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MAPS ON POSITIVE OPERATORS PRESERVING LEBESGUE DECOMPOSITIONS*

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Abstract. Let H be a complex Hilbert space. Denote by $B(H)^+$ the set of all positive bounded linear operators on H. A bijective map $\phi : B(H)^+ \to B(H)^+$ is said to preserve Lebesgue decompositions in both directions if for any quadruple A, B, C, D of positive operators, $B = C + D$ is an A-Lebesgue decomposition of B if and only if $\phi(B) = \phi(C)+\phi(D)$ is a $\phi(A)$-Lebesgue decomposition of $\phi(B)$. It is proved that every such transformation ϕ is of the form $\phi(A) = SAS^*$ $(A \in B(H)^+)$ for some invertible bounded linear or conjugate-linear operator S on H.

Key words. Positive operators, Lebesgue decomposition, Preservers.

AMS subject classifications. 47B49.

1. Introduction and statement of the results. In what follows, H denotes a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $B(H)$ stands for the algebra of all bounded linear operators on H. The space of all self-adjoint elements of $B(H)$ is denoted by $B_s(H)$. An operator $A \in B(H)$ is called positive if $\langle Ax, x\rangle \geq 0$ holds for every $x \in H$ in which case we write $A \geq 0$. (Observe that we use the expression "positive" in the operator algebraic sense. For matrices, this is the same as positive semi-definiteness.) The set of all positive elements of $B(H)$ is denoted by $B(H)^+$. The usual order among self-adjoint operators is defined by means of positivity as follows. For any $T, S \in B_s(H)$, we write $T \leq S$ if $S - T \geq 0$.

In analogy with the Lebesgue decomposition of positive measures, in [1], Ando defined a Lebesgue-type decomposition of positive operators, a concept which has proved to be very useful in operator theory. To explain that decomposition we need the following notions (for details, see [1]).

Given a positive operator $A \in B(H)^+$, the positive operator $C \in B(H)^+$ is said to be A-absolutely continuous if there is a sequence (C_n) of positive operators and a sequence (α_n) of nonnegative real numbers such that $C_n \uparrow C$ and $C_n \leq \alpha_n A$ for every n. Here, $C_n \uparrow C$ means that the sequence (C_n) is monotone increasing with

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respect to the usual order and it strongly converges to C. A positive operator C is called A-singular if for any $D \in B(H)^+$, the inequalities $D \le A$ and $D \le C$ imply $D = 0$. Now, for any pair $A, B \in B(H)^+$ of positive operators, by an A-Lebesgue decomposition of B we mean a decomposition $B = C + D$ where C, D are positive operators, C is A-absolutely continuous and D is A-singular. Ando proved in [1] that such decomposition exists for every pair A, B of positive operators.

In this paper, we study the problem of characterizing maps on positive operators which preserve Lebesgue decompositions. Investigations of this kind, i.e., the study of maps on different structures preserving important operations, quantities, relations, etc. corresponding to the underlying structures belong to the gradually enlarging field of so-called preserver problems. For important surveys on preservers in the classical sense, we refer to [3, 5, 6, 10]. For recent results concerning preservers in extended sense and defined on more general domains (especially on operator structures), we refer to [7] and its bibliography.

We say that the bijective map $\phi : B(H)^+ \to B(H)^+$ preserves Lebesgue decompositions in both directions if it has the following property. For any quadruple A, B, C, D of positive operators, $B = C + D$ is an A-Lebesgue decomposition of B if and only if $\phi(B) = \phi(C) + \phi(D)$ is a $\phi(A)$-Lebesgue decomposition of $\phi(B)$. It is rather clear from the definitions that any transformation of the form $A \longmapsto SAS^*$ for some invertible bounded linear or conjugate-linear[1] operator S on H preserves Lebesgue decompositions in both directions. The aim of this paper is to show that the reverse statement is also true: only transformations of this form have the above preserver property.

THEOREM 1.1. *Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map preserving Lebesgue decompositions in both directions. Then there is an invertible bounded linear or conjugate-linear operator S on H such that ϕ is of the form*

$$\phi(A) = SAS^* \qquad (A \in B(H)^+).$$

2. Proof. This section is devoted to the proof of the theorem. First we recall some of the results in [1] that we shall use in our arguments. In what follows, rng stands for the range of operators. Let $A, B \in B(H)^+$.

(A1) The operator B is A-singular if and only if $\operatorname{rng} A^{1/2} \cap \operatorname{rng} B^{1/2} = \{0\}$ (see [1, p. 256]).

[1]A transformation $S : H \to H$ is called conjugate-linear if it is additive and satisfies $S(\lambda x) = \bar{\lambda} Sx$ for every $x \in H$ and $\lambda \in \mathbb{C}$.

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(A2) The operator B is A-absolutely continuous if and only if the subspace $\{x \in H \,:\, B^{1/2}x \in \operatorname{rng} A^{1/2}\}$ is dense in H (see [1, Theorem 5]).

(A3) The operator B has an A-Lebesgue decomposition, which can be constructed in the following way. Define $[A]B = \lim_n (nA) : B$. Here : denotes the operation of parallel sum of positive operators.[2] The sequence $((nA) : B)_n$ of positive operators is monotone increasing and bounded by B from above. Hence, $[A]B$ is a well-defined positive bounded linear operator on H. Now, according to Theorem 2 in [1], we have that

$$B = [A]B + (B - [A]B)$$

is an A-Lebesgue decomposition of B and, further, $[A]B$ is the maximum of all A-absolutely continuous positive operators C with $C \le B$.

(A4) A-Lebesgue decomposition is not unique in general. Namely, according to [1, Corollary 7], for a given $A \in B(H)^+$, every positive operator admits a unique A-Lebesgue decomposition if and only if $\operatorname{rng} A$ is closed.

We begin the route leading to the proof of the theorem with the following simple lemma.

Lemma 2.1. *Let $A \in B(H)^+$. The range $\operatorname{rng} A$ of A is closed if and only if $\operatorname{rng} A^{1/2}$ is closed, and in this case, we have $\operatorname{rng} A = \operatorname{rng} A^{1/2}$.*

Proof. It is clear that

$$\operatorname{rng} A \subset \operatorname{rng} A^{1/2} \subset \overline{\operatorname{rng} A^{1/2}} = \overline{\operatorname{rng} A},$$

where the last equality follows from the easy fact that $\ker A^{1/2} = \ker A$. Therefore, we see that if $\operatorname{rng} A$ is closed, then so is $\operatorname{rng} A^{1/2}$ and they coincide. Conversely, if $\operatorname{rng} A^{1/2}$ is closed then we have

$$A(H) = A^{1/2}(A^{1/2}(H)) = A^{1/2}(\overline{\operatorname{rng} A^{1/2}}) = A^{1/2}((\ker A^{1/2})^{\perp}) = \operatorname{rng} A^{1/2}.$$

The proof is complete. ∎

By (A2), we immediately have the following.

Corollary 2.2. *Let $A, B \in B(H)^+$ be operators with closed ranges. Then B is A-absolutely continuous if and only if $\operatorname{rng} B \subset \operatorname{rng} A$. Therefore, we have $\operatorname{rng} B = \operatorname{rng} A$ if and only if B is A-absolutely continuous and A is B-absolutely continuous.*

In the proof of our theorem, we need the following additional corollary which gives a characterization of invertibility of positive operators.

[2] Recall that for any positive operators A and B, their parallel sum $A : B$ is the unique positive operator satisfying $\langle (A : B)z, z\rangle = \inf\{\langle Ax, x\rangle + \langle By, y\rangle \,:\, x + y = z\}$ for every $z \in H$.

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COROLLARY 2.3. *Let $A \in B(H)^+$. Then A is invertible if and only if* $\operatorname{rng} A$ *is closed and for every $B \in B(H)^+$ with closed range, we have that B is A-absolutely continuous.*

In the next lemma, we compute the Lebesgue decomposition of an arbitrary positive operator with respect to a rank-one element of $B(H)^+$ (recall that by (A4), in this case, we have unique Lebesgue decomposition). To do so, we need the concept of the strength of a positive operator A along a ray represented by a unit vector in H. This concept was originally introduced by Busch and Gudder in [2] for the so-called Hilbert space effects in the place of positive operators. Effects play a basic role in the mathematical foundations of the theory of quantum measurements. Mathematically, a Hilbert space effect is simply an operator $E \in B(H)$ that satisfies $0 \le E \le I$. Although in [2] the authors considered only effects, it is rather obvious that the following definition and result work also for arbitrary positive operators (the reason is simply that any positive operator can be multiplied by a positive scalar to obtain an effect). So, let $A \in B(H)^+$, consider a unit vector φ in H and denote by P_φ the rank-one projection onto the subspace generated by φ. The quantity

$$\lambda(A, P_\varphi) = \sup\{\lambda \in \mathbb{R}_+ \,:\, \lambda P_\varphi \le A\}$$

is called the strength of A along the ray represented by φ. ($\mathbb{R}_+$ stands for the set of all non-negative real numbers.) According to [2, Theorem 4], we have the following formula for the strength:

$$\lambda(A, P_\varphi) = \begin{cases} \|A^{-1/2}\varphi\|^{-2}, & \text{if } \varphi \in \operatorname{rng}(A^{1/2}) \\ 0, & \text{else} \end{cases}, \tag{2.1}$$

where $A^{-1/2}$ denotes the inverse of $A^{1/2}$ on its range.

Clearly, every positive rank-one operator can be written in the form μP, where P is a rank-one projection and μ is a positive real number.

LEMMA 2.4. *Let P be a rank-one projection, μ a positive real number and B an arbitrary positive operator. Then we have*

$$[\mu P]B = \lambda(B, P)P.$$

Therefore, the (μP)-Lebesgue decomposition of B is

$$B = \lambda(B, P)P + (B - \lambda(B, P)P).$$

In particular, the (μP)-Lebesgue decomposition of I is

$$I = P + (I - P).$$

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Proof. In paper [8], we presented structural results for the automorphisms of $B(H)^+$ with respect to the operation of the harmonic mean or that of the parallel sum. We recall that the harmonic mean $T!S$ of the positive operators T, S is the double of their parallel sum $T : S$. In [8, Lemma 2] we proved that for any $T \in B(H)^+$ and rank-one projection P, we have

$$T!P = \frac{2\lambda(T,P)}{\lambda(T,P)+1}P.$$

Using this, we compute

$$[\mu P]B = \lim_n (n\mu P) : B = \lim_n \frac{(n\mu P)!B}{2}$$

$$= \lim_n \frac{B!(n\mu P)}{2} = \lim_n n\mu \frac{(B/(n\mu))!P}{2} = \lim_n n\mu \frac{\lambda(B/(n\mu), P)}{\lambda(B/(n\mu), P)+1}P \tag{2.2}$$

$$= \lim_n n\mu \frac{(1/(n\mu))\lambda(B,P)}{(1/(n\mu))\lambda(B,P)+1}P = \lambda(B,P)P.$$

Here, we use the following properties of the harmonic mean and the strength function: for any $T, S \in B(H)^+$, rank-one projection P, and nonnegative number α, we have $T!S = S!T$, $(\alpha T)!(\alpha S) = \alpha(T!S)$, $\lambda(\alpha T, P) = \alpha\lambda(T,P)$. ◻

In the proof of our theorem, the solution of the following functional equation will play an important role.

Lemma 2.5. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a bijective function with $f(0) = 0$, and $\varphi : [0,1] \to [0,1]$ be a function such that*

$$f\left(\frac{1}{(1/\lambda)\alpha + (1/\mu)(1-\alpha)}\right) = \frac{1}{(1/f(\lambda))\varphi(\alpha) + (1/f(\mu))(1-\varphi(\alpha))}. \tag{2.3}$$

holds for every $0 < \lambda, \mu \in \mathbb{R}$ and $\alpha \in [0,1]$. If $f(1) = 1$, then f, φ are the identities on their domains.

Proof. First choose $\alpha = 1/2$. For $\alpha' = \varphi(1/2)$ and $\beta' = 1 - \alpha'$, we have

$$f\left(\frac{2}{(1/\lambda) + (1/\mu)}\right) = \frac{1}{(1/f(\lambda))\alpha' + (1/f(\mu))\beta'}. \tag{2.4}$$

Define $g(\lambda) = 1/f(1/\lambda)$ for every positive λ. Then g is a bijection of the set of all positive real numbers, and (2.4) turns into

$$\frac{1}{g\left(\frac{(1/\lambda)+(1/\mu)}{2}\right)} = \frac{1}{g(1/\lambda)\alpha' + g(1/\mu)\beta'}.$$

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Therefore, we have that

$$g\left(\frac{\lambda+\mu}{2}\right) = g(\lambda)\alpha' + g(\mu)\beta'$$

holds for all positive numbers λ, μ. Interchanging λ and μ, we get

$$g(\mu)\alpha' + g(\lambda)\beta' = g(\lambda)\alpha' + g(\mu)\beta'$$

$0 < \lambda, \mu$. Since g is injective, we infer that $\alpha' = \beta'$ and then it follows that $\alpha' = \beta' = 1/2$. Thus, we obtain that g satisfies the so-called Jensen equation

$$g\left(\frac{\lambda+\mu}{2}\right) = \frac{g(\lambda)+g(\mu)}{2}$$

on the set of all positive real numbers. From [4] we learn that every real-valued function defined on a convex subset of $\mathbb{R}^n$ with nonempty interior which satisfies the Jensen equation can be written as the sum of a real-valued additive function defined on the whole $\mathbb{R}^n$ and a real constant. This gives us that there exist an additive function $a : \mathbb{R} \to \mathbb{R}$ and a constant $d \in \mathbb{R}$ such that $g(\lambda) = a(\lambda) + d$ holds for every positive λ. As g takes only positive values, it follows that a is bounded from below on the set of positive real numbers. It is a classical result of Ostrowski from 1929 [9] that any additive function of $\mathbb{R}$ that is bounded from one side on a set of positive measure is necessarily a constant multiple of the identity. Hence, we have a constant $c \in \mathbb{R}$ such that $a(\lambda) = c\lambda$ for every $\lambda \in \mathbb{R}$. As g is a self-bijection of the set of all positive numbers with $g(1) = 1$, one can easily verify that $c = 1$ and $d = 0$. Clearly, this implies that f is the identity on $\mathbb{R}_+$. Finally, it immediately follows from (2.3) that φ is the identity on $[0, 1]$. □

After this preparation, we are now in a position to prove Theorem 1.1.

Proof. Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map which preserves Lebesgue decompositions in both directions.

First we show that ϕ sends 0 to 0. Indeed, $0 = 0+0$ is a 0-Lebesgue decomposition of 0. This implies that $\phi(0) = \phi(0) + \phi(0)$ is a $\phi(0)$-Lebesgue decomposition of $\phi(0)$. We have $\phi(0) = 0$.

Next, we assert that ϕ preserves absolutely continuity in both directions. This means that for any pair A, B of positive operators, B is A-absolutely continuous if and only if $\phi(B)$ is $\phi(A)$-absolutely continuous. In fact, this follows from the preserver property of ϕ and from the easy fact that B is A-absolutely continuous if and only if $B = B + 0$ is an A-Lebesgue decomposition of B. In a similar way, one can check that ϕ preserves singularity in both directions.

By the criterion (A4) of uniqueness of Lebesgue decompositions, ϕ preserves the

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elements of $B(H)^+$ with closed range in both directions. This means that the operator $A \in B(H)^+$ has closed range if and only if $\phi(A)$ has closed range.

Now, by Corollary 2.2, for arbitrary operators $A, B \in B(H)^+$ with closed ranges, we have

$$\operatorname{rng} B \subset \operatorname{rng} A \Longleftrightarrow \operatorname{rng} \phi(B) \subset \operatorname{rng} \phi(A),$$

and hence,

$$\operatorname{rng} B = \operatorname{rng} A \Longleftrightarrow \operatorname{rng} \phi(B) = \operatorname{rng} \phi(A).$$

We next prove that ϕ preserves the rank of finite rank operators. In fact, this follows from the following characterization of the rank. The positive operator A is of rank n $(1 \leq n \in \mathbb{N})$ if and only if it has closed range, there exists a sequence $A_0, A_0, \ldots, A_{n-1}$ of positive operators with closed range of length n such that

$$\operatorname{rng} A_0 \subsetneq \operatorname{rng} A_1 \subsetneq \ldots \subsetneq \operatorname{rng} A_{n-1} \subsetneq \operatorname{rng} A$$

and there is no similar sequence of length $n+1$. The already verified properties of ϕ imply that ϕ preserves the rank.

As ϕ preserves the positive operators with closed range in both directions, by Corollary 2.3, we obtain that ϕ preserves the invertible elements of $B(H)^+$ in both directions. Therefore, $\phi(I)$ is an invertible positive operator. Consider the transformation

$$A \longmapsto \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}. \tag{2.5}$$

Referring to the already mentioned fact that any transformation of the form $A \longmapsto SAS^*$ with some invertible bounded linear or conjugate-linear operator S on H preserves Lebesgue decompositions in both directions, we see that the transformation in (2.5) is a bijective map on $B(H)^+$ which has the same preserver property and, in addition, it sends I to I. Hence, there is no serious loss of generality in assuming that already ϕ satisfies $\phi(I) = I$.

We prove that ϕ preserves the rank-one projections in both directions. Let P be a rank-one projection. By Lemma 2.4, the P-Lebesgue decomposition of I is $I = P + (I - P)$. As ϕ preserves the rank, $\phi(P)$ is a rank-one operator. Hence, we have $\phi(P) = \mu Q$ with some rank-one projection Q and positive number μ. Now, on one hand, by the original preserver property of ϕ, the $\phi(P)$-Lebesgue decomposition of $\phi(I)$ is

$$I = \phi(I) = \phi(P) + \phi(I - P).$$

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But on the other hand, by Lemma 2.4, the (μQ)-Lebesgue decomposition of I is

$$I = Q + (I - Q).$$

Using the uniqueness of the Lebesgue decomposition with respect to positive operators having closed range, we obtain that $\phi(P) = Q$ and $\phi(I - P) = I - Q = I - \phi(P)$. Consequently, $\phi(P)$ is a rank-one projection, and we also have $\phi(I - P) = I - \phi(P)$.

We show that ϕ preserves the orthogonality among rank-one projections. Let P, Q be orthogonal rank-one projections. It is easy to see that the Q-Lebesgue decomposition of $(I - P)$ is $I - P = Q + (I - P - Q)$. Therefore, we have $\phi(I - P) = \phi(Q) + \phi(I - P - Q) \geq \phi(Q)$. But from the previous paragraph of the proof we know that $\phi(I - P) = I - \phi(P)$. Therefore, we obtain $I - \phi(P) \geq \phi(Q)$, which means that the projections $\phi(P)$ and $\phi(Q)$ are orthogonal to each other.

We assert that for any rank-one projection P, we have a bijective function f_P on $\mathbb{R}_+$ such that $\phi(\lambda P) = f_P(\lambda)\phi(P)$. This follows from the fact that for any positive λ, the ranges of the rank-one operators $\phi(\lambda P)$ and $\phi(P)$ coincide which is a consequence of $\operatorname{rng} \lambda P = \operatorname{rng} P$.

We next prove that the functions f_P are all the same. In order to verify this, first consider an arbitrary rank-one projection P. By Lemma 2.4, for any positive λ, the P-decomposition of λI is

$$\lambda I = \lambda P + \lambda(I - P).$$

Therefore, we obtain

$$\phi(\lambda I) = f_P(\lambda)\phi(P) + \phi(\lambda(I - P)). \tag{2.6}$$

The range of $\lambda(I - P)$ is equal to the range of $I - P$, and hence, we obtain that

$$\operatorname{rng} \phi(\lambda(I - P)) = \operatorname{rng} \phi(I - P) = \operatorname{rng}(I - \phi(P)) = \operatorname{rng} \phi(P)^{\perp}.$$

Consequently, the operators on the right hand side of (2.6) act on orthogonal subspaces. This means that the range of the rank-one projection $\phi(P)$ is an eigensubspace of $\phi(\lambda I)$. As P is an arbitrary rank-one projection, and hence, $\phi(P)$ runs through the set of all rank-one projections, we infer that $\phi(\lambda I)$ is a scalar operator. Again by (2.6), we see that this scalar is $f_P(\lambda)$. So, we have $\phi(\lambda I) = f_P(\lambda)I$. This shows that the bijection f_P of $\mathbb{R}_+$ in fact does not depend on P. We conclude that there is a bijective function f on $\mathbb{R}_+$ such that for every rank-one projection P and nonnegative real number λ, we have $\phi(\lambda P) = f(\lambda)\phi(P)$.

Let P, Q be orthogonal rank-one projections and λ, μ positive real numbers. Set $B = \lambda P + \mu Q$. The P-Lebesgue decomposition of B is $B = \lambda P + \mu Q$. Therefore, we

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have

$$\phi(\lambda P+\mu Q)=\phi(B)=\phi(\lambda P)+\phi(\mu Q)=f(\lambda)\phi(P)+f(\mu)\phi(Q).$$

In particular, we obtain that $\phi(P+Q)=\phi(P)+\phi(Q)$. Next, let R be an arbitrary rank-one subprojection of $P+Q$. Then we infer that $\phi(R)$ is a subprojection of $\phi(P)+\phi(Q)$ (ϕ preserves the inclusion of ranges of operators with closed range). As we have seen in Lemma 2.4, the absolutely continuous part in the R-Lebesgue decomposition of B is $\lambda(B,R)R$. We compute the quantity $\lambda(B,R)$ in the following way. Let r be a unit vector in the range of R. By (2.1), we have

$$\lambda(B,R)=\|B^{-1/2}r\|^{-2}=\frac{1}{\langle B^{-1}r,r\rangle}=\frac{1}{\langle((1/\lambda)P+(1/\mu)Q)r,r\rangle}$$

$$=\frac{1}{(1/\lambda)\langle Pr,r\rangle+(1/\mu)\langle Qr,r\rangle}=\frac{1}{(1/\lambda)\operatorname{tr}PR+(1/\mu)\operatorname{tr}QR}.$$

Therefore, we obtain

$$\lambda(B,R)R=\frac{1}{(1/\lambda)\operatorname{tr}PR+(1/\mu)\operatorname{tr}QR}R.$$

Similarly, the absolutely continuous part in the $\phi(R)$-Lebesgue decomposition of $\phi(B)=f(\lambda)\phi(P)+f(\mu)\phi(Q)$ is

$$\lambda(\phi(B),\phi(R))\phi(R)=\frac{1}{(1/f(\lambda))\operatorname{tr}\phi(P)\phi(R)+(1/f(\mu))\operatorname{tr}\phi(Q)\phi(R)}\phi(R).$$

As ϕ preserves Lebesgue decompositions, it follows that

$$\phi(\lambda(B,R)R)=\lambda(\phi(B),\phi(R))\phi(R).$$

Hence, using $\phi(\lambda(B,R)R)=f(\lambda(B,R))\phi(R)$, we have the following functional equation:

$$f\left(\frac{1}{(1/\lambda)\operatorname{tr}PR+(1/\mu)\operatorname{tr}QR}\right)=\frac{1}{(1/f(\lambda))\operatorname{tr}\phi(P)\phi(R)+(1/f(\mu))\operatorname{tr}\phi(Q)\phi(R)},$$

which can be rewritten as

$$f\left(\frac{1}{(1/\lambda)\alpha+(1/\mu)(1-\alpha)}\right)=\frac{1}{(1/f(\lambda))\alpha'+(1/f(\mu))(1-\alpha')},\tag{2.7}$$

where λ,μ are arbitrary positive numbers, $\alpha\in[0,1]$ is also arbitrary and $\alpha'\in[0,1]$. It is clear from the discussion above that α' does not depend on λ,μ, and thus, by (2.7), it depends only on α. Hence, we can write (2.7) into the following form

$$f\left(\frac{1}{(1/\lambda)\alpha+(1/\mu)(1-\alpha)}\right)=\frac{1}{(1/f(\lambda))\varphi(\alpha)+(1/f(\mu))(1-\varphi(\alpha))}$$

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($\lambda, \mu > 0$, $\alpha \in [0,1]$). Here, f is a bijective map on $\mathbb{R}_+$ sending 0 to 0 and 1 to 1, and $\varphi : [0,1] \to [0,1]$ is a function. We apply Lemma 2.5 and conclude that f, φ are the identities on their domains. What concerns φ, this gives us that

$$\operatorname{tr} PQ = \operatorname{tr} \phi(P)\phi(Q).$$

This means that the transformation ϕ, when restricted onto the set of all rank-one projections, is a bijective map preserving the trace of products. This latter quantity appears in the mathematical foundations of quantum mechanics and is usually called there transition probability. Transformations on the set of rank-one projections which preserve the transition probability are holding the name quantum mechanical symmetry transformations, and they play a fundamental role in the probabilistic aspects of quantum mechanics. A famous theorem of Wigner describes the structure of those transformations.[3] It says that every such map is implemented by a unitary or antiunitary operator on the underlying Hilbert space. This means that we have a unitary or antiunitary operator U on H such that

$$\phi(P) = UPU^*$$

holds for every rank-one projection P on H. (For generalizations of Wigner's theorem concerning different structures, we refer to the Sections 2.1-2.3 of [7]; see also the references therein.) Therefore, considering the transformation $A \mapsto U^*\phi(A)U$ if necessary, we can further assume without serious loss of generality that $\phi(P) = P$ holds for every rank-one projection P.

We complete the proof by showing that $\phi(B) = B$ holds for every positive operator B. Indeed, we already know that for an arbitrary rank-one projection P, the absolutely continuous part in the P-Lebesgue decomposition of B is $\lambda(B,P)P$. As ϕ preserves Lebesgue decompositions, we obtain that the absolutely continuous part in the $\phi(P)$-Lebesgue decomposition of $\phi(B)$ is $\phi(\lambda(B,P)P)$. Since f is the identity on $\mathbb{R}_+$ and $\phi(P) = P$, we have $\phi(\lambda(B,P)P) = \lambda(B,P)P$. On the other hand, the absolutely continuous part in the P-Lebesgue decomposition of $\phi(B)$ is $\lambda(\phi(B),P)P$. Therefore, we have

$$\lambda(\phi(B),P)P = \phi(\lambda(B,P)P) = \lambda(B,P)P.$$

This gives us that

$$\lambda(B,P) = \lambda(\phi(B),P)$$

holds for every rank-one projection P. Since according to [2, Corollary 1], every positive operator is uniquely determined by its strength function, we obtain that $\phi(B) = B$. This completes the proof of the theorem. $\square$

[3]There are in fact several equivalent formulations of Wigner's theorem; see, e.g., pp. 12-13 in [7]. The one we use here concerns transformations on rank-one projections.

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