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LARGEST EIGENVALUES OF THE DISCRETE *P*-LAPLACIAN OF TREES WITH DEGREE SEQUENCES*

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Abstract. Trees that have greatest maximum p-Laplacian eigenvalue among all trees with a given degree sequence are characterized. It is shown that such extremal trees can be obtained by breadth-first search where the vertex degrees are non-increasing. These trees are uniquely determined up to isomorphism. Moreover, their structure does not depend on p.

Key words. Discrete p-Laplacian, Largest eigenvalue, Eigenvector, Tree, Degree sequence, Majorization.

AMS subject classifications. 05C35, 05C75, 05C05, 05C50.

1. Introduction.

The eigenvalues of the combinatorial Laplacian have been intensively investigated during the last decades. Recently there is an increasing interest in the discrete p-Laplacian, a natural generalization of the Laplacian, which corresponds to $p = 2$; see e.g., [1–3, 9]. The related eigenvalue problems have been occasionally occurred in fields like network analysis [5, 10], pattern recognition [12], or image processing [6].

For a simple connected undirected graph $G = (V, E)$ with vertex set V and edge set E the *discrete p-Laplacian* $\Delta_p(G)$ of a function f on V $(1 < p < \infty)$ is given by

$$\Delta_p(G)f(v) = \sum_{u \in V, uv \in E} (f(v) - f(u))^{[p-1]} .$$

We use the symbol $t^{[q]}$ to denote a "power" function that preserves the sign of t, i.e., $t^{[q]} = \operatorname{sign}(t) \cdot |t|^q$. Notice that for $p = 2$, $\Delta_2(G)$ is the well-known combinatorial graph Laplacian, usually defined as $\Delta(G) = D(G) - A(G)$ where $A(G)$ denotes the adjacency matrix of G and $D(G)$ the diagonal matrix of vertex degrees. For $p \neq 2$, $\Delta_p(G)$ is a non-linear operator. We write Δ_p (and Δ) for short if there is no risk of confusion.

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A real number λ is called an *eigenvalue* of $\Delta_p(G)$ if there exists a function $f \neq 0$ on V such that

$$\Delta_p f(v) = \lambda\, f(v)^{[p-1]}\ . \tag{1.1}$$

The function f is then called the *eigenfunction* corresponding to λ.

In this paper we are interested in the largest eigenvalue of $\Delta_p(G)$, which we denoted by $\mu_p(G)$. In particular we investigate the structure of trees which have largest maximum eigenvalue $\mu_p(T)$ among all trees with a given degree sequence. We call such trees *extremal trees*. We show that for such trees the degree sequence is non-increasing with respect to an ordering of the vertices that is obtained by breadth-first search. Furthermore, we show that the largest maximum eigenvalue in such classes of trees is strictly monotone with respect to some partial ordering of degree sequences. Thus, we extend a result that has been independently shown by Zhang [11] and Bıyıkoğlu et al. [4]. It is remarkable that this result is independent from the value of p. Although there is still little known about the p-Laplacian our result shows that it shares at least some of the properties with the combinatorial Laplacian.

2. Degree Sequences and Largest Eigenvalue. Let $d(v)$ denote the degree of vertex v. We call a vertex v with $d(v) = 1$ a *pendant vertex* (or *leaf*) of a graph. Recall that a non-increasing sequence $\pi = (d_0, \dots, d_{n-1})$ of non-negative integers is called *degree sequence* if there exists a graph G for which $d_0, \dots, d_{n-1}$ are the degrees of its vertices. In particular, π is a tree sequence, i.e. a degree sequence of some tree, if and only if every $d_i > 0$ and $\sum_{i=0}^{n-1} d_i = 2\,(n-1)$. We refer the reader to [7] for relevant background on degree sequences. We denote the class of trees with a given degree sequence π as

$$\mathcal{T}_\pi = \{G \text{ is a tree with degree sequence } \pi\}\,.$$

We want to give a characterization of extremal trees in $\mathcal{T}_\pi$, i.e., those trees in $\mathcal{T}_\pi$ that have greatest maximum eigenvalue. For this task we introduce an ordering of the vertices $v_0, \dots, v_{n-1}$ of a graph G by means of breadth-first search: Select a vertex $v_0 \in G$ and create a sorted list of vertices beginning with v_0; append all neighbors $v_1, \dots, v_{d(v_0)}$ of v_0 sorted by decreasing degrees; then append all neighbors of v_1 that are not already in this list; continue recursively with $v_2, v_3, \dots$ until all vertices of G are processed. In this way we build layers where each v in layer i is adjacent to some vertex w in layer $i-1$ and vertices u in layer $i+1$. We then call the vertex w the *parent* of v and v a child of w.

Definition 2.1 (BFD-ordering). Let $G = (V, E)$ be a connected graph with root v_0. Then a well-ordering $\prec$ of the vertices is called *breadth-first search ordering with decreasing degrees* (*BFD*-ordering for short) if the following holds for all vertices in V:

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(B1) if $w_1 \prec w_2$ then $v_1 \prec v_2$ for all children v_1 of w_1 and v_2 of w_2;
(B2) if $v \prec u$, then $d(v) \geq d(u)$.

We call a connected graph that has a BFD-ordering of its vertices a *BFD-graph* (see Fig. 2.1 for an example).

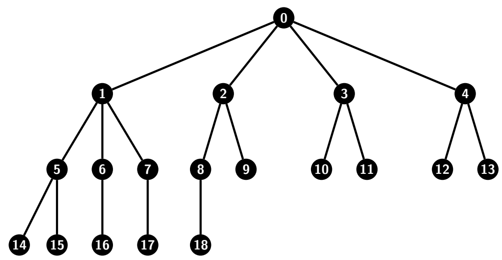

FIG. 2.1. *A BFD-tree with degree sequence* $\pi = (4^2, 3^4, 2^3, 1^{10})$

Every graph has for each of its vertices v an ordering with root v that satisfies (B1). This can be found by a breadth-first search as described above. However, not all trees have an ordering that satisfies both (B1) and (B2); consider the tree in Fig. 2.2.

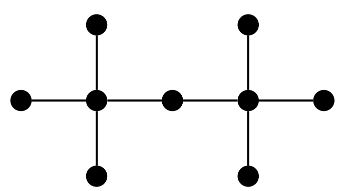

FIG. 2.2. *A tree with degree sequence* $\pi = (4^2, 2^1, 1^6)$ *where no BFD-ordering exists.*

THEOREM 2.2. *A tree T with degree sequence π is extremal in class $\mathcal{T}_\pi$ (i.e., has greatest maximum p-Laplacian eigenvalue) if and only if it is a BFD-tree. T is then uniquely determined up to isomorphism. The BFD-ordering is consistent with the corresponding eigenfunction f of G in such a way that $|f(u)| > |f(v)|$ implies $u \prec v$.*

For a tree with degree sequence π a sharp upper bound on the largest eigenvalue can be found by computing the corresponding BFD-tree. Obviously finding this tree can be done in $O(n)$ time if the degree sequence is sorted.

We now define a partial ordering on degree sequences $\pi = (d_0, \ldots, d_{n-1})$ and $\pi' = (d'_0, \ldots, d'_{n'-1})$ with $n \leq n'$ and $\pi \neq \pi'$ as follows: we write $\pi \lhd \pi'$ if and only if $\sum_{i=0}^{j} d_i \leq \sum_{i=0}^{j} d'_i$ for all $j = 0, \ldots, n-1$ (recall that the degree sequences are non-increasing). Such an ordering is called *majorization*.

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THEOREM 2.3. *Let π and π' be two distinct degree sequences of trees with $\pi \lhd \pi'$. Let T and T' be extremal trees in the classes $\mathcal{T}_\pi$ and $\mathcal{T}_{\pi'}$, respectively. Then we find for the corresponding maximum eigenvalues $\mu_p(T) < \mu_p(T')$.*

The following corollaries generalize results which are well-known for the case $p = 2$.

COROLLARY 2.4. *A tree T is extremal in the class of all trees with n vertices if and only if it is the star $K_{1,n-1}$.*

COROLLARY 2.5. *A tree G is extremal in the class of all trees with n vertices and k leaves if and only if it is a star $K_{1,k}$ with paths of almost the same lengths attached to each of its k leaves.* Almost *means that each path of two leaves in the tree G has length $l \in \left\{2 \cdot \left\lfloor \frac{n-1}{k} \right\rfloor, \left\lfloor \frac{n-1}{k} \right\rfloor + \left\lceil \frac{n-1}{k} \right\rceil, 2 \cdot \left\lceil \frac{n-1}{k} \right\rceil \right\}$.*

Proof. [of Cor. 2.4 and 2.5] The tree sequences $\pi_n = (n-1, 1, \ldots, 1)$ and $\pi_{n,k} = (k, 2, \ldots, 2, 1, \ldots, 1)$ are maximal w.r.t. ordering $\lhd$ in the respective classes of all trees with n vertices and all trees with n vertices and k pendant vertices. Thus the statement immediately follows from Theorems 2.2 and 2.3. □

3. Preliminaries. The *non-linear Rayleigh quotient* of $\Delta_p(G)$ is given as [2, 9]

$$\mathcal{R}_G^p(f) = \frac{\langle f, \Delta_p f \rangle}{\langle f, f^{[p-1]} \rangle} = \frac{\sum_{uv \in E} |f(u) - f(v)|^p}{\sum_{v \in V} |f(v)|^p} .$$

We can define a Banach space of real-valued functions on V using the p-norm of a function f, $||f||_p = \sqrt[p]{\sum_{v \in V} |f(v)|^p}$, and a Banach space of a function F on the directed edges $[uv]$ using the norm $|F|_p = \sqrt[p]{\sum_{e \in E} |F([uv])|^p}$. Thus the Rayleigh quotient can be written as [2]

$$\mathcal{R}_G^p(f) = \frac{|\nabla f|_p^p}{||f||_p^p} ,$$

where $(\nabla f)([uv]) = f(u) - f(v)$ for an oriented edge $[uv] \in E$. Calculus of variation then implies the existence of eigenvalues and eigenfunctions. These are exactly the critical points of $\mathcal{R}_G^p(f)$ constrained to $||f||_p^p = 1$ and can be found by means of Lagrange multipliers (with then are the eigenvalues).

An immediate consequence of these considerations is that Δ_p is a nonnegative operator, i.e., the smallest eigenvalue is 0 and all other eigenvalues are strictly positive (in case of a connected graph). We find the following characterization of the largest eigenvalue $\mu_p(G)$ of $\Delta_p(G)$ which generalizes the well-known Rayleigh-Ritz theorem.

PROPOSITION 3.1 ([2]).

$$\mu_p(G) = \max_{||f||_p = 1} \mathcal{R}_G^p(f) = \max_{||f||_p = 1} \sum_{uv \in E} |f(u) - f(v)|^p .$$

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Moreover, if $\mathcal{R}_G^p(f) = \mu_p(G)$ *for a function* f*, then* f *is an eigenfunction corresponding to the maximum eigenvalue* $\mu_p(G)$ *of* $\Delta_p(G)$*.*

Notice that every eigenfunction f corresponding to the maximum eigenvalue must fulfill the eigenvalue equation (1.1) for every $v \in V$.

An eigenfunction f corresponding to the largest eigenvalue $\mu_p(G)$ has alternate signs [2]. Thus the following observation simplifies our task. It generalizes a result of Merris [8]. Define the *signless p-Laplacian* $Q_p(G)$ analog to the linear case by

$$Q_p(G)f(v) = \sum_{u \in V, uv \in E} (f(v) + f(u))^{[p-1]} .$$

Its Rayleigh quotient is given by

$$\mathcal{Q}_G^p(f) = \frac{\langle f, Q_p f \rangle}{\langle f, f^{[p-1]} \rangle} = \frac{\sum_{uv \in E} |f(u) + f(v)|^p}{\sum_{v \in V} |f(v)|^p} .$$

Lemma 3.2. *Let* $G = (V_1 \dot\cup V_2, E)$ *be a bipartite graph. Then* $\Delta_p(G)$ *and* $Q_p(G)$ *have the same spectrum. Moreover,* f *is an eigenfunction of* $\Delta_p(G)$ *affording eigenvalue* λ *if and only if* f' *is an eigenfunction of* $Q_p(G)$ *affording* λ*, whereby* $f'(v) = f(v)$ *if* $v \in V_1$ *and* $f'(v) = -f(v)$ *if* $v \in V_2$*.*

Proof. Let f be an eigenfunction of $\Delta_p(G)$, i.e., $\Delta_p f(v) = \sum_{u \in V, uv \in E}(f(v) - f(u))^{[p-1]} = \lambda f(v)$. Define $s(v) = 1$ if $v \in V_1$ and $s(v) = -1$ if $v \in V_2$, hence $f'(v) = s(v)f(v)$. Then we find $(Q_p f')(v) = \sum_{u \in V, uv \in E}(f'(v) + f'(u))^{[p-1]} = \sum_{u \in V, uv \in E} s(v)(f(v) - f(u))^{[p-1]} = \lambda s(v) f(v)^{[p-1]} = \lambda f'(v)^{[p-1]}$ and hence f' is an eigenfunction of $Q_p(G)$ affording the same eigenvalue λ, as claimed. The necessity of the condition follows analogously. ∎

Hence we will investigate the eigenfunction to the maximum eigenvalue of $Q_p(G)$ in our proof. Prop. 3.1 holds completely analogously for the maximum eigenvalue.

Proposition 3.3.

$$\mu_p(G) = \max_{||f||_p = 1} \mathcal{Q}_G^p(f) = \max_{||f||_p = 1} \sum_{uv \in E} |f(u) + f(v)|^p .$$

Moreover, if $\mathcal{Q}_G^p(f) = \mu_p(G)$ *for a function* f*, then* f *is an eigenfunction corresponding to the maximum eigenvalue* $\mu_p(G)$ *of* $Q_p(G)$*.*

Corollary 3.4. *The maximum eigenvalue is strictly positive, i.e.,* $\mu_p(G) > 0$*. Moreover, if* f *is an eigenfunction of* $Q_p(G)$ *of a connected graph* G *corresponding to* $\mu_p(G)$*, then* f *is either strictly positive, i.e.,* $f(v) > 0$ *for all* $v \in V$*, or strictly negative.*

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We call a positive eigenfunction f corresponding to $\mu_p(G)$ a *Perron vector* of $Q(G)$. The following technical result will be useful.

LEMMA 3.5. *Let $0 \le \epsilon \le \delta \le z$ and $p > 1$. Then $(z+\epsilon)^p + (z-\epsilon)^p \le (z+\delta)^p + (z-\delta)^p$. Equality holds if and only if $\epsilon = \delta$.*

Proof. Obviously equality holds when $\epsilon = \delta$. Let $\epsilon < \delta$. Notice that t^p is strictly monotonically increasing and strictly convex for every $t \ge 0$. Thus we find $(z-\epsilon)^p - (z-\delta)^p < (\delta-\epsilon)\,p\,(z-\epsilon)^{p-1}$ and $(z+\epsilon)^p - (z+\delta)^p < -(\delta-\epsilon)\,p\,(z+\epsilon)^{p-1}$ by using tangents in $z-\epsilon$ and $z+\epsilon$, respectively. Consequently $(z-\epsilon)^p - (z-\delta)^p + (z+\epsilon)^p - (z+\delta)^p < p\,(\delta-\epsilon)\,((z-\epsilon)^{p-1} - (z+\epsilon)^{p-1}) \le 0$ and thus the statement follows. $\square$

4. Proof of the Theorems. Because of Lemma 3.2 we characterize trees that maximize the largest eigenvalue $\mu_p(G)$ of Q_p. In the following f always denotes a Perron vector of $Q_p(T)$ for some tree T. P_{uv} denotes the path between two vertices u and v.

The main techniques for proving our theorems is *rearranging* of edges. We need two types of rearrangement steps that we call *switching* and *shifting*, respectively, in the following.

LEMMA 4.1 (*Switching*). *Let $T \in \mathcal{T}_\pi$ and let u_1v_1, $u_2v_2 \in E(T)$ be edges such that the path $P_{v_1v_2}$ neither contains u_1 nor u_2, and $v_1 \ne v_2$. Then by replacing edges u_1v_1 and u_2v_2 by the respective edges u_1v_2 and u_2v_1 we get a new tree $T' \in \mathcal{T}_\pi$. Furthermore, $\mu(T') \ge \mu(T)$ whenever there exists a Perron vector f with $f(u_1) \ge f(u_2)$ and $f(v_2) \ge f(v_1)$, and $\mu(T') > \mu(T)$ if one of the two inequalities is strict.*

Proof. Since $P_{v_1v_2}$ neither contains u_1 nor u_2 by assumption, T' is again a tree. Since switching of two edges does not change degrees, T' also belongs to class $\mathcal{T}_\pi$. Let f be a Perron vector with $||f||_p = 1$. To verify the inequality we have to compute the effects of removing and inserting edges on the Rayleigh quotient.

$$\begin{aligned}
\mu_p(T') - \mu_p(T) &\ge \mathcal{Q}^p_{T'}(f) - \mathcal{Q}^p_T(f)\\
&= [(f(u_1)+f(v_2))^p + (f(v_1)+f(u_2))^p]\\
&\qquad - [(f(v_1)+f(u_1))^p + (f(u_2)+f(v_2))^p]\\
&\doteq D_1^p + D_2^p - D_3^p - D_4^p\\
&\ge 0\,.
\end{aligned}$$

The last inequality follows from Lemma 3.5 by setting $z+\delta = D_1$, $z-\delta = D_2$, $z+\epsilon = \max\{D_3, D_4\}$, and $z-\epsilon = \min\{D_3, D_4\}$. Notice that $D_1 + D_2 = D_3 + D_4$ and that $D_1 \ge D_3, D_4 \ge D_2$. If $f(u_1) > f(u_2)$ or $f(v_2) > f(v_1)$ then the eigenvalue equation (1.1) would not hold for v_1 or u_2. Thus f is not an eigenfunction corresponding to $\mu_p(T')$ and thus $\mu_p(T') > \mathcal{Q}^p_{T'}(f) \ge \mu_p(T)$ as claimed. $\square$

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Lemma 4.2 (*Shifting*). *Let $T \in \mathcal{T}_\pi$ and $u, v \in V(T)$ with $u \neq v$. Assume we have edges $ux_1, \ldots, ux_k \in E(T)$ such that none of the x_i is in P_{uv}. Then we get a new graph T' by replacing all edges $ux_1, \ldots, ux_k$ by the respective edges $vx_1, \ldots, vx_k$. Moreover, $\mu_p(T') > \mu_p(T)$ whenever there exists a Perron vector f with $f(u) \leq f(v)$.*

Proof. Let f be a Perron vector with $||f||_p = 1$. Then

$$\begin{aligned}\mu_p(T') - \mu_p(T) &\geq \mathcal{Q}^p_{T'}(f) - \mathcal{Q}^p_T(f)\\ &= \textstyle\sum_{i=1}^k \left[(f(v) + f(x_i))^p - (f(u) + f(x_i))^p\right]\\ &\geq 0\ .\end{aligned}$$

where the last inequality immediately follows from the strict monotonicity of t^p for every $t \geq 0$. Now if $\mu_p(T') = \mu_p(T)$ then f also must be an eigenfunction of T' by Lemma 3.3. Thus the eigenvalue equation (1.1) for vertex u in T and T' implies that $f(u) + f(x_i) = 0$, a contradiction, since f is strictly positive on each vertex. □

Lemma 4.3. *Let T be extremal in class $\mathcal{T}_\pi$ and f a Perron vector of $Q_p(T)$. If $d(u) > d(v)$, then $f(u) > f(v)$.*

Proof. Suppose $d(u) > d(v)$ but $f(u) \leq f(v)$. Then we construct a new graph $T' \in \mathcal{T}_\pi$ by shifting $k = d(u) - d(v)$ edges in T. For this task we can choose any k of the $d(u)-1$ edges that are not contained in P_{uv}. Thus $\mu_p(T') > \mu_p(T)$ by Lemma 4.2, a contradiction to our assumption that T is extremal. □

Lemma 4.4. *Each class $\mathcal{T}_\pi$ contains a BFD-tree T that is uniquely determined up to isomorphism.*

Proof. For a given tree sequence the construction of a BFD-tree is straightforward. To show that two BFD-trees T and T' in class $\mathcal{T}_\pi$ are isomorphic we use a function ϕ that maps the vertex v_i in the ith position in the BFD-ordering of T to the vertex w_i in the ith position in the BFD-ordering of T'. By the properties (B1) and (B2) ϕ is an isomorphism, as v_i and w_i have the same degree and the images of neighbors of v_i in the next layer are exactly the neighbors of w_i in the next layer. □

Now let T be an extremal tree in $\mathcal{T}_\pi$ with Perron vector f. Create an ordering $\prec$ of its vertices by breadth-first search starting with the maximum of f. For all children u_i of a vertex w we set $u_i \prec u_j$ whenever

(i) $f(u_i) > f(u_j)$ or
(ii) $f(u_i) = f(u_j)$ and $d(u_i) > d(u_j)$.

We enumerate the vertices of T with respect to this ordering, i.e., $v_i \prec v_j$ if and only if $i < j$. In particular, v_0 is a maximum of f.

Lemma 4.5. *Let T be extremal in class $\mathcal{T}_\pi$ with Perron vector f. Construct an ordering $\prec$ as described above. Then $f(v_i) > f(v_j)$ implies $v_i \prec v_j$.*

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Proof. Suppose we have two vertices $v_i \succ v_j$ with $f(v_i) > f(v_j)$. Let w_k be the first vertex (in the ordering $\prec$) which has such a child v_j with this property, and choose v_i ($\succ v_j$) as the first vertex with $f(v_i) > f(v_j)$. Since v_0 is a maximum of f such a w_k must exist. By construction of our breadth-first search we have $w_k \prec v_j \prec v_i$, $w_k v_i \notin E(T)$, and $f(w_k) \geq f(u)$ for all $u \succ w_k$. We have two cases:

(1) v_j is in the path $P_{w_k v_i}$: Then $f(v_i) > f(v_j)$ and Lemma 4.3 imply $d(v_i) \geq d(v_j) \geq 2$ and thus there exists a child w_m of v_i which cannot be adjacent to v_j.
(2) v_j is not in the path $P_{w_k v_i}$: Then the parent w_m of v_i cannot be adjacent to v_j. Moreover, $w_m \succ w_k$ since otherwise $v_i \prec v_j$.

In either case we find $f(w_k) \geq f(w_m)$ and $f(v_i) > f(v_j)$. Thus we can replace edges $w_k v_j$ and $w_m v_i$ by the edges $w_k v_i$ and $w_m v_j$ we get a new tree $T' \in \mathcal{T}_\pi$ with $\mu_p(T') > \mu_p(T)$ by Lemma 4.1, a contradiction to our assumption. □

Proof of Theorem 2.2. Again use the above ordering $\prec$. Thus (B1) holds. By Lemma 4.5 f is monotone with respect to this ordering. Thus Lemma 4.3 (together with construction rule (ii)) implies (B2). The sufficiency of our condition is a consequence of the uniqueness of BFD-trees as stated in Lemma 4.4. □

Proof of Theorem 2.3. Let $\pi = \{d_0, \ldots, d_{n-1}\}$ and $\pi' = \{d'_0, \ldots, d'_{n'-1}\}$ be two tree sequences with $\pi \lhd \pi'$ and $n = n'$. By Theorem 2.2 the maximum eigenvalue is largest for a tree T within class $\mathcal{T}_\pi$ when T is a BFD-tree. Again f denotes a Perron vector of $Q_p(T)$. We have to show that there exists a tree $T' \in \mathcal{T}_{\pi'}$ such that $\mu(T') > \mu(T)$. Therefore we construct a sequence of trees $T = T_0 \to T_1 \to \ldots \to T_s = T'$ by shifting edges and show that $\mu_p(T_j) > \mu_p(T_{j-1})$ for every $j = 1, \ldots, s$. We denote the degree sequence of T_j by $\pi^{(j)}$.

For a particular step in our construction, let k be the least index with $d'_k > d_k^{(j)}$. Let v_k be the corresponding vertex in tree T_j. Since $\sum_{i=0}^{k} d'_i > \sum_{i=0}^{k} d_i^{(j)}$ and $\sum_{i=0}^{n-1} d'_i = \sum_{i=0}^{n-1} d_i^{(j)} = 2(n-1)$ there must exist a vertex $v_l \succ v_k$ with degree $d_l^{(j)} \geq 2$. Thus it has a child u_l. By Lemma 4.2 we can replace edge $v_l u_l$ by edge $v_k u_l$ and get a new tree T_{j+1} with $\mu_p(T_{j+1}) > \mu_p(T_j)$. Moreover, $d_k^{(j+1)} = d_k^{(j)} + 1$ and $d_l^{(j+1)} = d_l^{(j)} - 1$, and consequently $\pi^{(j)} \lhd \pi^{(j+1)}$. By repeating this procedure we end up with degree sequence π' and the statement follows for the case where $n' = n$.

Now assume $n' > n$. Then we construct a sequence of trees T_j by the same procedure. However, now it happens that we arrive at some tree T_r where $d'_k > d_k^{(r)}$ but $d_l^{(r)} = 1$ for all $v_l \succ v_k$, i.e., they are pendant vertices. In this case we join a new pendant vertex to v_k. Then $d_k^{(r+1)} = d_k^{(r)} + 1$ and we have added a new vertex degree of value 1 to $\pi^{(r)}$ to obtain $\pi^{(r+1)}$. Thus $\pi^{(r+1)}$ is again a tree sequence with $\pi^{(r)} \lhd \pi^{(r+1)}$. Moreover, $\mu_p(T_{r+1}) > \mu_p(T_r)$ as $T_{r+1} \supset T_r$. By repeating this procedure we end up with degree sequence π' and the statement of Theorem 2.3. □

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