) have common solutions, $X_1 = X_2, Y_1 = Y_2$, if and only if there exist U_1 and U_2 in (2.14) and (2.15) such that

$$\min_{A_1X_1+Y_1B_1=C_1, A_2X_2+Y_2B_2=C_2} r(X_1 - X_2) = 0, \tag{2.16}$$

$$\min_{A_1X_1+Y_1B_1=C_1, A_2X_2+Y_2B_2=C_2} r(Y_1 - Y_2) = 0, \tag{2.17}$$

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which is equivalent to the existence of U_1 and U_2 such that

$$A_1^+C_1 - A_2^+C_2 + [U_1, U_2]\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + [L_{A_1}, -L_{A_2}]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0, \tag{2.18}$$

and

$$R_{A_1}C_1B_1^+ - R_{A_2}C_2B_2^+ + [-A_1, A_2]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + [W_1, W_2]\begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} = 0. \tag{2.19}$$

It follows from (2.16-2.17) and Lemma 2.3 that

$$\begin{aligned} &\min_{A_1X_1+Y_1B_1=C_1, A_2X_2+Y_2B_2=C_2} r\left(X_1 - X_2\right) \\ &= r\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ -C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} - r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0 \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &\min_{A_1X_1+Y_1B_1=C_1, A_2X_2+Y_2B_2=C_2} r\left(Y_1 - Y_2\right) \\ &= r\begin{bmatrix} A_1 & A_2 & -C_1 & C_2 \\ 0 & 0 & B_1 & B_2 \end{bmatrix} - r\left[A_1, A_2\right] - r\left[B_1, B_2\right] = 0 \end{aligned} \tag{2.21}$$

implying, from Lemma 2.3, that (2.18) and (2.19) are solvable for $[U_1, U_2]$ and $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, respectively, and

$$\begin{aligned} &[U_1, U_2] \\ &= R_{[L_{A_1}, -L_{A_2}]}\left(A_2^+C_2 - A_1^+C_1\right)\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ - [L_{A_1}, -L_{A_2}]\widetilde{U} + WR_{\widehat{B}}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} &\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &= [-A_1, A_2]^+\left(R_{A_2}C_2B_2^+ - R_{A_1}C_1B_1^+\right) + \widehat{U}\begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} + L_{[-A_1, A_2]}V, \end{aligned} \tag{2.23}$$

where $\widehat{U}, \widetilde{U}, W$ and V are any matrices over $\mathbb{H}$ with appropriate dimensions. Clearly,

$$[U_1, U_2]\begin{bmatrix} I_q \\ 0 \end{bmatrix} = [I_p, 0]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \tag{2.24}$$

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and

$$[U_1, U_2] \begin{bmatrix} 0 \\ I_q \end{bmatrix} = [0, I_p] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}. \tag{2.25}$$

Substituting (2.22) and (2.23) into (2.24) and (2.25) yields

$$\begin{aligned} &R_{\left[L_{A_1}, -L_{A_2}\right]} \left(A_2^+ C_2 - A_1^+ C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [I_p, 0]\, \alpha \\ &= [L_{A_1}, -L_{A_2}]\, \widetilde{U} \begin{bmatrix} I_q \\ 0 \end{bmatrix} + [I_p, 0]\, \widehat{U} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} - W R_{\widehat{B}} \begin{bmatrix} I_q \\ 0 \end{bmatrix} + [I_p, 0]\, L_{[-A_1, A_2]} V, \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} &R_{\left[L_{A_1}, -L_{A_2}\right]} \left(A_2^+ C_2 - A_1^+ C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} 0 \\ I_q \end{bmatrix} - [0, I_p]\, \alpha \\ &= [L_{A_1}, -L_{A_2}]\, \widetilde{U} \begin{bmatrix} 0 \\ I_q \end{bmatrix} + [0, I_p]\, \widehat{U} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} - W R_{\widehat{B}} \begin{bmatrix} 0 \\ I_q \end{bmatrix} + [0, I_p]\, L_{[-A_1, A_2]} V \end{aligned} \tag{2.27}$$

where

$$\alpha = [-A_1, A_2]^+ \left(R_{A_2} C_2 B_2^+ - R_{A_1} C_1 B_1^+\right), \widehat{B} = \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}.$$

Let

$$\widetilde{U} = \left[\widetilde{U}_1, \widetilde{U}_2\right], \widehat{U} = \begin{bmatrix} \widehat{U}_1 \\ \widehat{U}_2 \end{bmatrix}$$

in (2.26) and (2.27) where $\widetilde{U}_1, \widetilde{U}_2, \widehat{U}_1$ and $\widehat{U}_2$ are matrices over $\mathbb{H}$ with appropriate dimensions. Then it follows from (2.3) and Lemma 2.7 that (2.26) and (2.27) can be written as

$$\begin{aligned} &R_{\left[L_{A_1}, -L_{A_2}\right]} \left(A_2^+ C_2 - A_1^+ C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [I_p, 0]\, \alpha \\ &= [L_{A_1}, -L_{A_2}]\, \widetilde{U}_1 + \widehat{U}_1 \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} - W_1 R_{B_2 L_{B_1}} + V_1 R_{A_1}, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} &R_{\left[L_{A_1}, -L_{A_2}\right]} \left(A_2^+ C_2 - A_1^+ C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} 0 \\ I_q \end{bmatrix} - [0, I_p]\, \alpha \\ &= [L_{A_1}, -L_{A_2}]\, \widetilde{U}_2 + \widehat{U}_2 \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} - W_2 L_{B_1} + V_2 L_{R_{A_1} A_2}. \end{aligned} \tag{2.29}$$

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Therefore, the equations (2.10) and (2.11) have common solutions, $X_1 = X_2, Y_1 = Y_2$, if and only if there exist $W_1, V_1, \widetilde{U_1}, \widehat{U_1}; W_2, V_2, \widetilde{U_2}, \widehat{U_2}$ such that (2.28) and (2.29) hold, respectively. By Lemma 2.5, the equation (2.28) is solvable if and only if

$$
\begin{aligned}
(2.30)\qquad & r\begin{bmatrix} C & [L_{A_1}, -L_{A_2}] & [I_p, 0]\, L_{[-A_1, A_2]} \\ \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} & 0 & 0 \\ R_{\widehat{B}} \begin{bmatrix} -I_q \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix} \\
& = r\begin{bmatrix} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} \\ R_{\widehat{B}} \begin{bmatrix} -I_q \\ 0 \end{bmatrix} \end{bmatrix} + r([L_{A_1}, -L_{A_2}]\, , [I_p, 0]\, L_{[-A_1, A_2]}),
\end{aligned}
$$

where

$$
\begin{aligned}
C = & \left(I - [L_{A_1}, -L_{A_2}]\,[L_{A_1}, -L_{A_2}]^{+}\right) \left(A_2^{+} C_2 - A_1^{+} C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^{+} \begin{bmatrix} I_q \\ 0 \end{bmatrix} \\
& - [I_p, 0]\,[-A_1, A_2]^{+} \left(R_{A_2} C_2 B_2^{+} - R_{A_1} C_1 B_1^{+}\right).
\end{aligned}
$$

It follows from Lemma 2.2, (2.8) and block Gaussian elimination that

$$
\begin{aligned}
& r\begin{bmatrix} C & [L_{A_1}, -L_{A_2}] & [I_p, 0]\, L_{[-A_1, A_2]} \\ \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} & 0 & 0 \\ R_{\widehat{B}} \begin{bmatrix} -I_q \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix} \\
& = r\begin{bmatrix} C & L_{A_1} & -L_{A_2} & I_p & 0 & 0 \\ R_{B_1} & 0 & 0 & 0 & 0 & 0 \\ -R_{B_2} & 0 & 0 & 0 & 0 & 0 \\ -I_q & 0 & 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 & -B_2 \\ 0 & 0 & 0 & -A_1 & A_2 & 0 \end{bmatrix} - r\,[-A_1, A_2] - r\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} \\
& = r\begin{bmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ A_2 & 0 & F \end{bmatrix} + p + q - r\,[-A_1, A_2] - r\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix},
\end{aligned}
$$

$$
r\begin{bmatrix} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} \\ R_{\widehat{B}} \begin{bmatrix} -I_q \\ 0 \end{bmatrix} \end{bmatrix} = r\,[B_1, B_2] + q - r\,(B_1) - r\,(B_2)\,,
$$

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$$r\left[[L_{A_1}, -L_{A_2}], [I_p, 0] L_{[-A_1, A_2]}\right] = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + p - r(A_1) - r(A_2)$$

implying that (2.6) follows from (2.3) and (2.30).

Similarly, the equation (2.29) is solvable if and only if

$$r\begin{bmatrix} \widehat{C} & J & K \\ \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} & 0 & 0 \\ R_{\widehat{B}}\begin{bmatrix} 0 \\ I_q \end{bmatrix} & 0 & 0 \end{bmatrix} = r\begin{bmatrix} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} \\ R_{\widehat{B}}\begin{bmatrix} 0 \\ I_q \end{bmatrix} \end{bmatrix} + r(J, K), \tag{2.31}$$

where

$$\begin{aligned} J =& [L_{A_1}, -L_{A_2}], K = [0, I_p] L_{[-A_1, A_2]}, \\ \widehat{C} =& \Big(I - [L_{A_1}, -L_{A_2}][L_{A_1}, -L_{A_2}]^+\Big)\left(A_2^+ C_2 - A_1^+ C_1\right)\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} 0 \\ I_q \end{bmatrix} \\ & - [0, I_p][-A_1, A_2]^+\left(R_{A_2} C_2 B_2^+ - R_{A_1} C_1 B_1^+\right). \end{aligned}$$

Simplifying (2.31) yields (2.7) from (2.3) and (2.9). Moreover, (2.4) and (2.5) follow from (2.20) and (2.21), respectively. This proof is completed. □

Under an assumption, we have derived a necessary and sufficient condition for system (1.1) to have a pair of solutions X and Y over $\mathbb{H}$ by ranks. The open problem in [16] is, therefore, partially solved. By the way, we find that Corollary 2.3 in [16] is wrong.

Now we present a counterexample to illustrate the error. We first state the wrong corollary mentioned above: Suppose that the complex matrix equation $(A_0 + A_1 i) X + Y(B_0 + B_1 i) = (C_0 + C_1 i)$ is consistent. Then
(a) Equation $(A_0 + A_1 i) X + Y(B_0 + B_1 i) = (C_0 + C_1 i)$ has a pair of real solutions $X = X_0$ and $Y = Y_0$ if and only if

$$r\begin{bmatrix} B_0 & 0 \\ B_1 & 0 \\ C_0 & A_0 \\ C_1 & A_1 \end{bmatrix} = r\begin{bmatrix} A_0 \\ A_1 \end{bmatrix} + r\begin{bmatrix} B_0 \\ B_1 \end{bmatrix}, \tag{2.32}$$

$$r\begin{bmatrix} A_0 & A_1 & C_0 & C_1 \\ 0 & 0 & B_0 & B_1 \end{bmatrix} = r[A_0, A_1] + r[B_0, B_1]. \tag{2.33}$$

A counterexample is as follows. Let

$$A_0 = B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = B_1 = C_0 = 0, C_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

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Then we have

$$r\begin{bmatrix} B_0 & 0 \\ B_1 & 0 \\ C_0 & A_0 \\ C_1 & A_1 \end{bmatrix} = r\begin{bmatrix} A_0 & A_1 & C_0 & C_1 \\ 0 & 0 & B_0 & B_1 \end{bmatrix} = 4,$$

$$r\begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = r\begin{bmatrix} B_0 \\ B_1 \end{bmatrix} = r\,[A_0, A_1] = r\,[B_0, B_1] = 2,$$

i.e. (2.32) and (2.33) hold. However, the following matrix equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X + Y \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

has no real solution obviously.

Similarly, we can give a counterexample to illustrate that the part (c) of Corollary 2.3 in [16] is also wrong.

Using the methods in this paper, we can correct the mistakes mentioned above. We are planning to present these corrections in a separate article.

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