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ON THE MAXIMUM POSITIVE SEMI-DEFINITE NULLITY AND THE CYCLE MATROID OF GRAPHS*

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Abstract. Let $G = (V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$, in which we allow parallel edges but no loops, and let $\mathcal{S}_+(G)$ be the set of all positive semi-definite $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, $a_{i,j} \neq 0$ if $i \neq j$ and i and j are connected by exactly one edge, and $a_{i,j} \in \mathbb{R}$ if $i = j$ or i and j are connected by parallel edges. The maximum positive semi-definite nullity of G, denoted by $M_+(G)$, is the maximum nullity attained by any matrix $A \in \mathcal{S}_+(G)$. A k-separation of G is a pair of subgraphs (G_1, G_2) such that $V(G_1) \cup V(G_2) = V$, $E(G_1) \cup E(G_2) = E$, $E(G_1) \cap E(G_2) = \emptyset$ and $|V(G_1) \cap V(G_2)| = k$. When G has a k-separation (G_1, G_2) with $k \leq 2$, we give a formula for the maximum positive semi-definite nullity of G in terms of G_1, G_2, and in case of $k = 2$, also two other specified graphs. For a graph G, let c_G denote the number of components in G. As a corollary of the result on k-separations with $k \leq 2$, we obtain that $M_+(G) - c_G = M_+(G') - c_{G'}$ for graphs G and G' that have isomorphic cycle matroids.

Key words. Positive semi-definite matrices, Nullity, Graphs, Separation, Matroids.

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1. Introduction. Let $A = [a_{i,j}]$ be a symmetric matrix in which some of the off-diagonal entries are prescribed to be zero and some of the off-diagonal entries are prescribed to be nonzero. Can we give a reasonable upper bound for the multiplicity of the smallest eigenvalue of A? Let us formulate this in a different way. Let $G = (V, E)$ be a graph with vertex-set $V = \{1, 2, \ldots, n\}$. All graphs in this paper are allowed to have parallel edges but no loops. Let $\mathcal{S}(G)$ be the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

(i) $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent,
(ii) $a_{i,j} \neq 0$ if $i \neq j$ and i and j are connected by exactly one edge, and
(iii) $a_{i,j} \in \mathbb{R}$ if $i = j$ or i and j are connected by multiple edges.

Let $\mathcal{S}_+(G)$ be the set of all positive semi-definite $A \in \mathcal{S}(G)$. It is clear how to adjust the definition of $\mathcal{S}_+(G)$ for the case that the vertex-set of G is not of the form $\{1, 2, \ldots, n\}$ but a subset thereof. We denote for any matrix A the nullity of A by $\mathrm{nul}(A)$. What is the largest possible nullity attained by any $A \in \mathcal{S}_+(G)$? In other

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words, what is

$$\max\{\text{nul}(A) \mid A \in \mathcal{S}_+(G)\}? \tag{1.1}$$

We call this number the maximum positive semi-definite nullity of G and denote it by $M_+(G)$.

We could also pose the question of finding the smallest possible rank attained by any matrix $A \in \mathcal{S}_+(G)$. We denote the smallest rank attained by any $A \in \mathcal{S}_+(G)$ by $\text{mr}_+(G)$, and call this number the minimum positive semi-definite rank of G. If G has n vertices, then $M_+(G) + \text{mr}_+(G) = n$. Hence, the problem of finding the maximum positive semi-definite nullity of a graph G is the same as the problem of finding the minimum positive semi-definite rank of G.

Without the requirement that the matrices in (1.1) are positive semi-definite, we obtain the maximum nullity of a graph G. This, which is denoted by $M(G)$, is defined as

$$\max\{\text{nul}(A) \mid A \in \mathcal{S}(G)\}.$$

The minimum rank of a graph G, denoted by $\text{mr}(G)$, is defined as

$$\min\{\text{rank}(A) \mid A \in \mathcal{S}(G)\}.$$

See Fallat and Hogben [2] for a survey on the minimum rank and the minimum positive semi-definite rank of a graph.

A *separation* of G is a pair of subgraphs (G_1, G_2) such that $V(G_1) \cup V(G_2) = V$, $E(G_1) \cup E(G_2) = E$, $E(G_1) \cap E(G_2) = \emptyset$; the order of a separation is $|V(G_1) \cap V(G_2)|$. A k-separation is a separation of order k, and a $(\leq k)$-separation is a separation of order $\leq k$. A 1-separation (G_1, G_2) of a graph G corresponds to a vertex-sum of G_1 and G_2 at the vertex v of $V(G_1) \cap V(G_2)$. Let G be a graph which has a (≤ 2)-separation (G_1, G_2). The author gave in [5] a formula for the maximum nullity of G in terms of G_1, G_2, and other specified graphs. In this paper, we give a formula for the maximum positive semi-definite nullity of G in terms of G_1, G_2, and in case that the separation has order 2, also two other specified graphs. The positive semi-definiteness makes the proof of this formula in part different from the formula for the maximum nullity of graphs with a 2-separation.

If $G = (V, E)$ and $G' = (V', E')$ are graphs such that the cycle matroid of G is isomorphic to the cycle matroid of G', then there is a bijection $f : E \to E'$ such that for each circuit C of G, the edges in $f(E(C))$ span a circuit of G', and for each circuit C' of G', the edges in $f^{-1}(E(C'))$ span a circuit of G. See Oxley [3] for an introduction to Matroid Theory. As a corollary of the result on (≤ 2)-separations, we

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obtain that $M_+(G) - c_G = M_+(G') - c_{G'}$ for graphs G and G' that have isomorphic cycle matroids. Here c_G denotes the number of components in G.

Although we state our results for graphs that may have parallel edges, it is easy to translate them to graphs without parallel edges. One way to do this is as follows: Let G' be obtained from a graph G by removing all edges in the parallel class of an edge e, and let G'' be obtained from G by removing all edges but e in the parallel class of e. Then $M_+(G) = \max\{M_+(G'), M_+(G'')\}$. Another way to translate results for graphs that may have parallel edges to graphs without parallel edges is stated in Lemma 2.11.

The outline of the paper is as follows. In the next section, we give formulas relating $M_+(G)$ to $M_+(G_1)$, $M_+(G_2)$ if G has a 1-separation (G_1, G_2), and to $M_+(G_1)$, $M_+(G_2)$, and two other graphs, if G has a 2-separation (G_1, G_2). We do this for graphs in which we allow multiple edges as well as for graphs in which we do not allow multiple edges. As a corollary, we obtain that the graph G' obtain from identifying a vertex in a graph G and a vertex in some tree satisfies $M_+(G') = M_+(G)$. In Section 3, we show that $M_+(G) - c_G$ is invariant on the class of graphs that have the same cycle matroid. We also show that suspended trees G have $M_+(G) \leq 2$, from which we obtain the corollary that $M_+(G) - c_G \leq 2$ if G has a cycle matroid isomorphic to the cycle matroid of a suspended tree.

2. 1- and 2-separations of graphs. Let (G_1, G_2) be a (≤ 2)-separation of a graph G. In this section, we give formulas for $M_+(G)$ in terms of $M_+(G_1)$, $M_+(G_2)$, and, in case that (G_1, G_2) is a 2-separation, the maximum positive semi-definite nullity of two other specified graphs.

The proofs of the following lemma and theorem are standard.

LEMMA 2.1. *Let (G_1, G_2) be a k-separation of $G = (V, E)$. Then $M_+(G) \geq M_+(G_1) + M_+(G_2) - k$.*

THEOREM 2.2. *Let G be the disjoint union of G_1 and G_2. Then $M_+(G) = M_+(G_1) + M_+(G_2)$.*

Let R and C be finite sets. An $R \times C$ matrix $A = [a_{i,j}]$ is one whose set of row indices is R and set of column indices is C. An ordinary $m \times n$ matrix is then a $\{1, \ldots, m\} \times \{1, \ldots, n\}$ matrix.

Let A be a symmetric $V \times V$ matrix, where V is a finite set. If $S \subseteq V$ such that $A[S]$ is nonsingular, the Schur complement of $A[S]$ is defined as the $(V \setminus S) \times (V \setminus S)$ matrix

$$A/A[S] = A[V \setminus S] - A[V \setminus S, S]A[S]^{-1}A[S, V \setminus S].$$

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If A is a positive semi-definite matrix, then $A/A[S]$ is also a positive semi-definite matrix.

To obtain theorems similar to Theorem 2.2 for 1- and 2-separations, we will use the following lemma.

LEMMA 2.3. *Let V be a finite set and let $R \subseteq V$. Let A be a positive semi-definite $V \times V$ matrix. Then there exists an $S \subseteq V \setminus R$ such that $N = [n_{i,j}] = A/A[S]$ satisfies $N[V \setminus (S \cup R), V \setminus S] = 0$.*

Proof. Take $S \subset V \setminus R$ such that $A[S]$ is positive definite and $|S|$ is as large as possible. Let $N = [n_{i,j}] = A/A[S]$. If $n_{i,i} \neq 0$ for some $i \in V \setminus (R \cup S)$, then $\det(A[S \cup \{i\}]) = \det(A[S]) \det(A[S \cup \{i\}]/A[S]) = \det(A[S]) n_{i,i} \neq 0$ and $|S \cup \{i\}| > |S|$, contradicting that we had chosen S such that $|S|$ is as large as possible. Hence, $n_{i,i} = 0$ for $i \in V \setminus (R \cup S)$. Since A is positive semi-definite, $n_{i,j} = 0$ for $i, j \in V \setminus (S \cup R)$. Hence, $N[V \setminus (S \cup R), V \setminus S] = 0$. □

THEOREM 2.4. *Let (G_1, G_2) be a 1-separation of $G = (V, E)$. Then*

$$M_+(G) = M_+(G_1) + M_+(G_2) - 1.$$

Proof. From Lemma 2.1 it follows that $M_+(G) \geq M_+(G_1) + M_+(G_2) - 1$.

To see that $M_+(G) \leq M_+(G_1) + M_+(G_2) - 1$, let $A = [a_{i,j}] \in \mathcal{S}_+(G)$ with $\mathrm{nul}(A) = M_+(G)$. Let $\{v\} = V(G_1) \cap V(G_2)$. By Lemma 2.3, there exists an $S \subseteq V$ with $v \notin S$ such that $N = [n_{i,j}] = A/A[S]$ is zero everywhere except possibly for entry $n_{v,v}$. If $n_{v,v} \neq 0$, then, by subtracting $n_{v,v}$ from $a_{v,v}$, we obtain a positive semi-definite matrix A' with $\mathrm{nul}(A') = \mathrm{nul}(A) + 1$. This contradiction shows that $n_{v,v} = 0$, and so $M_+(G) = |V \setminus S|$.

We claim that $M_+(G_1) \geq |V(G_1) \setminus S|$ and $M_+(G_2) \geq |V(G_2) \setminus S|$. From this the lemma follows. The matrix $K = [k_{i,j}] = A[V(G_1)]$ belongs to $\mathcal{S}_+(G_1)$. By Lemma 2.3, $L = [l_{i,j}] = K/K[V(G_1) \cap S]$ is zero everywhere except possibly $l_{v,v}$. If $l_{v,v} \neq 0$, then subtracting $l_{v,v}$ from $k_{v,v}$ yields a matrix that belongs to $\mathcal{S}_+(G_1)$ and whose nullity is equal to $|V(G_1) \setminus S|$. Hence, $M_+(G_1) \geq |V(G_1) \setminus S|$. The case $M_+(G_2) \geq |V(G_2) \setminus S|$ can be done similarly. □

COROLLARY 2.5. *Let (G_1, G_2) be a 1-separation of a graph G. Then $mr_+(G) = mr_+(G_1) + mr_+(G_2)$.*

A different proof of the next theorem can be found in van der Holst [4].

THEOREM 2.6. *If G is a tree, then $M_+(G) = 1$.*

Proof. Use Theorem 2.4, that $M_+(K_1) = 1$ and $M_+(K_2) = 1$, and induction on

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the number of vertices in G to show that $M_+(G) = 1$. □

From Theorems 2.4 and 2.6, we obtain:

THEOREM 2.7. *Let G_1 be a graph and let T be a tree disjoint from G_1. If G is obtained from identifying a vertex in G_1 with a vertex in T, then $M_+(G) = M_+(G_1)$.*

Let $G = (V, E)$ be a graph, let (G_1, G_2) be a k-separation of G, and let $R = \{r_1, r_2, \ldots, r_k\} = V(G_1) \cap V(G_2)$. If $B = [b_{i,j}] \in \mathcal{S}_+(G_1)$ and $C = [c_{i,j}] \in \mathcal{S}_+(G_2)$, then we denote by $B \oplus_{r_1,r_2,\ldots,r_k} C$ the matrix $A = [a_{i,j}] \in \mathcal{S}_+(G)$ with

1. $a_{i,j} = b_{i,j}$ if $i, j \in V(G_1)$ and at least one of i and j does not belong to $\{r_1, r_2, \ldots, r_k\}$,
2. $a_{i,j} = c_{i,j}$ if $i, j \in V(G_2)$ and at least one of i and j does not belong to $\{r_1, r_2, \ldots, r_k\}$, and
3. $a_{i,j} = b_{i,j} + c_{i,j}$ if $i, j \in \{r_1, r_2, \ldots, r_k\}$.

This matrix operation is also called sub-direct sum of B and C; see [1]. The matrix A is positive semi-definite and belongs to $\mathcal{S}_+(G)$.

Let $A = [a_{i,j}]$ be a positive semi-definite $n \times n$ matrix. If we multiply simultaneously the vth row and column by a nonzero scalar α, then we obtain a matrix $B = [b_{i,j}]$ that is also positive semi-definite. To see this, let UU^T be the Cholesky decomposition of A, and let W be obtained from U by multiplying its vth column by α. Then $B = WW^T$.

THEOREM 2.8. *Let (G_1, G_2) be a 2-separation of a graph $G = (V, E)$, and let H_1 and H_2 be obtained from $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, by adding an edge between the vertices of $R = \{r_1, r_2\} = V_1 \cap V_2$. Then*

$$M_+(G) = \max\{M_+(G_1) + M_+(G_2) - 2, M_+(H_1) + M_+(H_2) - 2\}.$$

Proof. From Lemma 2.1 it follows that $M_+(G) \geq M_+(G_1) + M_+(G_2) - 2$.

Next we show that $M_+(G) \geq M_+(H_1) + M_+(H_2) - 2$. Let $B = [b_{i,j}] \in \mathcal{S}_+(H_1)$ and $C = [c_{i,j}] \in \mathcal{S}_+(H_2)$ be matrices with $\text{nul}(B) = M_+(H_1)$ and $\text{nul}(C) = M_+(H_2)$. If $b_{r_1,r_2} = c_{r_1,r_2} = 0$, then both G_1 and G_2 have at least one edge between r_1 and r_2. Hence, G has multiple edges between r_1 and r_2, and so $A = B \oplus_{r_1,r_2} C \in \mathcal{S}_+(G)$. If $b_{r_1,r_2} = 0$ and $c_{r_1,r_2} \neq 0$, then G_1 has at least one edge between r_1 and r_2. Hence, G has at least one edge between r_1 and r_2, and therefore $A = B \oplus_{r_1,r_2} C \in \mathcal{S}_+(G)$. The case with $b_{r_1,r_2} \neq 0$ and $c_{r_1,r_2} = 0$ is similar. If $b_{r_1,r_2} \neq 0$, $c_{r_1,r_2} \neq 0$ and there is no edge in G between r_1 and r_2, then, by multiplying simultaneously the r_1th row and column of B by a nonzero scalar if necessary, we may assume that $b_{r_1,r_2} = -c_{r_1,r_2}$. Multiplying simultaneously the r_1th row and column of a positive semi-definite matrix

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by a nonzero scalar yields a positive semi-definite matrix. Then $A = B \oplus_{r_1,r_2} C \in \mathcal{S}_+(G)$. If $b_{r_1,r_2} \neq 0$, $c_{r_1,r_2} \neq 0$ and there is at least one edge in G between r_1 and r_2, then, by multiplying simultaneously the r_1th row and column of B by a scalar if necessary, we may assume that $b_{r_1,r_2} \neq -c_{r_1,r_2}$. Then $A = B \oplus_{r_1,r_2} C \in \mathcal{S}_+(G)$. Since $\mathrm{nul}(A) \geq \mathrm{nul}(B) + \mathrm{nul}(C) - 2$, we obtain $M_+(G) \geq \mathrm{nul}(A) \geq M_+(H_1) + M_+(H_2) - 2$.

We show now that $M_+(G) \leq \max\{M_+(G_1)+M_+(G_2)-2, M_+(H_1)+M_+(H_2)-2\}$. For this, we must show that at least one of the following holds:

1. $M_+(G) \leq M_+(G_1) + M_+(G_2) - 2$, or
2. $M_+(G) \leq M_+(H_1) + M_+(H_2) - 2$.

Let $A = [a_{i,j}] \in \mathcal{S}_+(G)$ be a matrix with $\mathrm{nul}(A) = M_+(G)$. By Lemma 2.3, there exists an $S \subseteq V \setminus R$ such that $A[S]$ is positive definite and $L = (l_{i,j}) = A/A[S]$ satisfies $L[V \setminus (R \cup S), V \setminus S] = 0$. Then $M_+(G) = \mathrm{nul}(A) \leq |V \setminus S|$.

We use the following notation. For $t = 1, 2$, let $S_t = V_t \cap S$, let

$$p_t = A[\{r_1\}, S_t]A[S_t]^{-1}A[S_t, \{r_2\}],$$

and let f_t be the number of edges between r_1 and r_2 in G_t. To shorten the remainder of the proof, we set, for $t = 1, 2$, $q_t = 0$ if $p_t = 0$ and $q_t = 1$ if $p_t \neq 0$.

For $t = 1, 2$, we define the symmetric $V_t \times V_t$ matrix $B = [b_{i,j}]$ by $b_{i,j} = a_{i,j}$ if $i \in V_t \setminus \{r_1, r_2\}$ or $j \in V_t \setminus \{r_1, r_2\}$, $b_{r_1,r_2} = 0$ and $b_{u,u} = A[\{u\}, S_t]A[S_t]^{-1}A[S_t, \{u\}]$ for $u = r_1, r_2$. Then $\mathrm{nul}(B) = |V_t \setminus S_t|$. If $q_t + f_t \neq 1$, then $B \in \mathcal{S}_+(G_t)$, hence $M_+(G_t) \geq |V_t \setminus S_t|$. If $q_t + f_t \geq 1$, then $B \in \mathcal{S}_+(H_t)$, hence $M_+(H_t) \geq |V_t \setminus S_t|$.

If $q_1 + f_1 \neq 1$ and $q_2 + f_2 \neq 1$, then $M_+(G_1) \geq |V_1 \setminus S_1|$ and $M_+(G_2) \geq |V_2 \setminus S_2|$, and so

$$\begin{aligned} M_+(G) &\leq |V \setminus S| \\ &= |V_1 \setminus S_1| + |V_2 \setminus S_2| - 2 \\ &\leq M_+(G_1) + M_+(G_2) - 2. \end{aligned}$$

If $q_1 + f_1 \geq 1$ and $q_2 + f_2 \geq 1$, then $M_+(H_1) \geq |V_1 \setminus S_1|$ and $M_+(H_2) \geq |V_2 \setminus S_2|$, and so

$$\begin{aligned} M_+(G) &\leq |V \setminus S| \\ &= |V_1 \setminus S_1| + |V_2 \setminus S_2| - 2 \\ &\leq M_+(H_1) + M_+(H_2) - 2. \end{aligned}$$

If $q_1 + f_1 = 1$ and $q_2 + f_2 = 0$, then one of the following holds:

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1. $p_1 = 0$, $p_2 = 0$, there is exactly one edge between r_1 and r_2 in G_1, and there are no edges between r_1 and r_2 in G_2, or
2. $p_1 \neq 0$, $p_2 = 0$, and there are no edges between r_1 and r_2 in G_1 and in G_2.

In the first case, $p_1 + p_2 = 0$ and there is exactly one edge between r_1 and r_2 in G. Hence, $M_+(G) = \text{nul}(A) = \text{nul}(A/A[S]) \leq |V \setminus S| - 1$, as $L = [l_{i,j}] = A/A[S]$ has nonzero entries only if $i, j \in \{r_1, r_2\}$. Define the symmetric $V_1 \times V_1$ matrix $B = [b_{i,j}]$ by $b_{i,j} = a_{i,j}$ if $i \in V_1 \setminus \{r_1, r_2\}$ or $j \in V_1 \setminus \{r_1, r_2\}$, $b_{r_1,r_2} = 1$, and $b_{u,u} = 1 + A[\{u\}, S_1]A[S_1]^{-1}A[S_1, \{u\}]$ for $u = r_1, r_2$. Then $B \in \mathcal{S}_+(G_1)$ and $\text{nul}(B) = |V_1 \setminus S_1| - 1$. So $M_+(G_1) \geq |V_1 \setminus S_1| - 1$. Define the symmetric $V_2 \times V_2$ matrix $C = [c_{i,j}]$ by $c_{i,j} = a_{i,j}$ if $i \in V_2 \setminus \{r_1, r_2\}$ or $j \in V_2 \setminus \{r_1, r_2\}$, $c_{r_1,r_2} = 0$, and $c_{u,u} = A[\{u\}, S_2]A[S_2]^{-1}A[S_2, \{u\}]$ for $u = r_1, r_2$. Then $C \in \mathcal{S}_+(G_2)$ and $\text{nul}(C) = |V_2 \setminus S_2|$. So $M_+(G_2) \geq |V_2 \setminus S_2|$. Hence,

$$\begin{aligned} M_+(G) &\leq |V \setminus S| - 1 \\ &= |V_1 \setminus S_1| - 1 + |V_2 \setminus S_2| - 2 \\ &\leq M_+(G_1) + M_+(G_2) - 2. \end{aligned}$$

In the second case, $p_1 + p_2 \neq 0$ and there are no edges between r_1 and r_2 in G. Then $M_+(G) = \text{nul}(A) = |V \setminus S| - 1$. Since $A[V_1] \in \mathcal{S}_+(G_1)$ and $\text{nul}(A[V_1]) = |V_1 \setminus S_1| - 1$, $M_+(G_1) \geq |V_1 \setminus S_1| - 1$. Since $A[V_2] \in \mathcal{S}_+(G_2)$ and $\text{nul}(A[V_2]) = |V_2 \setminus S_2|$, $M_+(G_2) \geq |V_2 \setminus S_2|$. Hence, $M_+(G) \leq M_+(G_1) + M_+(G_2) - 2$.

The case with $q_1 + f_1 = 0$ and $q_2 + f_2 = 1$ is similar. □

COROLLARY 2.9. *Let (G_1, G_2) be a 2-separation of a graph G, and let H_1 and H_2 be obtained from G_1 and G_2, respectively, by adding an edge between the vertices of $S = \{s_1, s_2\} = V(G_1) \cap V(G_2)$. Then $mr_+(G) = \min\{mr_+(G_1) + mr_+(G_2), mr_+(H_1) + mr_+(H_2)\}$.*

We will use the following lemma in the proof of Lemma 2.11.

LEMMA 2.10. *Let $G = (V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$ and let r_1, r_2 be distinct vertices of G. Let H be obtained from G adding an edge between r_1 and r_2. Then $M_+(G) \leq M_+(H) + 1$.* □

LEMMA 2.11. *Let $G = (V, E)$ be a graph and let v be a vertex with exactly two neighbors r_1, r_2. If v is connected to both neighbors by single edges, then $M_+(G) = M_+(H)$, where H is the graph obtained from $G - v$ by connecting r_1 and r_2 by an additional edge.*

Proof. Let $G_1 = G - v$ and let G_2 be a path of length two connecting r_1 and r_2. Then (G_1, G_2) is a 2-separation of G. Let H_1 and H_2 be the graphs obtained from G_1 and G_2, respectively, by adding an edge between r_1 and r_2. From Theorem 2.8, it

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follows that $M_+(G) = \max\{M_+(G_1) + M_+(G_2) - 2, M_+(H_1) + M_+(H_2) - 2\}$. Since G_2 is a path and H_2 is a triangle, $M_+(G) = \max\{M_+(G_1) - 1, M_+(H_1)\}$. From Lemma 2.10, it follows that $M_+(G_1) - 1 \leq M_+(H_1)$. Hence, $M_+(G) = M_+(H_1)$. □

Lemma 2.11 shows us that if G is a graph and G' is obtained from G by subdividing some of its edges, then $M_+(G) = M_+(G')$.

We state now the formula for 2-separations for simple graphs.

COROLLARY 2.12. *Let (G_1, G_2) be a 2-separation of a simple graph G, and let H_1 and H_2 be obtained from G_1 and G_2, respectively, by adding a path of length two between the vertices of $R = \{r_1, r_2\} = V(G_1) \cap V(G_2)$. Then*

$$M_+(G) = \max\{M_+(G_1) + M_+(G_2) - 2, M_+(H_1) + M_+(H_2) - 2\}.$$

In case v is a vertex in G with two neighbors and v is connected to exactly one of its neighbors by a single edge, we have the following proposition.

PROPOSITION 2.13. *Let G be a graph and let v be a vertex with exactly two neighbors r_1, r_2. If v is connected to exactly one of its neighbors by a single edge, then $M_+(G) = M_+(H)$, where H is the graph obtained from $G - v$ by connecting r_1 and r_2 by two edges in parallel.*

Proof. Let $G_1 = G - v$ and let G_2 be the induced subgraph of G spanned by $\{v, r_1, r_2\}$. Then (G_1, G_2) is a 2-separation of G. Let H_i for $i = 1, 2$ be obtained from G_i by adding an edge between r_1 and r_2. Since $M_+(G_2) = 2$ and $M_+(H_2) = 2$, it follows from Theorem 2.8 that $M_+(G) = \max\{M_+(G_1), M_+(H_1)\}$. Hence, $M_+(G) = M_+(H)$. □

3. Cycle matroid of graphs. In this section, we show that graphs G and G' that have isomorphic cycle matroids satisfy $M_+(G) - c_G = M_+(G') - c_{G'}$. For the proof we will use a result of Whitney, which shows that the cycle matroid of a graph G' is isomorphic to the cycle matroid of G if G' can be obtained from G by a sequence of the following three operations:

1. Let G be obtained from G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2. We say that G is obtained from G_1 and G_2 by *vertex identification*.
2. The converse operation of vertex identification is *vertex cleaving*.
3. Let G be obtained from disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2, and identifying the vertices v_1 of G_1 and v_2 of G_2. A *twisting* of G *about* $\{u, v\}$ is the graph G' obtained from G_1 and G_2 by identifying u_1 and v_2, and u_2 and v_1.

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THEOREM 3.1 (Whitney's 2-Isomorphism Theorem [6]). *Let G and H be graphs. Then G and H have isomorphic cycle matroids if and only if H can be transformed into a graph isomorphic to G by a sequence of vertex identifications, vertex cleavings, and twistings.*

See also [3] for a proof of Theorem 3.1.

THEOREM 3.2. *Let G be a graph. If G' is a graph that has the same cycle matroid as G, then $M_+(G') - c_{G'} = M_+(G) - c_G$.*

Proof. By Whitney's 2-Isomorphism Theorem, G' can be obtained from G by a sequence of vertex identifications, vertex cleavings, and twistings. To prove the theorem, it suffices to show the theorem for the case where G' is obtained from G by one of these operations.

We assume first that the operation is vertex identification. Let G_1 and G_2 be vertex-disjoint graphs such that G' is obtained from identifying u_1 of G_1 and u_2 of G_2. By Theorem 2.4, $M_+(G') = M_+(G_1) + M_+(G_2) - 1$. Since G is the disjoint union of G_1 and G_2, $M_+(G) = M_+(G_1) + M_+(G_2)$. Hence, $M_+(G) - 1 = M_+(G')$. Since G has one component more than G', $M_+(G) - c_G = M_+(G') - c_{G'}$. The proof for vertex cleaving is similar.

We assume now that the operation is twisting. Let G_1 and G_2 be graphs such that G is obtained by identifying u_1 of G_1 and u_2 of G_2, and identifying the vertices v_1 of G_1 and v_2 of G_2, and G' is obtained by identifying u_1 and v_2, and u_2 and v_1. For $i = 1, 2$, let H_i be the graph obtained from G_i by adding an additional edge between u_i and v_i. By Theorem 2.8, $M_+(G) = \max\{M_+(G_1) + M_+(G_2) - 2, M_+(H_1) + M_+(H_2) - 2\}$ and $M_+(G') = \max\{M_+(G_1) + M_+(G_2) - 2, M_+(H_1) + M_+(H_2) - 2\}$. Hence, $M_+(G') = M_+(G)$. □

A *suspended tree* is a graph obtained from a tree T by adding a new vertex v and connecting this vertex to some of the vertices in T by edges, possibly by parallel edges. We call v a *suspended vertex*.

LEMMA 3.3. *If G is a suspended tree, then $M_+(G) \leq 2$.*

Proof. We prove the lemma by induction on the number of vertices in G. By Theorem 2.7, we may assume that G is 2-connected. If G has at most three vertices, then clearly $M_+(G) \leq 2$. If G has more than three vertices, let (G_1, G_2) be a 2-separation such that the suspended vertex belongs to $V(G_1) \cap V(G_2)$, and $V(G_1) \setminus (V(G_1) \cap V(G_2)) \neq \emptyset$, and $V(G_2) \setminus (V(G_1) \cap V(G_2)) \neq \emptyset$. Then G_1 and G_2 are suspended trees with fewer vertices, and so $M_+(G_1) \leq 2$ and $M_+(G_2) \leq 2$. Let H_1 and H_2 be obtained from G_1 and G_2, respectively, by adding an additional edge between the vertices in $V(G_1) \cap V(G_2)$. Then H_1 and H_2 are suspended trees with

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fewer vertices, and so $M_+(H_1) \leq 2$ and $M_+(H_2) \leq 2$. As

$$M_+(G) = \max\{M_+(G_1) + M_+(G_2) - 2, M_+(H_1) + M_+(H_2) - 2\},$$

by Theorem 2.8, we obtain $M_+(G) \leq 2$. □

A different proof of the next corollary for the case that G is connected can be found in [4].

COROLLARY 3.4. *If the cycle matroid of G is isomorphic to the cycle matroid of a suspended tree, then $M_+(G) - c_G \leq 1$.*

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