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TOTALLY POSITIVE COMPLETIONS FOR MONOTONICALLY LABELED BLOCK CLIQUE GRAPHS*

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Abstract. It is shown that if the connected graph of the specified entries of a combinatorially symmetric, partial totally positive matrix is monotonically labeled block clique, then there is a totally positive completion. Necessarily the completion strategy is very different from and more complicated than the known totally nonnegative one. The completion preserves symmetry and can be used to solve some non-connected or rectangular, nonsymmetric completion problems.

Key words. Totally positive matrix completion problems, Partial matrix, Monotonically labeled block clique graph.

AMS subject classifications. 15A15, 15A57, 05C50.

1. Introduction.

An m-by-n matrix A is *totally positive*, TP (*totally nonnegative*, TN), if *all* of its minors are positive (nonnegative) [FJ]. A is TP_k (TN_k), if *all* of its minors of size less than or equal to k are positive (nonnegative) [FJ]. A *partial matrix* is a rectangular array in which some entries are specified while the remaining, unspecified, entries may be freely chosen. A *completion* of a partial matrix is a particular choice of values for the unspecified entries, resulting in a conventional matrix. A *matrix completion problem* asks which partial matrices have a completion with a desired property. Of course, in order for a partial matrix to have a TP (TN) completion, every submatrix, consisting entirely of specified entries, must be TP (TN). We call such a partial matrix a partial TP (TN) matrix.

In the study of the combinatorially symmetric TN completion problem [JKL], it is shown that for every partial TN matrix with specified entries corresponding to a monotonically labeled block clique graph, there is a TN completion. (A combinatorially symmetric partial matrix is square, has specified diagonal, and has specified i, j entry iff it has specified j, i entry). Among connected graphs, these are the only ones that assure that a partial TN matrix has a TN completion. Recall that the graph of an n-by-n combinatorially symmetric partial matrix is a graph on n vertices with an

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edge between i and j iff the i, j entry is specified. A connected labeled graph is a *monotonically labeled block clique graph* if it is a collection of ordered cliques in which all the labels of one clique are less than or equal to all those in the next clique. Of course, consecutive cliques overlap in exactly one vertex. Since TP and TN matrices are not permutation similarity invariant it is especially important that these graphs are labeled.

Since [JKL], it has been an open question as to whether a partial TP matrix with specified entries corresponding to a monotonically labeled block clique graph has a TP completion. Of course, there is a TN completion. That completion is quite natural, but is only TN, not TP. Finding a TP completion of these partial matrices has proven quite elusive. It would seem that there should be some TP completion near the standard TN completion, but the problem has resisted numerous attempts to show this is true in any simple explicit or implicit way. Here, we show that there is a TP completion in this case, and the proof has several nice by-products, including some non-combinatorially symmetric and some non-connected completion results for TP.

A combinatorially symmetric completion problem, with monotonically labeled block clique graph, easily reduces inductively to the case of two blocks (or cliques). In that event the standard TN completion is easily described (though the verification requires some effort). Let

$$A = \begin{bmatrix} A_{1,1} & a_{1,2} & ? \\ a_{2,1}^T & a_{2,2} & a_{2,3}^T \\ ? & a_{3,2} & A_{3,3} \end{bmatrix} \tag{1.1}$$

in which $A_{1,1}$ ($A_{3,3}$) is an n_1-by-n_1 (n_3-by-n_3) matrix, $a_{1,2}, a_{2,1} \in \mathbb{R}^{n_1}$ ($a_{2,3}, a_{3,2} \in \mathbb{R}^{n_2}$), $a_{2,2} > 0$ is a scalar, $n = n_1 + 1 + n_3$, and

$$\begin{bmatrix} A_{1,1} & a_{1,2} \\ a_{2,1}^T & a_{2,2} \end{bmatrix}, \quad \begin{bmatrix} a_{2,2} & a_{2,3}^T \\ a_{3,2} & A_{3,3} \end{bmatrix}$$

are TN. Then

$$\tilde{A} = \begin{bmatrix} A_{1,1} & a_{1,2} & \frac{a_{1,2}a_{2,3}^T}{a_{2,2}} \\ a_{2,1}^T & a_{2,2} & a_{2,3}^T \\ \frac{a_{3,2}a_{2,1}^T}{a_{2,2}} & a_{3,2} & A_{3,3} \end{bmatrix} \tag{1.2}$$

is a TN completion of A. In later work the assumption that $a_{2,2} > 0$ was relaxed to $a_{2,2} \geq 0$ [JKD]. Of course, $\tilde{A}$ has relatively large rank 1 submatrices. As a result, $\tilde{A}$ is not TP, even if A is partial TP.

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2. Main result and strategy. A key idea in the present work for the 2-block TP case is to complete the partial matrix indirectly by choosing certain contiguous 2-by-2 minors that involve unspecified entries. By the *k-th contiguous compound* of an m-by-n matrix A, $\mathfrak{C}_k(A)$ with $k \leq \min\{m, n\}$, we mean the $(m-k+1)$-by-$(n-k+1)$ matrix of k-by-k contiguous minors of A, with the index sets ordered lexicographically. For the proof of our theorem we will be dealing only with contiguous compounds of square matrices. We label the entries of $\mathfrak{C}_k(A)$ as $\mathfrak{c}_{i,j|k}$ with i, j specifying the upper left entry of the submatrix in A from which the minor is derived, and k specifying the size of the minor. *Note that A may be reconstructed from $\mathfrak{C}_2(A)$ if certain entries, for example, the diagonal and superdiagonal entries of A, are known. This means that in the 2-block TP case, the completion of A may be recovered from the completion of $\mathfrak{C}_2(A)$. (The standard TN completion is just the one in which $\mathfrak{C}_2(A)$ is completed with 0's). This is important because our strategy implicitly produces a TP completion via targeting of minors in a particular order.* We also note, and will use the fact that, A is TP (TP$_k$) iff all its contiguous compounds, from the 1-*st* to n-*th* (1-*st* to k-*th*), are positive, i.e. if all contiguous minors are positive [FJ]. It is actually true that to prove total positivity the initial minors will suffice [FJ], but we will not use this fact. Our main result is the following.

THEOREM 1. *If A is a n-by-n combinatorially symmetric partial TP matrix, with specified entries corresponding to a monotonically labeled block clique graph, then there is a TP completion, $\hat{A}$, of A. Moreover, if the blocks of A are symmetric, the completion $\hat{A}$ may be taken to be symmetric.*

As mentioned earlier, by induction on the number of cliques, the proof of this theorem reduces to the case of just two cliques. In a graph of k cliques, completing any two cliques reduces the total number of cliques to $k-1$. So our proof simply focuses on the 2-block case. The statement about symmetry will be clear from the completion strategy. Nonetheless, the proof is rather lengthy (it seems necessarily so, based upon a number of attempts at simple proofs). Before beginning, we mention a number of background facts that are used in the proof.

The determinantal inequalities of Koteljanskii and the special case named after Fischer are known to hold for TP matrices. In this case, the inequalities are strict [FJ]. In particular, if $B \in \mathbb{M}_n(\mathbb{R})$ is TP and is partitioned as

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & \dots & B_{1,k} \\ \vdots & & & \vdots \\ B_{k,1} & \dots & & B_{k,k} \end{bmatrix}$$

with square diagonal blocks $B_{i,i}$, then

$$\det B < \det B_{1,1} \dots \det B_{k,k}.$$

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In the event that B has diagonal entries no greater than 1, this means that, for any diagonal block $B_{i,i}$,

$$\det B < \det B_{i,i}.$$

If a partial matrix A of the form (1.1) is partial TP, we may assume without loss of generality that the diagonal entries of A are all 1, by (symmetric, if necessary) positive diagonal scaling. This changes neither the partial TP property nor whether a completion is TP. In addition, since the specified entries of A are positive, we may also assume without loss of generality, via positive diagonal similarity, that the tridiagonal part of A is symmetric (i.e. $a_{i,i+1} = a_{i+1,i},\ i = 1, ..., n-1$). We assume both. Then, by the positivity of contiguous 2-by-2 minors, every off-diagonal entry (and every completing entry as well) will be less than 1. This permits the sort of (frequent) use of Koteljanskii/Fischer mentioned above.

Another fact we frequently use is the special case of Sylvester's determinantal identity [HJ1], which says the following. For matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2}^T & a_{1,3} \\ a_{2,1} & A_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2}^T & a_{3,3} \end{bmatrix}$$

with A n-by-n, $A_{2,2}$ $(n-2)$-by-$(n-2)$ and nonsingular, $a_{1,2}, a_{2,1}, a_{2,3}, a_{3,2} \in \mathbb{R}^{n-2}$, and $a_{1,1}, a_{1,3}, a_{3,1}, a_{3,3}$ scalars,

$$\det A = \frac{\det \begin{bmatrix} a_{1,1} & a_{1,2}^T \\ a_{2,1} & A_{2,2} \end{bmatrix} \det \begin{bmatrix} A_{2,2} & a_{2,3} \\ a_{3,2}^T & a_{3,3} \end{bmatrix} - \det \begin{bmatrix} a_{1,2}^T & a_{1,3} \\ A_{2,2} & a_{2,3} \end{bmatrix} \det \begin{bmatrix} a_{2,1} & A_{2,2} \\ a_{3,1} & a_{3,2}^T \end{bmatrix}}{\det A_{2,2}}.$$

Note, by Sylvester, if A is TP_{n-2}, then A is TP_n iff $\mathfrak{C}_{n-1}(A)$ is TP_2. This allows us to give a very simple proof of the case when $n = 3$. This observation is used repeatedly in the general proof. The 3-by-3 case has also been thoroughly examined in [JKD].

3. Supporting results. From here on we will adopt the convention of labeling unspecified entries of A above and below the main diagonal as $v_{i,j}$ and $u_{j,i}$ respectively. Also, we label all entries of the *k-th* contiguous compound involving *any* unspecified entries as $d_{i,j|k}$ with i, j specifying the upper left entry of the submatrix in A from which the minor is derived, and k specifying the size of the minor.

LEMMA 3.1. *For any partial TP matrix*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & ? \\ a_{2,1} & a_{2,2} & a_{2,3} \\ ? & a_{3,2} & a_{3,3} \end{bmatrix}$$

there exists a real number $\eta > 0$ *such that for all* $d_{1,2|2} = d_{2,1|2} = d$ *with* $d \in (0, \eta]$

$$v_{1,3} = (a_{1,2}a_{2,3} - d)/a_{2,2} \qquad \textit{and} \qquad u_{3,1} = (a_{2,1}a_{3,2} - d)/a_{2,2}$$

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forms a TP completion of A.

Proof. We define $\eta \equiv \min\{a_{1,2}a_{2,3},\ a_{2,1}a_{3,2},\ \mathfrak{c}_{1,1|2},\ \mathfrak{c}_{2,2|2}\}/2$. Note that η is positive. Then

$$v_{1,3} = (a_{1,2}a_{2,3} - d)/a_{2,2} \ > \ (a_{1,2}a_{2,3} - a_{1,2}a_{2,3})/a_{2,2} = 0$$
$$u_{3,1} = (a_{2,1}a_{3,2} - d)/a_{2,2} \ > \ (a_{2,1}a_{3,2} - a_{2,1}a_{3,2})/a_{2,2} = 0.$$

So A is TP_1. Also

$$\mathfrak{C}_2(A) = \begin{bmatrix} \mathfrak{c}_{1,1|2} & d \\ d & \mathfrak{c}_{2,2|2} \end{bmatrix}$$

has all entries positive, and

$$\det \mathfrak{C}_2(A) = \mathfrak{c}_{1,1|2}\ \mathfrak{c}_{2,2|2} - d^2 \ > \ \mathfrak{c}_{1,1|2}\ \mathfrak{c}_{2,2|2} - \mathfrak{c}_{1,1|2}\ \mathfrak{c}_{2,2|2} = 0.$$

So $\mathfrak{C}_2(A)$ is TP_2. Then A is TP_3, and, thus TP. $\square$

LEMMA 3.2. *For any partial TP matrix A or A' with specified entries*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & ? \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}, \qquad A' = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ ? & a_{3,2} & a_{3,3} \end{bmatrix}$$

there exist real numbers $\eta_{1,2},\ \eta_{2,1} > 0$ such that for all $d_{1,2|2} \in (0, \eta_{1,2}]$ and $d_{2,1|2} \in (0, \eta_{2,1}]$

$$v_{1,3} = (a_{1,2}a_{2,3} - d_{1,2|2})/a_{2,2} \quad \text{and} \quad u_{3,1} = (a_{2,1}a_{3,2} - d_{2,1|2})/a_{2,2}$$

form a TP completion of A and A' respectively.

Proof. We define $\eta_{1,2} \equiv \min\{a_{1,2}a_{2,3},\ \mathfrak{c}_{1,1|2}\mathfrak{c}_{2,2|2}/\mathfrak{c}_{2,1|2}\}/2$, which is positive. Then

$$v_{1,3} = (a_{1,2}a_{2,3} - d_{1,2|2})/a_{2,2} \ > \ (a_{1,2}a_{2,3} - a_{1,2}a_{2,3})/a_{2,2} = 0.$$

So A is TP_1. Also

$$\mathfrak{C}_2(A) = \begin{bmatrix} \mathfrak{c}_{1,1|2} & d_{1,2|2} \\ \mathfrak{c}_{2,1|2} & \mathfrak{c}_{2,2|2} \end{bmatrix}$$

has all entries positive, and

$$\det \mathfrak{C}_2(A) = \mathfrak{c}_{1,1|2}\ \mathfrak{c}_{2,2|2} - d_{1,2|2}\ \mathfrak{c}_{2,1|2} \quad > \quad \mathfrak{c}_{1,1|2}\ \mathfrak{c}_{2,2|2} - \mathfrak{c}_{1,1|2}\ \mathfrak{c}_{2,2|2} = 0.$$

So $\mathfrak{C}_2(A)$ is TP_2. Then A is TP_3, and, thus, TP. The case with A', $\eta_{2,1}$, and $u_{3,1}$ is proven via transposition. $\square$

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Note that in Lemmas 3.1 and 3.2 the values $\eta_{1,2}$ and $\eta_{2,1}$ are functions of the specified entries (in Lemma 3.1 $\eta_{1,2} = \eta_{2,1} = \eta$). Subscripted η's bounding corresponding d's will occur throughout. Along with these η's, we will define one other quantity of importance to the following proofs. We define $\omega_{1,1}$ as the value of $d_{1,1|3}$ when $d_{1,2|2} = \eta_{1,2}$ and/or $d_{2,1|2} = \eta_{2,1}$. For example, when only $v_{1,3}$ is unspecified,

$$\omega_{1,1} = (\mathfrak{c}_{1,1|2}\ \mathfrak{c}_{2,2|2} - \eta_{1,2}\ \mathfrak{c}_{2,1|2})/a_{2,2}.$$

If both $v_{1,3}$ and $u_{3,1}$ are unspecified, then

$$\omega_{1,1} = (\mathfrak{c}_{1,1|2}\ \mathfrak{c}_{2,2|2} - \eta^2)/a_{2,2}.$$

Therefore, if we place $d_{1,2|2}$ and $d_{2,1|2}$ in the open intervals $(0, \eta_{1,2})$ and $(0, \eta_{2,1})$ it is *always* true that $0 < \omega_{1,1} < d_{1,1|3}$. In general, if a 3-by-3 submatrix $A(\{i, ..., i+2\}, \{j, ..., j+2\})$ of a partial TP matrix A has specified entries as described in Lemmas 3.1 or 3.2, we define $\eta_{i,j+1}$, $\eta_{i+1,j}$ and $\omega_{i,j}$ correspondingly, i.e. $\eta_{i,j+1}$ and $\eta_{i+1,j}$ represent the maximum allowed values of $d_{i,j+1|2}$ and $d_{i+1,j|2}$, and $\omega_{i,j}$ represents the corresponding minimum value of $d_{i,j|3}$. If $d_{i,j+1|2} \in (0, \eta_{i,j+1})$ and/or $d_{i+1,j|2} \in (0, \eta_{i+1,j})$ then $0 < \omega_{i,j} < d_{i,j|3}$.

Before turning to the general 2-block proof, we mention the case in which one of the blocks is 2-by-2. Here, essentially because of the column/row linearity of the determinant, there is a much simpler proof, in which the TP completion may be viewed as a simple perturbation of the TN completion. Take the partial TP matrix A in the form (1.1), with $A_{1,1}$ 1-by-1, and $\tilde{A}$ in the form (1.2), the TN completion of A. For any TP matrix, an exterior row or column may be added in a simple way so that the result is TP [JS]. Suppose that $b = (b_{1,2},\ b_{1,3}^T)$ is a row and $c = (c_{2,1},\ c_{3,1})$ is a column such that

$$\begin{bmatrix} b_{1,2} & b_{1,3}^T \\ a_{2,2} & a_{2,3}^T \\ a_{3,2} & A_{3,3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_{2,1} & a_{2,2} & a_{2,3}^T \\ c_{3,1} & a_{3,2} & A_{3,3} \end{bmatrix}$$

are TP. By positive diagonal scaling we may assume that $b_{1,2} = a_{1,2}$ and $c_{2,1} = a_{2,1}$. Consider the completion

$$\tilde{\tilde{A}} = \begin{bmatrix} A_{1,1} & a_{1,2} & b_{1,3}^T \\ a_{2,1} & a_{2,2} & a_{2,3}^T \\ c_{3,1} & a_{3,2} & A_{3,3} \end{bmatrix}$$

of A. $\tilde{\tilde{A}}$ need not be TP. For $\varepsilon > 0$, consider the completion

$$\hat{A} = (1-\varepsilon)\tilde{A} + \varepsilon\tilde{\tilde{A}}.$$

of A. Because the principal minors of $\tilde{A}$ are positive, for sufficiently small ε all principal minors of $\hat{A}$ are positive. Also, all non-principal minors are positive in $\tilde{\tilde{A}}$ and

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nonnegative in $\tilde{A}$. Then by the coumn/row linearity of the determinant all non-principal minors of $\hat{A}$ are positive. Thus $\hat{A}$ is a totally positive completion of A. This seems to be the only situation in which such a proof works. Examples show that the same proof does not apply when each block is at least 3-by-3.

Finally, to begin the general proof of the theorem, and make it easier to follow, we describe the proof when A is 5-by-5 and the two cliques are composed of 3 vertices each. This is the first case in which the above proof does not apply. Here we use the following phrase "$\hat{A}$ is TP_k between diagonals $\pm r$" to mean all minors of size less than or equal to k and involving only entries in the i-th row and j-th column with $|i-j| \leq r$ are positive.

Proposition 3.3. *If A is a 5-by-5 partial TP matrix of form (1.1) with $n_1 = n_3 = 2$, then there is a TP completion, $\hat{A}$, of A. Moreover, if the blocks of A are symmetric, the completion $\hat{A}$ may be taken to be symmetric.*

Proof. Consider a matrix A as described and call the proposed completing entries $v_{i,j}$ and $u_{j,i}$. Without loss of generality assume A has 1's on the diagonal and has symmetric tridiagonal part. We define

$$\begin{aligned}
&\eta_1 \equiv \min\{\eta_{2,3}, \eta_{3,2}\}.\\
&\text{For specified } d_{2,3|2},\ d_{3,2|2},\\
&\eta_2 \equiv \min\{\eta_{1,3}, \eta_{3,1}, \eta_{2,4}, \eta_{4,2}\}.\\
&\text{For specified } d_{2,3|2},\ d_{3,2|2},\ d_{1,3|2},\ d_{3,1|2},\ d_{2,4|2}\ d_{4,2|2},\\
&\eta_3 \equiv \min\{\eta_{1,4}, \eta_{4,1}\}.
\end{aligned}$$

$$\begin{aligned}
&\omega_0 \equiv \omega_{2,2}.\\
&\text{For specified } d_{2,3|2},\ d_{3,2|2},\\
&\omega_1 \equiv \min\{\omega_{1,2}, \omega_{2,3}, \omega_{2,1}, \omega_{3,2}\}.\\
&\text{For specified } d_{2,3|2},\ d_{3,2|2},\ d_{1,3|2},\ d_{3,1|2},\ d_{2,4|2}\ d_{4,2|2},\\
&\omega_2 \equiv \min\{\omega_{1,3}, \omega_{3,1}\}.
\end{aligned}$$

Take

$$\begin{array}{ll}
 & d_{2,3|2} = d_{3,2|2} \equiv d_1 \in (0, \eta_1).\\
\text{For specified } d_1, & d_{1,3|2} = d_{3,1|2} = d_{2,4|2} = d_{4,2|2} \equiv d_2 \in (0, \eta_2).\\
\text{For specified } d_1,\ d_2, & d_{1,4|2} = d_{4,1|2} \equiv d_3 \in (0, \eta_3).
\end{array}$$

Then

$$\hat{A} = \begin{bmatrix} 1 & a_{1,2} & a_{1,3} & v_{1,4} & v_{1,5} \\ a_{2,1} & 1 & a_{2,3} & v_{2,4} & v_{2,5} \\ a_{3,1} & a_{3,2} & 1 & a_{3,4} & a_{3,5} \\ u_{4,1} & u_{4,2} & a_{4,3} & 1 & a_{4,5} \\ u_{5,1} & u_{5,2} & a_{5,3} & a_{5,4} & 1 \end{bmatrix}, \quad \mathfrak{C}_2(\hat{A}) = \left[\begin{array}{cc|cc} \mathfrak{c}_{1,1|2} & \mathfrak{c}_{1,2|2} & d_2 & d_3 \\ \mathfrak{c}_{2,1|2} & \mathfrak{c}_{2,2|2} & d_1 & d_2 \\ \hline d_2 & d_1 & \mathfrak{c}_{3,3|2} & \mathfrak{c}_{3,4|2} \\ d_3 & d_2 & \mathfrak{c}_{4,3|2} & \mathfrak{c}_{4,4|2} \end{array}\right].$$

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By Lemmas 3.1 and 3.2 this choice of d_1, d_2, and d_3, will always make $\hat{A}$ TP$_3$. With this structure in mind we seek to show that there exist scalars e_0, dependent on ω_0, and e_1, dependent on ω_1, with $0 < e_0, e_1 < 1$, such that if

$$d_1 \in (0,\ e_0\omega_0/2), \quad d_2 \in (0,\ e_1\omega_1), \quad d_3 \in (0, \eta_3)$$

then $\hat{A}$ is TP.

Take

$$e_0 = \min\{\eta_1,\ \mathfrak{c}_{11|3}^2\omega_0,\ \mathfrak{c}_{33|3}^2\omega_0\} \quad \text{and} \quad e_1 = \min\{\eta_2,\ \omega_1\}.$$

Then

$$d_1 \in (0,\ e_0\omega_0/2) \subset (0, \eta_1), \quad d_2 \in (0,\ e_1\omega_1) \subset (0, \eta_2), \quad d_3 \in (0, \eta_3)$$

We will simply check all minors of size > 3 to prove $\hat{A}$ is TP.

Consider, first, the principal 4-by-4 minors. By Fischer's inequality $d_{1,2|3} < a_{1,2}d_1 < d_1 < 1$ and $d_{2,1|3} < a_{2,1}d_1 < d_1 < 1$, and, for $k > 1$, all $\mathfrak{c}_{i,j|k} < \mathfrak{c}_{i,j|2} < 1$. Then, using Sylvester's inequality,

$$\begin{aligned} d_{1,1|4} &= (\mathfrak{c}_{1,1|3}\ d_{2,2|3} - d_{1,2|3}\ d_{2,1|3})/\mathfrak{c}_{2,2|2} \\ &> \mathfrak{c}_{1,1|3}\ d_{2,2|3} - d_{1,2|3}\ d_{2,1|3} \\ &> \mathfrak{c}_{1,1|3}\ \omega_0 - d_{1,2|3}\ d_{2,1|3} \\ &> \mathfrak{c}_{1,1|3}\ \omega_0 - d_1 d_1 \\ &> \mathfrak{c}_{1,1|3}\ \omega_0 - d_1 \\ &> \mathfrak{c}_{1,1|3}\ \omega_0 - \mathfrak{c}_{1,1|3}^2\ \omega_0 \ > 0. \end{aligned} \tag{3.1}$$

Similarly

$$\begin{aligned} d_{2,2|4} &= (\mathfrak{c}_{3,3|3}\ d_{2,2|3} - d_{2,3|3}\ d_{3,2|3})/\mathfrak{c}_{3,3|2} \\ &> \mathfrak{c}_{3,3|3}\ \omega_0 - d_1 \\ &> \mathfrak{c}_{3,3|3}\ \omega_0 - \mathfrak{c}_{3,3|3}^2\ \omega_0 \ > 0. \end{aligned} \tag{3.2}$$

Thus $\hat{A}$ is TP$_4$ between diagonals ± 3. We now proceed to show that between diagonals ± 4 $\hat{A}$ is TP$_5$, or rather, $\hat{A}$ is TP. We begin with the non-principal 4-by-4 minors, and then verify the determinant, $d_{1,1|5}$. By Fischer, $d_{1,3|3} < d_2 d_2 < d_2$ and $d_{2,2|3} < 1$. Then

$$\begin{aligned} d_{1,2|4} &= (d_{1,2|3}\ d_{2,3|3} - d_{1,3|3}\ d_{2,2|3})/d_{2,3|2} \\ &> d_{1,2|3}\ d_{2,3|3} - d_{1,3|3}\ d_{2,2|3} \\ &> \omega_1\omega_1 - d_{1,3|3}\ d_{2,2|3} \\ &> \omega_1\omega_1 - d_{1,3|3} \\ &> \omega_1\omega_1 - d_2 \\ &\geq e_1\omega_1 - d_2 \ > 0. \end{aligned}$$

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Similarly

$$\begin{aligned} d_{2,1|4} &= (d_{2,1|3}\ d_{3,2|3} - d_{3,1|3}\ d_{2,2|3})/d_{3,2|2} \\ &> \omega_1\omega_1 - d_{3,1|3}\ d_{2,2|3} \\ &> \omega_1\omega_1 - d_{3,1|3} \\ &> \omega_1\omega_1 - d_2 \\ &\geq e_1\omega_1 - d_2\ > 0. \end{aligned}$$

Thus $\hat{A}$ is TP_4. Now consider $\det\hat{A}$. Recall, from the inequalities (3.1) and (3.2), $d_{1,1|4} > \mathfrak{c}_{1,1|3}\ \omega_0 - d_1$ and $d_{2,2|4} > \mathfrak{c}_{3,3|3}\ \omega_0 - d_1$. Therefore

$$\begin{aligned} d_{1,1|5} &= (d_{1,1|4}\ d_{2,2|4} - d_{1,2|4}\ d_{2,1|4})/d_{2,2|3} \\ &> d_{1,1|4}\ d_{2,2|4} - d_{1,2|4}\ d_{2,1|4} \\ &> (\mathfrak{c}_{1,1|3}\ \omega_0 - d_1)(\mathfrak{c}_{3,3|3}\ \omega_0 - d_1) - d_{1,2|4}\ d_{2,1|4} \\ &= (\mathfrak{c}_{1,1|3}\ \mathfrak{c}_{3,3|3}\ \omega_0)\omega_0 - d_1(\mathfrak{c}_{1,1|3}\ \omega_0 + \mathfrak{c}_{3,3|3}\ \omega_0) + d_1^2 - d_{1,2|4}\ d_{2,1|4}. \end{aligned}$$

Also, $d_{1,2|4} < a_{1,2}d_1a_{4,5} < d_1$ and $d_{2,1|4} < a_{2,1}d_1a_{5,4} < d_1$. Then

$$\begin{aligned} d_{1,1|5} &> (\mathfrak{c}_{1,1|3}\ \mathfrak{c}_{3,3|3}\ \omega_0)\omega_0 - d_1(\mathfrak{c}_{1,1|3}\ \omega_0 + \mathfrak{c}_{3,3|3}\ \omega_0) + d_1^2 - d_1^2 \\ &= (\mathfrak{c}_{1,1|3}\ \mathfrak{c}_{3,3|3}\ \omega_0)\omega_0 - d_1(\mathfrak{c}_{1,1|3}\ \omega_0 + \mathfrak{c}_{3,3|3}\ \omega_0) \\ &> e_0\omega_0 - d_1(\mathfrak{c}_{1,1|3}\ \omega_0 + \mathfrak{c}_{3,3|3}\ \omega_0) \\ &> e_0\omega_0 - 2d_1\ > 0. \end{aligned}$$

Therefore $\hat{A}$ is TP. $\square$

It may be helpful at this point to consider an example. Suppose we are given the following partial TP matrix:

$$A = \begin{bmatrix} 1 & 1/2 & 1/6 & & \\ 1/2 & 1 & 1/2 & & \\ 1/6 & 1/2 & 1 & 1/2 & 1/8 \\ & & 1/2 & 1 & 1/3 \\ & & 1/9 & 1/3 & 1 \end{bmatrix}.$$

We may follow the proof of Proposition 3.3 directly to produce a totally positive completion $\hat{A}$. We first consider entries $v_{2,4}$ and $u_{4,2}$. We have $\eta_1 = 1/8$, $\mathfrak{c}_{11|3} = 5/9$, $\mathfrak{c}_{33|3} = 287/432$, and $\omega_0 = 35/64$. Then

$$e_0 = \min\{\eta_1,\ \mathfrak{c}_{11|3}^2\omega_0,\ \mathfrak{c}_{33|3}^2\omega_0\} = 1/8$$

and we must take $d_1 \in (0,\ e_0\omega_0/2) = (0, 35/1024)$. We may certainly choose $d_1 = \frac{e_0\omega_0}{2}\frac{4}{5} = 7/256$. Then $v_{2,4} = u_{4,2} = 57/256$.

Now that we have chosen appropriate values for the unspecified entries in diagonals ± 2 we may proceed to diagonals ± 3. Again, following the proof of Proposition

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3.3 we get $\eta_2 = 7/9216$, $\omega_1 = 7/6144$, and $e_1 = 7/9216$. Therefore we must take $d_2 \in (0,\ e_1\omega_1) = (0, 49/56623104)$. We choose $d_2 = e_1\omega_1\frac{32}{49} = 1/1769472$. Then $v_{1,4} = u_{4,1} = 65663/884736$, $v_{2,5} = 49247/884736$, and $u_{5,2} = 65663/884736$.

Lastly, we wish to complete diagonals ± 4. We find $\eta_3 = 1/171228266496$ and choose $d_3 = 1/171228271599 \in (0, \eta_3)$. Then $v_{1,5} = \frac{759536142020577767}{40936455195404009472}$ and $u_{5,1} = \frac{675141523566415751}{40936455195404009472}$. One may check to find that the resulting completion

$$\hat{A} = \begin{bmatrix} 1 & 1/2 & 1/6 & \frac{65663}{884736} & \frac{759536142020577767}{40936455195404009472} \\ 1/2 & 1 & 1/2 & \frac{57}{256} & \frac{49247}{884736} \\ 1/6 & 1/2 & 1 & 1/2 & 1/8 \\ \frac{65663}{884736} & \frac{57}{256} & 1/2 & 1 & 1/3 \\ \frac{675141523566415751}{40936455195404009472} & \frac{43775}{884736} & 1/9 & 1/3 & 1 \end{bmatrix}$$

is indeed totally positive.

4. Proof of main result.

We now prove Theorem 1.

Proof. We need only prove the statement for a graph consisting of two cliques. We seek to prove the theorem by induction on the number of incomplete diagonals of matrix A. As described, A has the form (1.1).

Call $A_1 = \begin{bmatrix} A_{1,1} & a_{1,2} \\ a_{2,1}^T & a_{2,2} \end{bmatrix} \in \mathbb{M}_{k_1,k_1}(\mathbb{R})$ and call $A_2 = \begin{bmatrix} a_{2,2} & a_{2,3}^T \\ a_{3,2} & A_{3,3} \end{bmatrix} \in \mathbb{M}_{k_2,k_2}(\mathbb{R})$.

In our induction step we wish to complete a total of $2r$ diagonals (r above and r below the main diagonal) from an existing completion of $2(r-1)$ diagonals. Since the $\pm 1st$ diagonals are already specified, this is equivalent to completing the $\pm(r+1)st$ diagonals from the existing $\pm r$ diagonals. We assume A has $1's$ on the diagonal and has symmetric tridiagonal part. We also take $k_1 = k_2 = (r+1)$. If k_1 or $k_2 < (r+1)$ then we can extend matrix A_1 or A_2 to an $(r+1)$-by-$(r+1)$ TP matrix. The completion $\hat{A}$ of A will then be a submatrix of the completion of the extended matrix. We will express $v_{i,j}$, $u_{j,i}$ in the same manner as previously.

$$\begin{aligned} v_{i,j} &= (v_{i,j-1}v_{i+1,j} - d_{i,j-1|2})/v_{i+1,j-1} \\ u_{j,i} &= (u_{j-1,i}u_{j,i+1} - d_{j-1,i|2})/u_{j-1,i+1}. \end{aligned}$$

Let us begin with some definitions. Take η_l as the minimum of all values $\eta_{i,i+l}$, $\eta_{i+l,i}$ (all $\eta_{i,j}$ falling on diagonals $\pm l$), take ω_l as the minimum of all values $\omega_{i,i+l}$, $\omega_{i+l,i}$, and normalize all values $d_{i,i+l|2} = d_{i+l,i|2} \equiv d_l \in (0,\ \eta_l)$. We have

$$\begin{aligned} \eta_l &\equiv \min\{\eta_{i,i+l},\ \eta_{i+l,i} | (r+1) - l \le i \le r\} \\ \omega_l &\equiv \min\{\omega_{i,i+l},\ \omega_{i+l,i} | r - l \le i \le r\} \\ d_{i,i+l|2} &\equiv d_l \ \in (0,\ \eta_l) \quad \text{for } (r+1) - l \le i \le r. \end{aligned}$$

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By Fischer's inequality we know $0 < \eta_l,\ \omega_l,\ d_l < 1$.

We wish to show there exists $\varepsilon_l > 0$, dependent on ω_l, and $\rho_l \in \mathbb{N}$, such that if

$$d_{l+1} \in (0,\ \varepsilon_l \omega_l/\rho_l) \quad \text{for} \quad 1 \le l+1 \le r,$$

then matrix $\hat{A}$, completed with corresponding $v_{i,j},\ u_{j,i}$, is totally positive. For our base case we take a completion of diagonals ± 2 and ± 3 similar to the one outlined in the proof of Proposition 3.3. Take

$$\begin{gathered} e_0 = \min\{\mathfrak{c}_{r-1,r-1|3},\ \mathfrak{c}_{r+1,r+1|3},\ \eta_1\} \\ p_0 = 1 \end{gathered}$$

and

$$\begin{gathered} d_1 \in (0,\ e_0\omega_0/p_0) \\ d_2 \in (0,\ \eta_2). \end{gathered}$$

Then, from equations (3.1) and (3.2) we get

$$\begin{aligned} d_{r-1,r-1|4} \quad &> \quad \mathfrak{c}_{r-1,r-1|3}\, \omega_0 - d_1 \quad \ge \quad e_0\, \omega_0 - p_0\, d_1 > 0 \\ d_{r,r|4} \quad &> \quad \mathfrak{c}_{r+1,r+1|3}\, \omega_0 - d_1 \quad \ge \quad e_0\, \omega_0 - p_0\, d_1 > 0. \end{aligned}$$

Therefore $\hat{A}$, completed with corresponding $v_{i,j},\ u_{j,i}$, is totally positive between diagonals ± 3. As a formality, let $e_1 = 1$ and $p_1 = 0$.

For our induction hypothesis assume there exists e_l, dependent on ω_l, and $p_l \in \mathbb{N}$ for $0 < l \le (r-2)$ such that if

$$\begin{gathered} d_{l+1} \in (0,\ e_l\omega_l/p_l) \subset (0,\ \eta_{l+1}) \quad \text{for } l+1 \le r-2 \\ d_{r-1} \in (0,\ \eta_{r-1}) \end{gathered}$$

then matrix $\hat{A}$ is TP between diagonals $\pm r$. Also, for all contiguous minors of size $b > 3$

$$\left.\begin{matrix} d_{i,i+l|b} \\ d_{i+l,i|b} \end{matrix}\right\} \quad > \quad e_l\, \omega_l - p_l\, d_{l+1} > 0.$$

All contiguous submatrices of size $b \le 3$ are totally positive by our definition of the intervals. As a formality, let $e_{r-2} = 1$ and $p_{r-2} = 0$.

We seek similarly chosen e'_l and p'_l for $l \le (r-1)$ such that if

$$\begin{gathered} d_{l+1} \in (0,\ e'_l\omega_l/p'_l) \subset (0,\ \eta_{l+1}) \quad \text{for } l+1 \le r-1 \\ d_r \in (0,\ \eta_r) \end{gathered}$$

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then, matrix $\hat{A}$ is TP between diagonals $\pm(r+1)$ and for contiguous minors of size $b > 3$

$$\left.\begin{array}{l} d_{i,i+l|b} \\ d_{i+l,i|b} \end{array}\right\} \quad > \quad e'_l\ \omega_l - p'_l\ d_{l+1} > 0.$$

For $l \leq (r-2)$, take

$$\begin{gathered} e'_l = \\ \min\{\eta_{l+1},\ \mathfrak{c}_{1,1+l|(r-l+1)}e_l,\ \mathfrak{c}_{(r+1),(r+1)+l|(r-l+1)}e_l, \\ \mathfrak{c}_{1+l,1|(r-l+1)}e_l,\ \mathfrak{c}_{(r+1)+l,(r+1)|(r-l+1)}e_l,\ e_l^2\omega_l\} \end{gathered}$$

$$p'_l = 2p_l + 1.$$

Take $e'_{r-1} = 1$ and $p'_{r-1} = 0$.

If $d_{l+1} \in (0,\ e'_l\omega_l/p'_l)$ for $l+1 \leq (r-2)$ then $d_{l+1} \in (0,\ e_l\omega_l/p_l)$. Therefore we may apply the induction hypothesis. For contiguous minors of size $b > 3$ not involving the $\pm(r+1)st$ diagonals

$$\left.\begin{array}{l} d_{i,i+l|b} \\ d_{i+l,i|b} \end{array}\right\} \quad > \quad e_l\ \omega_l - p_l\ d_{l+1} > 0.$$

We now wish to prove that for contiguous minors of size $b > 3$

$$\left.\begin{array}{l} d_{i,i+l|b} \\ d_{i+l,i|b} \end{array}\right\} \quad > \quad e'_l\ \omega_l - p'_l\ d_{l+1} > 0.$$

All minors independent of the $\pm(r+1)st$ diagonals are positive by hypothesis, therefore all that is left to check is the positivity of all minors involving the $\pm(r+1)st$ diagonals. Minors of size $b \leq 3$ are positive by choice. We seek to prove the positivity of all remaining contiguous minors at once, but, to be explicit, let b run from 4 to $(r+2)$. This will preserve the structure necessary to use Fischer's inequality. In general, we have two cases to consider.

$$\text{case } (i) \qquad \left.\begin{array}{l} d_{i,i+l|b} \\ d_{i+l,i|b} \end{array}\right\} \quad \text{for } i \in \{2, ..., r-1\}$$

$$\text{case } (ii) \qquad \left.\begin{array}{l} d_{i,i+l|b} \\ d_{i+l,i|b} \end{array}\right\} \quad \text{for } i \in \{1,\ r\}.$$

For case (i)

$$d_{i,i+l|b} \quad = \quad (d_{i,i+l|b-1}\ d_{i+1,i+1+l|b-1} - d_{i,i+(l+1)|b-1}\ d_{i+1,i+l|b-1})/d_{i+1,i+l+1|b-2}.$$

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By induction hypothesis $d_{i,i+l|b-1},\ d_{i+1,i+1+l|b-1} > e_l\ \omega_l - p_l\ d_{l+1}$, and by Fischer's inequality $d_{i,i+(l+1)|b-1} < d_{l+1}$. Then

$$
\begin{array}{rcl}
d_{i,i+l|b} & > & (e_l\ \omega_l - p_l\ d_{l+1})^2 - d_{i,i+(l+1)|b-1}\ d_{i+1,i+l|b-1} \\
& > & e_l^2\ \omega_l^2 - 2e_l\ \omega_l\ p_l\ d_{l+1} + p_l^2 d_{l+1}^2 - d_{i,i+(l+1)|b-1}\ d_{i+1,i+l|b-1} \\
& > & e_l^2\ \omega_l^2 - 2p_l\ d_{l+1} + p_l^2 d_{l+1}^2 - d_{l+1}\ d_{i+1,i+l|b-1} \\
& > & e_l^2\ \omega_l^2 - 2p_l\ d_{l+1} - d_{l+1}\ d_{i+1,i+l|b-1} \\
& > & e_l^2\ \omega_l^2 - 2p_l\ d_{l+1} - d_{l+1} \\
& > & (e_l^2\ \omega_l)\omega_l - d_{l+1}(2p_l + 1) \\
& > & e_l'\ \omega_l - p_l'\ d_{l+1} > 0.
\end{array}
$$

Similarly for $d_{i+l,i|b}$,

$$
\begin{array}{rcl}
d_{i+l,i|b} & > & d_{i+l,i|b-1}\ d_{i+l+1,i+1|b-1} - d_{i+(l+1),i|b-1}\ d_{i+l,i+1|b-1} \\
& > & (e_l\ \omega_l - p_l\ d_{l+1})^2 - d_{l+1} \\
& > & e_l^2\ \omega_l^2 - 2p_l\ d_{l+1} - d_{l+1} \\
& > & e_l'\ \omega_l - p_l'\ d_{l+1} > 0.
\end{array}
$$

In case (ii) $b = r - l + 2$. Again, by Fischer's inequality $d_{1,1+(l+1)|b-1} < d_{l+1}$.

$$
\begin{array}{rcl}
d_{1,1+l|b} & = & (\mathfrak{c}_{1,1+l|b-1}\ d_{2,2+l|b-1} - d_{1,1+(l+1)|b-1}\ \mathfrak{c}_{2,1+l|b-1})/\mathfrak{c}_{2,2+l|b-2} \\
& > & \mathfrak{c}_{1,1+l|b-1}\ d_{2,2+l|b-1} - d_{1,1+(l+1)|b-1}\ \mathfrak{c}_{2,1+l|b-1} \\
& > & \mathfrak{c}_{1,1+l|b-1}(e_l\ \omega_l - p_l\ d_{l+1}) - d_{1,1+(l+1)|b-1}\ \mathfrak{c}_{2,1+l|b-1} \\
& > & \mathfrak{c}_{1,1+l|b-1}(e_l\ \omega_l - p_l\ d_{l+1}) - d_{l+1} \\
& > & \mathfrak{c}_{1,1+l|b-1}\ e_l\ \omega_l - \mathfrak{c}_{1,1+l|b-1}\ p_l\ d_{l+1} - d_{l+1} \\
& > & \mathfrak{c}_{1,1+l|b-1}\ e_l\ \omega_l - 2p_l\ d_{l+1} - d_{l+1} \\
& > & \mathfrak{c}_{1,1+l|b-1}\ e_l\ \omega_l - d_{l+1}\ (2p_l + 1) \\
& > & e_l'\ \omega_l - p_l'\ d_{l+1} > 0.
\end{array}
$$

Similarly

$$
\begin{array}{rcl}
d_{1+l,1|b} & > & \mathfrak{c}_{1+l,1|b-1}(e_l\ \omega_l - p_l\ d_{l+1}) - d_{1+(l+1),1|b-1}\ \mathfrak{c}_{1+l,2|b-1} \\
& > & e_l'\omega_l - 2p_l\ d_{l+1} - d_{l+1} \\
& > & e_l'\omega_l - p_l'\ d_{l+1} > 0
\end{array}
$$

$$
\begin{array}{rcl}
d_{r,r+l|b} & > & \mathfrak{c}_{(r+1),(r+1)+l|b-1}(e_l\ \omega_l - p_l\ d_{l+1}) - d_{r,r+(l+1)|b-1}\ \mathfrak{c}_{(r+1),(r+l)|b-1} \\
& > & e_l'\ \omega_l - 2p_l\ d_{l+1} - d_{l+1} \\
& > & e_l'\ \omega_l - p_l'\ d_{l+1} > 0
\end{array}
$$

$$
\begin{array}{rcl}
d_{r+l,r|b} & > & \mathfrak{c}_{(r+1)+l,(r+1)|b-1}(e_l\ \omega_l - p_l\ d_{l+1}) - d_{r+(l+1),r|b-1}\ \mathfrak{c}_{r+l,r+1|b-1} \\
& > & e_l'\ \omega_l - 2p_l\ d_{l+1} - d_{l+1} \\
& > & e_l'\ \omega_l - p_l'\ d_{l+1} > 0.
\end{array}
$$

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Thus, we arrive at our desired result.

There exists e'_l, dependent on ω_l, and $p'_l \in \mathbb{N}$ for $l \leq (r-1)$ such that if

$$\begin{gathered} d_{l+1} \in (0,\ e'_l\omega_l/p_l) \subset (0,\ \eta_{l+1}) \quad \text{ for } l+1 \leq r-1 \\ d_r \in (0,\ \eta_r). \end{gathered}$$

Matrix $\hat{A}$ is totally positive between diagonals $\pm(r+1)$ and for contiguous minors of size $b > 3$

$$\left.\begin{matrix} d_{i,i+l|b} \\ d_{i+l,i|b} \end{matrix}\right\} \quad > \quad e'_l\ \omega_l - p'_l\ d_{l+1} > 0.$$

$$\left[\begin{array}{cccc|cccc} & & & & d_r & & & \\ & \mathfrak{C}_2(A_1) & & & \vdots & \ddots & & \\ & & & & d_2 & & \ddots & \\ & & & & d_1 & d_2 & \dots & d_r \\ \hline d_r & \dots & d_2 & d_1 & & & & \\ & \ddots & & d_2 & & & & \\ & & \ddots & \vdots & & & \mathfrak{C}_2(A_2) & \\ & & & d_r & & & & \end{array}\right].$$

Then, by induction, there exists $\varepsilon_l > 0$, dependent on ω_l, and $\rho_l \in \mathbb{N}$, such that if

$$d_{l+1} \in (0,\ \varepsilon_l\omega_l/\rho_l) \quad \text{ for } 1 \leq l+1 \leq n-2$$

then matrix $\hat{A}$, completed with corresponding $v_{i,j},\ u_{j,i}$, is totally positive. Since all d_i's were chosen symmetrically we see that if A is symmetric then $\hat{A}$ is symmetric. $\square$

5. Additional remarks. Given the theorem and its proof, it is worth noting that several TP matrix completion results follow from it. First, consider the case of combinatorially symmetric partial TP matrices in which the graph is not connected, but each component is monotonically labeled block clique and the components are ordered so that all the labels in any component are greater than all the labels in all following components. Completability to TP, using the theorem, comes down to completion when there are two non-overlapping cliques.

$$\begin{bmatrix} A_1 & ? \\ ? & A_2 \end{bmatrix}$$

In this case there is a TP completion.

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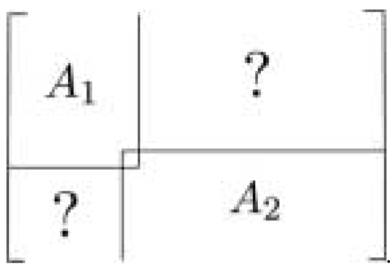
Begin the completion by exterior bordering of A_1 with one column on the right and one row below to produce a TP matrix A_1' and a new matrix of the form (1.1). This new matrix is partial TP and therefore has a TP completion. Thus we may imagine extending the idea of monotonically labeled block clique graphs to those that are not connected. The completion theory remains the same.

The theorem also implies the solvability of a class of non-combinatorially-symmetric, rectangular, TP completion problems. Suppose that the specified entries of a rectangular partial TP matrix comprise exactly two contiguous, possibly overlapping, rectangular submatrices (blocks).



If the two blocks have, at most, one row index and one column index in common, then there is a TP completion. If the overlap is not one in both directions, it may be made so by bordering, as discussed above. Then, if one block is northwest of the other, the matrix may be embedded in one of the form (1.1) via exterior bordering and completed to TP, using the theorem. Of course, the appropriate submatrix of this completion is the TP completion for the original, asymmetric problem.

If, on the other hand, one block is northeast of the other, simply using exterior bordering on each block leads to a TP completion. Start in the upper left corner of the lower right, unspecified portion and work outward (say, all the way down, then back to the top of the next column, etc.). Each time, choose the new entry large enough so that every minor it completes is positive. In the upper left unspecified portion, start in the lower right and work outward via a similar strategy.

Lemma 3.2 (and the above strategy) shows that, in some cases, TP completion may still be possible when the overlaps are greater than one index. If there are more overlapping blocks, as long as they do not overlap by more than one in either direction and are ordered so that they proceed in one direction (southwest or southeast) completion is possible, inductively.

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