Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 274-280, May 2009

ELA

LEFT EIGENVALUES OF 2×2 SYMPLECTIC MATRICES*

E. MACíAS-VIRGÓS† AND M.J. PEREIRA-SÁEZ†

Abstract. A complete characterization is obtained of the 2×2 symplectic matrices that have an infinite number of left eigenvalues. Also, a new proof is given of a result of Huang and So on the number of eigenvalues of a quaternionic matrix. This is achieved by applying an algorithm for the resolution of equations due to De Leo et al.

Key words. Quaternions, Quadratic equation, Left eigenvalues, Symplectic matrix.

AMS subject classifications. 15A04, 11R52, 15A33.

1. Introduction. Left eigenvalues of $n \times n$ quaternionic matrices are still not well understood. For $n = 2$, Huang and So gave in [6] a characterization of those matrices having an infinite number of left eigenvalues. Their result (see Theorem 2.3 below) is based on previous explicit formulae by the same authors for solving some quadratic equations [7]. Later, De Leo et al. proposed in [4] an alternative method of resolution, which reduces the problem to finding the *right* eigenvalues of a matrix associated to the equation.

In this paper, we firstly give a new proof of Huang-So's result, based on the method of De Leo et al. Secondly, we completely characterize those symplectic matrices having an infinite number of left eigenvalues (see Theorem 6.2). The application we have in mind is to compute in a simple way the so-called Lusternik-Schnirelmann category of the symplectic group $Sp(2)$. This will be done in a forthcoming paper [5].

2. Left eigenvalues of quaternionic matrices. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 matrix with coefficients in the quaternion algebra $\mathbb{H}$. We shall always consider $\mathbb{H}^2$ as a right vector space over $\mathbb{H}$.

DEFINITION 2.1. A left eigenvalue of A is any quaternion $q \in \mathbb{H}$ such that there exists a nonzero $u \in \mathbb{H}^2$ with $Au = qu$.

*Received by the editors December 11, 2008. Accepted for publication May 5, 2009. Handling Editor: Joao Filipe Queiro.

†Institute of Mathematics, Department of Geometry and Topology, University of Santiago de Compostela, Spain (quique.macias@usc.es). Partially supported by FEDER and Research Project MTM2008-05861 MICINN, Spain.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 274-280, May 2009

Clearly, if $bc = 0$, then the eigenvalues are the diagonal entries. For a non-triangular matrix A, the following result appears in [6] (with a different proof).

Proposition 2.2. *If $bc \neq 0$, then the left eigenvalues of A are given by $q = a+bp$, where p is any solution of the unilateral quadratic equation*

$$p^2 + a_1 p + a_0 = 0, \tag{2.1}$$

with $a_1 = b^{-1}(a-d)$ and $a_0 = -b^{-1}c$.

Proof. Let us write $A = X + jY$, where X, Y are complex 2×2 matrices, and let

$$\tilde{A} = \begin{bmatrix} X & -\bar{Y} \\ Y & \bar{X} \end{bmatrix}$$

be the complexification of A. Let $\mathrm{Sdet}(A) = |\det(\tilde{A})|^{1/2}$ be Study's determinant. By using the axiomatic properties of Sdet (see [1, 3]), we can triangularize the matrix $A - qI$, obtaining that

$$\mathrm{Sdet}(A - qI) = |a - q| \cdot |(d-q) - c(a-q)^{-1}b|.$$

It is easy to see that $q = a$ would imply $bc = 0$. Hence, $A - qI$ is not invertible if and only if $(d-q) - c(a-q)^{-1}b = 0$. By putting $p = b^{-1}(q-a)$, we obtain the desired equation. ☐

In [6], Huang and So also proved the following theorem.

Theorem 2.3. *The matrix A has either one, two or infinite left eigenvalues. The infinite case is equivalent to the conditions $a_1, a_0 \in \mathbb{R}$, $a_0 \neq 0$ and $\Delta = a_1^2 - 4a_0 < 0$.*

The original proof is based on a case by case study guided by the explicit formulae that the same authors obtained in [7] for solving an equation like (2.1). In particular, they prove that in the infinite case, the eigenvalues are given by the formula $(a + d + b\xi)/2$, where ξ runs over the quaternions $\xi \in \langle i, j, k\rangle$ with $|\xi|^2 = |\Delta|$.

It is easy to see that the conditions in Theorem 2.3 above are sufficient. In fact, if $a_0 = s$ and $a_1 = t$, $s, t \in \mathbb{R}$, then the definition of eigenvalue leads to the equation $q^2 + tq + s = 0$ that, after the change $p = q + t/2$, gives $p^2 = \Delta/4 < 0$ which has infinite solutions $p = (\sqrt{-\Delta}/2)\omega$, $\omega \in \langle i, j, k\rangle$, $|\omega| = 1$.

For the necessity of the conditions, we shall give an alternative proof. It exploits an elegant method for the resolution of equations, proposed by De Leo et al. in [4] as an improvement of a previous algorithm by Serôdio et al. [8]

3. The eigenvectors method. In order to facilitate the understanding of this paper, we explicitly discuss in this section the algorithm of De Leo et al. cited above.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 274-280, May 2009

Let

$$M = \begin{bmatrix} -a_1 & -a_0 \\ 1 & 0 \end{bmatrix}$$

be the so-called *companion matrix* of equation (2.1). Then

$$M \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} -a_1 p - a_0 \\ p \end{bmatrix} = \begin{bmatrix} p^2 \\ p \end{bmatrix} = \begin{bmatrix} p \\ 1 \end{bmatrix} p,$$

which shows that in order to find the solutions, we have to look for *right* eigenvalues p of M corresponding to eigenvectors of the precise form $(p, 1)$. Accordingly to [4], we shall call p a *privileged* right eigenvalue.

Right eigenvalues. The theory of right eigenvalues is well established [2, 9]. A crucial point is that the eigenvectors associated to a given right eigenvalue do not form a (right) $\mathbb{H}$-vector space.

PROPOSITION 3.1. *Let λ be a right eigenvalue, and let v be a λ-eigenvector. Then vq^{-1} is a $q\lambda q^{-1}$-eigenvector for any nonzero $q \in \mathbb{H}$.*

Proof. Since $Mv = v\lambda$, we have $M(vq^{-1}) = v\lambda q^{-1} = vq^{-1}(q\lambda q^{-1})$. ∎

As a consequence, each eigenvector gives rise to a similarity class $[\lambda] = \{q\lambda q^{-1} : q \in \mathbb{H}, q \neq 0\}$ of right eigenvalues. Recall that two quaternions λ', λ are similar if and only if they have the same norm, $|\lambda'| = |\lambda|$, and the same real part, $\Re(\lambda') = \Re(\lambda)$. In particular, any quaternion λ is similar to a complex number and to its conjugate $\bar{\lambda}$.

Eigenvectors. So, in order to solve the equation (2.1), we first need to find the *complex* right eigenvalues of the companion matrix M. These correspond to the eigenvalues of the complexified 4×4 matrix $\tilde{M}$ and can be computed by solving the characteristic equation $\det(\tilde{M} - \lambda I) = 0$. Due to the structure of $\tilde{M}$, its eigenvalues appear in pairs $\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2$ [9].

In order to compute the eigenvectors, let us consider the $\mathbb{C}$-isomorphism

$$(z', z) \in \mathbb{C}^2 \mapsto z' + jz \in \mathbb{H}. \tag{3.1}$$

PROPOSITION 3.2. *$(x', x, y', y) \in \mathbb{C}^4$ is a λ-eigenvector of the complexified matrix $\tilde{M}$ if and only if $(x' + jy', x + jy)$ is a λ-eigenvector of M.*

Equation solutions. Let M be the companion matrix of equation (2.1). Once we have found a complex right eigenvalue λ of M and some λ-eigenvector (q', q), we observe that $q' = q\lambda$, due to the special form of M. Hence, by Proposition 3.1, the

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 274-280, May 2009

vector

$$\begin{bmatrix} q' \\ q \end{bmatrix} q^{-1} = \begin{bmatrix} q\lambda q^{-1} \\ 1 \end{bmatrix}$$

is a $q\lambda q^{-1}$-eigenvector, that is, $p = q\lambda q^{-1}$ is a privileged eigenvalue in the similarity class $[\lambda]$, and hence, it is the desired solution.

Notice that, by Proposition 3.1, two $\mathbb{H}$-linearly dependent eigenvectors give rise to the same privileged eigenvalue.

4. Number of solutions. Now we are in a position to discuss the number of solutions of equation (2.1). This will give a new proof of Theorem 2.3.

Let $V(\lambda) \subset \mathbb{C}^4$ be the eigenspace associated to the eigenvalue λ of the complexified matrix $\tilde{M}$. By examining the possible complex dimensions of the spaces $V(\lambda_k)$ and $V(\bar{\lambda}_k)$, $1 \leq k \leq 2$, we see that:

1. If the four eigenvalues $\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2$ are different, then each $V(\lambda_k)$ has dimension 1 and gives just one privileged eigenvalue p_k. Since $\bar{\lambda}_k$ gives p_k too, it follows that there are exactly two solutions.
2. If one of the eigenvalues is real, say $\lambda_1 \in \mathbb{R}$, then all its similar quaternions equal $p_1 = \lambda_1$, independently of the dimension of $V(\lambda_1)$. So, there are one or two solutions, depending on whether $\lambda_1 = \lambda_2$ or not.
3. The only case where infinite different privileged eigenvalues may appear is when $\lambda_1 = \lambda_2 \notin \mathbb{R}$, which implies $\dim_{\mathbb{C}} V(\lambda_1) = 2 = \dim_{\mathbb{C}} V(\bar{\lambda}_1)$.

The infinite case. So, we focus on case 3, when $\tilde{M}$ has exactly two different eigenvalues $\lambda_1, \bar{\lambda}_1$. The following proposition states that we actually have an infinite number of solutions (recall that $\lambda_1 \notin \mathbb{R}$).

PROPOSITION 4.1. *In case 3, all the quaternions similar to λ_1 are privileged right eigenvalues of M.*

Proof. Take a $\mathbb{C}$-basis $\tilde{u}, \tilde{v}$ of $V(\lambda_1) \subset \mathbb{C}^4$ and the corresponding vectors u, v in $\mathbb{H}^2$ by the isomorphism (3.1). Since the latter are of the form $(q\lambda, q)$, it follows that the second coordinates u_2, v_2 of u and v are $\mathbb{C}$-independent in $\mathbb{H}$, hence a $\mathbb{C}$-basis. This means that the privileged eigenvalues $p = q\lambda_1 q^{-1}$, where q is a $\mathbb{C}$-linear combination of u_2 and v_2, run over all possible quaternions similar to λ_1. □

The Huang-So conditions. It remains to verify that in case 3, the conditions of Theorem 2.3 are satisfied.

Let $\mathbb{H}_0 \cong \mathbb{R}^3$ be the real vector space of quaternions with null real part. The scalar product is given by $\langle q, q' \rangle = -\Re(qq')$. An orthonormal basis is $\langle i, j, k \rangle$. If

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 274-280, May 2009

$\xi \in \mathbb{H}_0$, then we have $\bar{\xi} = -\xi$ and $-\xi^2 = |\xi|^2$. Let $\Omega = S^3 \cap \mathbb{H}_0$ be the set of vectors in $\mathbb{H}_0$ with norm 1. It coincides with the quaternions similar to the imaginary unit i.

Let $\lambda_1 = x + iy$, $y \neq 0$, be one of the two complex eigenvalues of $\tilde{M}$ and let $p \in [\lambda_1]$ be any privileged eigenvalue of M. Since $\Re(p) = \Re(\lambda_1)$ and $|p| = |\lambda_1|$, we can write $p = x + |y|\omega$, where ω is an arbitrary element of Ω.

Put $a_1 = t + \xi_1$, with $t \in \mathbb{R}$ and $\xi_1 \in \mathbb{H}_0$. ¿From equation (2.1) written in the form $a_0 = -(p + a_1)p$, we deduce that

$$\Re(a_0) = xt + x^2 - |y|^2 + |y|\Re(\xi_1\omega).$$

Hence, $|y|\langle \xi_1, \omega\rangle$ does not depend on $\omega \in \Omega$. Since $y \neq 0$, the following Lemma 4.2 ensures that $\xi_1 = 0$, i.e. $a_1 \in \mathbb{R}$.

Lemma 4.2. *Let $\xi \in \mathbb{H}_0$ satisfy $\langle \xi, \omega - \omega'\rangle = 0$ for any pair ω, ω' of vectors in Ω. Then $\xi = 0$.*

Proof. Let $\xi = xi + yj + zk \neq 0$, and suppose for instance that $x \neq 0$ (the other cases are analogous). Take any $\omega \in \Omega$ orthogonal to ξ and $\omega' = i$. Then $\langle \xi, \omega - \omega'\rangle = x \neq 0$. $\square$

Now, a consequence of Proposition 4.1 is that $p = \lambda_1$ is a privileged eigenvalue of M, and thus, by equation (2.1), a_0 is a complex number. Since $\bar{\lambda}_1$ is a solution too, we deduce that $a_0 = \bar{a}_0$ is a real number. Finally, from $\lambda_1^2 + a_1\lambda_1 + a_0 = 0$, it follows that $a_1^2 - 4a_0 < 0$ because $\lambda_1 \notin \mathbb{R}$.

This ends the verification of the Huang-So conditions given in Theorem 2.3.

5. Symplectic matrices. Let us consider the 10-dimensional Lie group $Sp(2)$ of 2×2 symplectic matrices, that is, quaternionic matrices A such that $A^*A = I$. Geometrically, they correspond to the (right) $\mathbb{H}$-linear endomorphisms of $\mathbb{H}^2$ which preserve the hermitian product

$$\langle u, v\rangle = u^*v = \bar{u}_1v_1 + \bar{u}_2v_2. \tag{5.1}$$

Thus, a matrix is symplectic if and only if its columns form an orthonormal basis for this hermitian product.

Let us find a general expression for any symplectic matrix.

Proposition 5.1. *A symplectic matrix $A \in Sp(2)$ is either diagonal or of the form*

$$A = \begin{bmatrix} \alpha & -\overline{\beta}\gamma \\ \beta & \beta\overline{\alpha}\overline{\beta}\gamma/|\beta|^2 \end{bmatrix}, \quad \beta \neq 0, |\alpha|^2 + |\beta|^2 = 1, |\gamma| = 1. \tag{5.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 274-280, May 2009

ELA

Proof. By definition, the two columns A_1, A_2 of A form an orthonormal basis of $\mathbb{H}^2$ for the hermitian product. Let the first one be

$$A_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad \alpha, \beta \in \mathbb{H}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

If $\beta = 0$, then A is a diagonal matrix $\operatorname{diag}(\alpha, \delta)$, with $|\alpha| = 1 = |\delta|$.

Suppose $\beta \neq 0$, and consider the (right) $\mathbb{H}$-linear map $\langle A_1, - \rangle \colon \mathbb{H}^2 \to \mathbb{H}$, which is onto because $|A_1| = 1$. Then its kernel $K = (A_1)^\perp$ has dimension $\dim_{\mathbb{H}} K = 1$. Clearly, the vector

$$u = \begin{bmatrix} -\overline{\beta} \\ \beta\overline{\alpha}\overline{\beta}/|\beta|^2 \end{bmatrix} \neq 0,$$

where $|u| = 1$, is orthogonal to A_1, so any other vector in K must be a quaternionic multiple of u. Since A_2 has norm 1, we have

$$A_2 = u\gamma = \begin{bmatrix} -\overline{\beta}\gamma \\ \beta\overline{\alpha}\overline{\beta}\gamma/|\beta|^2 \end{bmatrix}, \quad \gamma \in \mathbb{H}, \quad |\gamma| = 1. \qquad \square$$

6. Left eigenvalues of a symplectic matrix. In this section, we apply Theorem 2.3 to the symplectic case. We begin with a result that is also true for right eigenvalues.

PROPOSITION 6.1. *The left eigenvalues of a symplectic matrix have norm* 1.

Proof. For the hermitian product $\langle , \rangle$ defined in (5.1), the usual euclidean norm in $\mathbb{H}^2 \cong \mathbb{R}^4$ is given by $|u|^2 = u^*u = \langle u, u \rangle$. Let $Au = qu$, $u \neq 0$. Then

$$|u|^2 = \langle u, u \rangle = \langle Au, Au \rangle = \langle qu, qu \rangle = u^*\bar{q}qu = |q|^2|u|^2. \qquad \square$$

THEOREM 6.2. *The only symplectic matrices with an infinite number of left eigenvalues are those of the form*

$$\begin{bmatrix} q\cos\theta & -q\sin\theta \\ q\sin\theta & q\cos\theta \end{bmatrix}, \qquad |q| = 1, \quad \sin\theta \neq 0.$$

Such a matrix corresponds to the composition $L_q \circ R_\theta$ of a real rotation $R_\theta \neq \pm\mathrm{id}$ with a left translation L_q, $|q| = 1$.

Proof. Clearly, the matrix above verifies the Huang-So conditions of Theorem 2.3. Conversely, by taking into account Proposition 5.1, we must check those conditions for the values $a = \alpha$, $b = -\bar{\beta}\gamma$, $c = \beta$, $d = \beta\bar{\alpha}\bar{\beta}\gamma/|\beta|^2$, $\beta \neq 0$. Since

$$a_0 = -b^{-1}c = \bar{\gamma}\beta^2/|\beta|^2 = s \in \mathbb{R},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 274-280, May 2009

ELA

it follows that $s = |\gamma| = 1$ (notice that the condition $a_1^2 - 4a_0 < 0$ implies $a_0 > 0$). Then

$$\gamma = (\beta/|\beta|)^2 .$$

Substituting γ, we obtain $b = -\beta$ and $d = \beta\bar{\alpha}\beta/|\beta|^2$. Then $a_0 = -b^{-1}c = -\beta^{-1}\beta = 1$.

We now compute

$$a_1 = b^{-1}(a - d) = -\beta^{-1}(\alpha - \beta\bar{\alpha}\beta/|\beta|^2) = \frac{-1}{|\beta|^2}(\bar{\beta}\alpha - \bar{\alpha}\beta).$$

Hence, $\Re(a_1) = 0$, so the condition $a_1 \in \mathbb{R}$ implies $a_1 = 0$. This means that $\bar{\beta}\alpha$ equals its conjugate $\bar{\alpha}\beta$, i.e. it is a real number. Denote $r = \bar{\beta}\alpha \in \mathbb{R}$.

Since $|\alpha|^2 + |\beta|^2 = 1$ and $\beta \neq 0$, it is $0 < |\beta| \leq 1$. Take any angle θ such that $|\beta| = \sin\theta$, $\sin\theta \neq 0$. Define $q = \beta/|\beta|$, so we shall have $\beta = q\sin\theta$ with $|q| = 1$. On the other hand, the relationships $|\alpha| = |\cos\theta|$ and $|r| = |\bar{\beta}||\alpha|$ imply that $r = \pm\sin\theta\cos\theta$. By changing the angle if necessary, we can assume that $r = \sin\theta\cos\theta$ without changing $\sin\theta$. Then

$$\alpha = r(\bar{\beta})^{-1} = r\beta/|\beta|^2 = q\cos\theta.$$

Finally, $d = \beta\bar{\alpha}\beta/|\beta|^2 = q\cos\theta$, and the proof is complete. □

Notice that the companion equation of a symplectic matrix is always $p^2 + 1 = 0$.

REFERENCES

[1] H. Aslaksen. Quaternionic determinants. *Math. Intelligencer*, 18(3):57–65, 1996.
[2] J.L. Brenner. Matrices of quaternions. *Pacific J. Math.*, 1:329–335, 1951.
[3] N. Cohen and S. De Leo. The quaternionic determinant. *Electron. J. Linear Algebra*, 7:100–111, 2000.
[4] S. De Leo, G. Ducati, and V. Leonardi. Zeros of unilateral quaternionic polynomials. *Electron. J. Linear Algebra*, 15:297–313, 2006.
[5] A. Gómez-Tato, E. Macías-Virgós, and M.J. Pereira-Sáez. Trace map, Cayley transform and LS category of Lie groups. Preprint, 2008.
[6] L. Huang and W. So. On left eigenvalues of a quaternionic matrix. *Linear Algebra Appl.*, 323:105–116, 2001.
[7] L. Huang and W. So. Quadratic formulas for quaternions. *Appl. Math. Lett.*, 15(5):533–540, 2002.
[8] R. Serôdio, E. Pereira, and J. Vitória. Computing the zeros of quaternion polynomials. *Comput. Math. Appl.*, 42(8–9):1229–1237, 2001.
[9] F. Zhang. Quaternions and matrices of quaternions. *Linear Algebra Appl.*, 251:21–57,1997.