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ON THE MINIMUM RANK OF NOT NECESSARILY SYMMETRIC MATRICES: A PRELIMINARY STUDY[*]

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Abstract. The minimum rank of a directed graph Γ is defined to be the smallest possible rank over all real matrices whose ijth entry is nonzero whenever (i,j) is an arc in Γ and is zero otherwise. The symmetric minimum rank of a simple graph G is defined to be the smallest possible rank over all symmetric real matrices whose ijth entry (for $i \neq j$) is nonzero whenever $\{i,j\}$ is an edge in G and is zero otherwise. Maximum nullity is equal to the difference between the order of the graph and minimum rank in either case. Definitions of various graph parameters used to bound symmetric maximum nullity, including path cover number and zero forcing number, are extended to digraphs, and additional parameters related to minimum rank are introduced. It is shown that for directed trees, maximum nullity, path cover number, and zero forcing number are equal, providing a method to compute minimum rank for directed trees. It is shown that the minimum rank problem for any given digraph or zero-nonzero pattern may be converted into a symmetric minimum rank problem.

1. Introduction. The *symmetric minimum rank problem for a simple graph* (the *symmetric minimum rank problem* for short) asks us to determine the minimum rank among all real symmetric matrices whose zero-nonzero pattern of off-diagonal

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entries is described by a given simple graph G (the diagonal of the matrix is free). Minimum rank has also been studied over fields other than the real numbers. This problem arose from the study of possible eigenvalues of real symmetric matrices described by a graph and has received considerable attention over the last ten years (see [7] for a survey with an extensive bibliography).

For a not necessarily symmetric (square) matrix, the zero-nonzero pattern of entries can be described by a directed graph (digraph). Here the absence or presence of loops in the digraph describes the zero-nonzero pattern of the diagonal entries of the matrix. The *asymmetric minimum rank problem for a digraph* (the *asymmetric minimum rank problem* for short) asks us to determine the minimum rank among all real matrices whose zero-nonzero pattern of entries is described by a given digraph Γ.

We adopt the convention that a graph is simple (no loops), is denoted $G = (V_G, E_G)$ where V_G and E_G are the sets of vertices and edges of G, and describes a family of symmetric matrices with free diagonal, whereas a digraph allows loops (but not multiple copies of the same arc), is denoted by $\Gamma = (V_\Gamma, E_\Gamma)$ where V_Γ and E_Γ are the sets of vertices and arcs of Γ, and describes a family of (not necessarily symmetric) matrices with constrained diagonal. Occasionally we will refer to a graph with loops that describes a family of symmetric matrices with constrained diagonal, using the term 'loop graph' or 'loop tree.'

For a symmetric matrix $A \in F^{n \times n}$, the *graph* of A, denoted $\mathcal{G}(A)$, is the (simple) graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0 \text{ and } 1 \leq i < j \leq n\}$. Note that a graph does not have loops and the main diagonal of A plays no role in the determination of $\mathcal{G}(A)$. The *minimum rank* (over field F) of a graph G is

$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) : A \in F^{n \times n}, A^T = A, \mathcal{G}(A) = G\},$$

and the *maximum nullity* of a graph G (over F) is defined to be

$$\mathrm{M}^F(G) = \max\{\mathrm{null}(A) : A \in F^{n \times n}, A^T = A, \mathcal{G}(A) = G\}.$$

Clearly $\mathrm{mr}^F(G) + \mathrm{M}^F(G) = |G|$, where the *order* $|G|$ is the number of vertices of G. In case $F = \mathbb{R}$, the superscript $\mathbb{R}$ may be omitted, so $\mathrm{mr}(G) = \mathrm{mr}^{\mathbb{R}}(G)$, etc. The *positive semidefinite minimum rank* of G is

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}(A) : A \in \mathbb{R}^{n \times n}, A \text{ is positive semidefinite}, \mathcal{G}(A) = G\}.$$

For $B \in F^{n \times n}$, the *digraph* of B, denoted $\Gamma(B)$, is the digraph with vertices $\{1, \ldots, n\}$ and arcs $\{(i, j) : b_{ij} \neq 0\}$. Note that a digraph may have loops and the diagonal entries of B determine the presence or absence of loops in $\Gamma(B)$. The *minimum rank* (over F) of a digraph Γ is

$$\mathrm{mr}^F(\Gamma) = \min\{\mathrm{rank}(B) : B \in F^{n \times n}, \Gamma(B) = \Gamma\},$$

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and the *maximum nullity* of a digraph Γ (over F) is defined to be

$$\mathrm{M}^F(\Gamma) = \max\{\mathrm{null}(B): \ B \in F^{n \times n}, \Gamma(B) = \Gamma\}.$$

Clearly $\mathrm{mr}^F(\Gamma) + \mathrm{M}^F(\Gamma) = |\Gamma|$.

Section 2 contains necessary graph, digraph, and pattern terminology. In Section 3 we show that any asymmetric minimum rank problem can be converted into a (larger) symmetric minimum rank problem. This result gives added weight to the importance of solving the symmetric minimum rank problem. However, until that problem is solved, it is desirable to investigate the asymmetric minimum rank problem, which has natural connections to minimum rank problems for sign patterns. At this time the result of converting an asymmetric minimum rank problem to a symmetric minimum rank problem is usually harder to solve, not only because the order is increased but also because some important properties, such as being a directed tree, are lost in the conversion.

A *tree* is a connected acyclic graph and a *directed tree* or *ditree* is a digraph whose underlying simple graph (see Section 2) is a tree. For a tree T, two readily computable parameters $P(T)$ and $\Delta(T)$ were defined and shown to be equal to $\mathrm{M}(T)$ in [9]. The *path cover number* $P(T)$ is the minimum number of vertex disjoint paths that cover all the vertices of T. In [5] a generalization of Δ was used because the obvious extension of the definition of path cover number, namely the minimum number of vertex disjoint paths that cover all the vertices of T, need not be equal to maximum nullity for a loop tree T. Here we introduce a different definition of path cover number, which coincides with that in [9] for trees, and show in Section 5 that using our Definition 4.19, path cover number, maximum nullity, and another readily computable parameter, the zero forcing number, are equal for any ditree. Based on this result, software currently available can compute the minimum rank of a ditree. Section 4 discusses the parameters used to obtain the results in Section 5.

Since many parameters will be discussed, we provide a list of parameter names, symbols, and definition numbers in Table 1.1.

2. Graph, Digraph, and Pattern Terminology. A *path* is a graph or digraph $P_n = (\{v_1, \ldots, v_n\}, E)$ such that $E = \{\{v_i, v_{i+1}\} : i = 1, \ldots, n-1\}$ or $E = \{(v_i, v_{i+1}) : i = 1, \ldots, n - 1\}$. A *cycle* is a graph or digraph $C_n = (\{v_1, \ldots, v_n\}, E)$ such that $E = \{\{v_i, v_{i+1}\} : i = 1, \ldots, n-1\} \cup \{\{v_n, v_1\}\}$ or $E = \{(v_i, v_{i+1}) : i = 1, \ldots, n - 1\} \cup \{(v_n, v_1)\}$. The *length* of a path or cycle is the number of edges or arcs. Note that $(\{v\}, \{(v, v)\})$ is a digraph cycle of length one and $(\{v, w\}, \{(v, w), (w, v)\})$ is a digraph cycle of length two, whereas the minimum length of a graph cycle is three.

Let Γ be a digraph. To *reverse* arc (v, w) means to replace it by arc (w, v). The digraph obtained from Γ by reversing all the arcs of Γ will be denoted by Γ^T. Since

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TABLE 1.1
Summary of digraph parameter definitions

Parameter symbol	Parameter name	Definition # or Section #
$\mathrm{mr}(\Gamma)$	minimum rank	§1
$\mathrm{tri}(\Gamma)$	triangle number	§4.1
$\mathrm{M}(\Gamma)$	maximum nullity	§1
$\delta(\Gamma)$	minimum degree	§2
$\mathrm{ED}(\Gamma)$	edit distance to nonsingularity	4.7
$\mathrm{Z}_o(\Gamma)$	zero forcing number	4.11
$\mathcal{P}(\Gamma)$	path cover number	4.19

for any $B \in F^{n \times n}$, $\mathrm{rank}(B^T) = \mathrm{rank}(B)$, $\mathrm{mr}^F(\Gamma^T) = \mathrm{mr}^F(\Gamma)$. We say Γ is *symmetric* if $\Gamma = \Gamma^T$ (note this is equality, not isomorphism).

For a digraph Γ, the *underlying simple graph* of Γ is the simple graph G obtained from Γ by deleting loops and then replacing every arc (v, w) or pair of arcs $(v, w), (w, v)$ by the edge $\{v, w\}$. Even if Γ is a symmetric digraph there are two significant differences between the family of matrices described by Γ and its underlying simple graph G: When we write $\Gamma(B) = \Gamma$, the diagonal of B is constrained by the presence or absence of loops but B need not be symmetric (even though Γ is symmetric), whereas when we write $\mathcal{G}(A) = G$, the diagonal of A is free but A must be symmetric.

A vertex w is an *out-neighbor* (*in-neighbor*) of vertex u in Γ if (u, w) $((w, u))$ is an arc of Γ. Note that v is an out-neighbor of itself if and only if the loop (v, v) is an arc of Γ. The notation $u \to w$ means that w is an out-neighbor of u. In a symmetric digraph, an out-neighbor is called a *neighbor*. The *out-degree* $\deg_o(v)$ of a vertex v of Γ is the number of distinct arcs (v, w); note that the arc (v, v) contributes one to the out-degree of v. The *minimum out-degree* over all vertices of a digraph Γ will be denoted by $\delta_o(\Gamma)$. The *minimum degree* of Γ is $\delta(\Gamma) = \max\{\delta_o(\Gamma), \delta_o(\Gamma^T)\}$. For a graph G, the *minimum degree* of G is denoted by $\delta(G)$.

A digraph Γ *allows singularity* (over a field F) if $\mathrm{mr}^F(\Gamma) < |\Gamma|$; otherwise Γ *requires nonsingularity* (over F). A *permutation digraph* of a digraph Γ is a spanning subdigraph that consists of a (vertex) disjoint union of cycles. A digraph Γ requires nonsingularity if and only if Γ has a unique permutation digraph.

Note that a digraph is being used to describe the zero-nonzero pattern of a square matrix. While the digraph has some visual advantages, there are also advantages to

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working with the pattern itself, and a pattern could be rectangular. A *zero-nonzero pattern matrix* (a *pattern* for short) is an $m \times n$ matrix Y whose entries are elements of $\{*, 0\}$. For $B = [b_{ij}] \in F^{m \times n}$, the *pattern of B*, $\mathcal{Y}(B) = [y_{ij}]$, is the $m \times n$ pattern with

$$y_{ij} = \begin{cases} * & \text{if } b_{ij} \neq 0; \\ 0 & \text{if } b_{ij} = 0. \end{cases}$$

An $n \times n$ pattern is called *square*. If Γ is a digraph, $\mathcal{Y}(\Gamma) = \mathcal{Y}(B)$ where $\Gamma(B) = \Gamma$, and if Y is a square pattern, $\Gamma(Y) = \Gamma(B)$ where $\mathcal{Y}(B) = Y$. All terminology from digraphs is applied to square patterns and vice versa. The definitions of *minimum rank* and *maximum nullity* are also extended to a rectangular pattern Y (over a field F):

$$\mathrm{mr}^F(Y) = \min\{\mathrm{rank}(B) : \ B \in F^{m \times n}, \mathcal{Y}(B) = Y\}.$$

$$\mathrm{M}^F(Y) = \max\{\mathrm{null}(B) : \ B \in F^{m \times n}, \mathcal{Y}(B) = Y\}.$$

Note that the minimum rank of a pattern is invariant under an arbitrary permutation of rows and/or columns of the pattern.

If R is a subset of row indices and C is a subset of column indices, the *subpattern* $Y[R|C]$ is the pattern consisting of the entries in rows indexed by R and columns indexed by C. In the case that Y is square, $Y[R|R]$ is called a *principal subpattern* and is denoted by $Y[R]$. The subpattern $Y[\overline{R}|\overline{C}]$ is also denoted by $Y(R|C)$, and $Y[\overline{R}|C]$ is also denoted by $Y(R|C]$, etc. When R or C is $\{1, \ldots, n\}$, it can be denoted by a colon, e.g., $Y[: | \{j\}]$ denotes the jth column of Y. We can abbreviate $Y(R|R)$ to $Y(R)$, $Y(\{s\})$ to $Y(s)$, or $Y(\{s\}|\{t\})$ to $Y(s|t)$.

For a digraph $\Gamma = (V_\Gamma, E_\Gamma)$ and $R \subseteq V_\Gamma$, the *induced subdigraph* $\Gamma[R]$ is the digraph with vertex set R and arc set $\{(v, w) \in E_\Gamma \mid v, w \in R\}$. The induced subdigraph $\Gamma[R]$ is naturally associated with the principal subpattern $\mathcal{Y}(\Gamma)[R]$. The subdigraph induced by $\overline{R}$ is also denoted by $\Gamma - R$, or in the case R is a single vertex v, by $\Gamma - v$.

3. Conversion of Asymmetric Minimum Rank to Symmetric Minimum Rank. There are substantial connections between the asymmetric diagonal constrained minimum rank problem (with matrices described by a digraph or pattern) and the symmetric diagonal free minimum rank problem (with matrices described by a graph). In fact, over the real numbers (or any infinite field), an asymmetric minimum rank problem may be converted to a symmetric one.

THEOREM 3.1. *Suppose Y is an $m \times n$ pattern such that every row and column of Y has a nonzero entry. Define Γ_Y to be the symmetric digraph having pattern*

$\begin{bmatrix} * & Y \\ Y^T & * \end{bmatrix}$ *(where $*$ denotes all-nonzero patterns of appropriate size), and G_Y to be the underlying simple graph of Γ_Y. Then*

$$\mathrm{mr}(Y) = \mathrm{mr}(\Gamma_Y) = \mathrm{mr}(G_Y) = \mathrm{mr}_+(G_Y).$$

Proof. Clearly $\mathrm{mr}(\Gamma_Y) \geq \mathrm{mr}(Y)$ and $\mathrm{mr}_+(G_Y) \geq \mathrm{mr}(G_Y) \geq \mathrm{mr}(Y)$. We construct a positive semidefinite matrix A such that $\Gamma(A) = \Gamma_Y$ and $\mathrm{rank}(A) = \mathrm{mr}(Y)$.

Let $k = \mathrm{mr}(Y)$ and let M be an $m \times n$ matrix such that $\mathcal{Y}(M) = Y$ and $\mathrm{rank}(M) = k$. There exist $k \times m$ and $k \times n$ matrices S, T such that $M = S^T T$. There exists an invertible $k \times k$ matrix P such that $S^T P^{-1} P^{-1T} S$ and $T^T P^T P T$ both have all entries nonzero. Let C be the $k \times (m + n)$ matrix $[P^{-1T} S \quad PT]$. Then $C^T C = \begin{bmatrix} S^T P^{-1} P^{-1T} S & M \\ M^T & T^T P^T P T \end{bmatrix}$, so $\Gamma(C^T C) = \Gamma_Y$. □

REMARK 3.2. *The only place where properties of the real numbers were used (in addition to statements about positive semidefiniteness) was the assertion about the existence of P such that $S^T P^{-1} P^{-1T} S$ and $T^T P^T P T$ both have all entries nonzero. This statement is true for any infinite field, so over an infinite field any asymmetric minimum rank problem can be converted a symmetric minimum rank problem.*

In a special case, Theorem 3.1 can be used to convert a symmetric minimum rank problem to a (smaller) asymmetric minimum rank problem. A bipartite graph G having bipartition $V(G) = U \cup W$ is *undominated* if no vertex of U is adjacent to every vertex of W and no vertex of W is adjacent to every vertex of U; the complement of an undominated bipartite graph is exactly the type of graph G_Y in Theorem 3.1. In [3] it was conjectured that for any graph G and any infinite field F, $\delta(G) \leq \mathrm{M}^F(G)$. Theorem 3.1 can be used to establish this for the complement of an undominated bipartite graph.

Proposition 3.3 and the resulting direct consequences below originally appeared in [3] in a slightly different form. They are included here because they represent important tools for the asymmetric minimum rank problem.

PROPOSITION 3.3. [3, Proposition 3.5] *Let Y be an $m \times n$ pattern such that each row has at least r nonzero entries. Then over any infinite field there exists $B \in F^{m \times n}$ such that $\mathcal{Y}(B) = Y$ and $\mathrm{null}(B) \geq r - 1$ and thus $\mathrm{rank}(B) \leq n - r + 1$.*

Using Proposition 3.3 it is evident that if Y is an $m \times n$ pattern such that each row has at least r nonzero entries and if F is an infinite field, then

$$r - 1 \leq \mathrm{M}^F(Y) \quad \text{and} \quad \mathrm{mr}^F(Y) \leq n - r + 1. \tag{3.1}$$

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Hence taking this further, by examining both Γ and Γ^T, we see that for any digraph Γ and infinite field F, we have

$$\delta(\Gamma) - 1 \leq \mathrm{M}^F(\Gamma) \quad \text{and} \quad \mathrm{mr}^F(\Gamma) \leq |\Gamma| - \delta(\Gamma) + 1. \tag{3.2}$$

PROPOSITION 3.4. *Let G be the complement of an undominated bipartite graph and let F be an infinite field. Then*

$$\delta(G) \leq \mathrm{M}^F(G) \quad \text{and} \quad \mathrm{mr}^F(G) \leq |G| - \delta(G).$$

Proof. Let $Y_G = [y_{uw}]$ be the $|U| \times |W|$ pattern defined by

$$y_{uw} = \begin{cases} * & \text{if } \{u, w\} \in E_G; \\ 0 & \text{if } \{u, w\} \notin E_G. \end{cases}$$

By Theorem 3.1, $\mathrm{mr}^F(Y_G) = \mathrm{mr}^F(G)$. Clearly Y_G has at least $\delta(G) - |U| + 1$ nonzero entries in each row, so by (3.1), $\mathrm{mr}^F(Y_G) \leq |W| - (\delta(G) - |U| + 1) + 1 = |G| - \delta(G)$. □

REMARK 3.5. *As was done in [3, Theorem 3.1], Proposition 3.4 can be improved be noting that only $\delta_U(G)$, the minimum degree over vertices in U, has been used. The result is also valid using a $|W| \times |U|$ pattern and $\delta_W(G)$. Thus $\mathrm{mr}^F(G) \leq |G| - \max\{\delta_U(G), \delta_W(G)\}$.*

4. Parameters for Asymmetric Minimum Rank and Maximum Nullity.
In this section we establish relationships between several parameters related to minimum rank and maximum nullity. Here we focus on parameters that will be used in Section 5 to establish a computational method for determining the minimum rank of a directed tree.

4.1. Triangle Number. A *t-triangle* of an $m \times n$ pattern Y is a $t \times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero. The *triangle number* of pattern Y, denoted $\mathrm{tri}(Y)$, is the maximum size of a triangle in Y. For a digraph Γ, $\mathrm{tri}(\Gamma) = \mathrm{tri}(\mathcal{Y}(\Gamma))$. The triangle number has been used as a lower bound for minimum rank in both the symmetric and asymmetric minimum rank problems, see e.g., [2], [4], [8].

OBSERVATION 4.1. *For any pattern Y and field F, $\mathrm{tri}(Y) \leq \mathrm{mr}^F(Y)$.*

Triangles can sometimes be found through a sequence of eliminations, as described in the next proposition.

PROPOSITION 4.2. *Let Y be a pattern having a row s (or column t) that has exactly one nonzero entry, y_{st}. Then for any field F,*

$$\mathrm{mr}^F(Y) = \mathrm{mr}^F(Y(s|t)) + 1$$

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and

$$\text{tri}(Y) = \text{tri}(Y(s|t)) + 1.$$

Proof. The statement about minimum rank is obvious. To find a triangle, if row s has its only nonzero entry in column t, move both row s and column t to the last position; for the only nonzero entry in the column being y_{st}, move both row s and column t to the first position. □

It is known that triangle number can be strictly less than minimum rank for a pattern (or for a digraph). The classic illustration is obtained from the Fano projective plane, as in the next example.
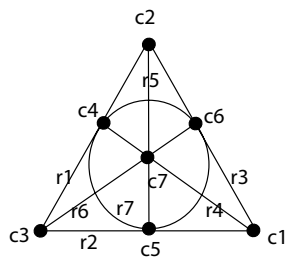

FIG. 4.1. *The Fano projective plane*

EXAMPLE 4.3. Let

$$X_F = \begin{bmatrix} * & 0 & 0 & 0 & * & * & * \\ 0 & * & 0 & * & 0 & * & * \\ 0 & 0 & * & * & * & 0 & * \\ 0 & * & * & 0 & * & * & 0 \\ * & 0 & * & * & 0 & * & 0 \\ * & * & 0 & * & * & 0 & 0 \\ * & * & * & 0 & 0 & 0 & * \end{bmatrix}$$

be the pattern constructed as the complement of the incidence pattern of the Fano projective plane shown in Figure 4.1, where line ri represents row i and point cj represents column j. Then $\text{tri}(X_F) = 3 < 4 = \text{mr}(X_F)$. Note that X_F is a square pattern associated with a symmetric digraph.

It is possible to use this example to construct an acyclic digraph that has $\text{mr}(T) > \text{tri}(T)$.

EXAMPLE 4.4. Let X_F be the pattern in Example 4.3 that has triangle number

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3 and minimum rank 4 and let O be a 7×7 zero matrix. The digraph Γ that has pattern $\mathcal{Y}(\Gamma) = \begin{bmatrix} O & X_F \\ O & O \end{bmatrix}$ is acyclic, and has $\operatorname{tri}(\Gamma) = 3 < 4 = \operatorname{mr}(\Gamma)$.

THEOREM 4.5. *Let Y be a pattern and let F be an infinite field. If $\operatorname{tri}(Y) \leq 2$, then $\operatorname{mr}^F(Y) = \operatorname{tri}(Y)$.*

Proof. If $\operatorname{tri}(Y) = 0$ then Y is an all-0 pattern and $\operatorname{mr}^F(Y) = 0 = \operatorname{tri}(Y)$. For $\operatorname{tri}(Y) > 0$, delete any all-0 rows and all-0 columns from Y (this does not affect either triangle number or minimum rank). If $\operatorname{tri}(Y) = 1$ then Y is an all-$*$ pattern and $\operatorname{mr}^F(Y) = 1 = \operatorname{tri}(Y)$.

Suppose $\operatorname{tri}(Y) = 2$. We show that Y can be permuted to the form

$$
Y' = \begin{bmatrix}
O & * & \dots & * & * \\
* & O & \dots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & \dots & O & * \\
* & * & \dots & * & *
\end{bmatrix}
\tag{4.1}
$$

where each O represents an all-0 block of any size (not necessarily square), each $*$ represents an all-$*$ block whose size is determined by the diagonal blocks, and the last row and column (of all-$*$ blocks) are independently optional. To obtain such a form:

1. Permute the columns to put any all-$*$ columns last.
2. Permute the rows to put all the 0s of column 1 at the top.
3. Permute the columns to put all columns having a 0 in row 1 first.

Observe that for distinct indices p, q, r, s, if $y_{pq} = y_{rq} = y_{rs} = 0$, then $y_{ps} = 0$ (otherwise there is a nonzero entry y_{iq} in column q and a nonzero entry y_{rj} in row r, so $Y[\{i, p, r\}|\{q, s, j\}]$ would be a 3-triangle). Thus, the result of the permutations in steps $(1) - (3)$ above is a matrix of the form $\begin{bmatrix} O & * \\ * & Y_{22} \end{bmatrix}$. Repeat on Y_{22} as needed to obtain a matrix in the form (4.1).

To complete the proof, we exhibit a rank 2 matrix having form (4.1), where Y' is a $k \times \ell$ block pattern with u all-0 blocks (note $k, \ell \in \{u, u + 1\}$). Let $\alpha_3, \dots, \alpha_{u+1}$ be distinct elements of F that are different from 1. Let

$$
M = \begin{bmatrix} 0 & 1 & -\alpha_3 & \dots & -\alpha_u & -1 \\ 1 & 0 & 1 & \dots & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \alpha_3 & \dots & \alpha_u & \alpha_{u+1} \\ 1 & 0 & 1 & \dots & 1 & 1 \end{bmatrix},
$$

where the last row of the transposed first matrix in the product (respectively, the last column of the third matrix in the product) is omitted if $k = u$ (if $\ell = u$). Then $\mathcal{Y}(M) = Y'$ and clearly $\operatorname{rank}(M) = 2$. Let B be the block matrix conformal with Y' having all the entries in block B_{ij} equal to m_{ij}. $\square$

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PROPOSITION 4.6. *Let Y be an $m \times n$ pattern and let F be an infinite field. If $\mathrm{mr}^F(Y) = n$, then $\mathrm{mr}^F(Y) = \mathrm{tri}(Y)$.*

Proof. Note first that we can delete any all-0 rows without affecting minimum rank. Since $\mathrm{mr}^F(Y) = n$, there must be a row that has exactly one $*$ (otherwise, by (3.1), $\mathrm{mr}^F(Y) \leq n - 1$). Apply Proposition 4.2 and induction. □

4.2. Edit Distance. In this subsection we introduce a new parameter for the study of maximum nullity and minimum rank.

DEFINITION 4.7. Let Y be a square pattern. The *(row) edit distance to nonsingularity*, $\mathrm{ED}(Y)$, of Y is the minimum number of rows that must be changed to obtain a pattern that requires nonsingularity. The edit distance to nonsingularity of a digraph Γ is by definition equal to $\mathrm{ED}(\mathcal{Y}(\Gamma))$ and will be denoted by $\mathrm{ED}(\Gamma)$.

Editing row v of $\mathcal{Y}(\Gamma)$ is equivalent to editing the out-neighborhood of v in Γ.

OBSERVATION 4.8. *Let Y' be obtained from Y by deleting one row. Then $\mathrm{tri}(Y') \geq \mathrm{tri}(Y) - 1$.*

THEOREM 4.9. *For any digraph Γ, $\mathrm{tri}(\Gamma) + \mathrm{ED}(\Gamma) = |\Gamma|$.*

Proof. Observe that $\mathrm{ED}(\Gamma) \leq |\Gamma| - \mathrm{tri}(\Gamma)$, because we can edit the $|\Gamma| - \mathrm{tri}(\Gamma)$ rows not in a $\mathrm{tri}(\Gamma)$-triangle of $\mathcal{Y}(\Gamma)$ to get a pattern that requires nonsingularity.

To show $\mathrm{tri}(\Gamma) \geq |\Gamma| - \mathrm{ED}(\Gamma)$, let $Y = \mathcal{Y}(\Gamma)$ and $e = \mathrm{ED}(Y)$. Perform edits on rows $r_1, \ldots, r_e$ to obtain a pattern $\widetilde{Y}$ that requires nonsingularity. Note that $\widetilde{Y}$ is a $|\Gamma| \times |\Gamma|$ pattern that requires nonsingularity and thus is a $|\Gamma|$-triangle by Proposition 4.6. Let Y' be obtained from Y (or equivalently from $\widetilde{Y}$) by deleting rows $r_1, \ldots, r_e$. By applying Observation 4.8 repeatedly, Y' has a $|\Gamma| - e$ triangle, so Y has a $|\Gamma| - e$ triangle. □

COROLLARY 4.10. *For any digraph Γ and any field F, $\mathrm{M}^F(\Gamma) \leq \mathrm{ED}(\Gamma)$.*

4.3. Zero Forcing Sets. Although the underlying concept had been used previously, zero forcing sets and the zero forcing number $\mathrm{Z}(G)$ were introduced in [1]. Here we extend Z and its properties from simple graphs to digraphs.

DEFINITION 4.11.

- *(out) color change rule*: If Γ is a digraph with each vertex colored either white or black, u is a vertex of Γ, and exactly one out-neighbor w of u is white, then change the color of w to black.
- Given a coloring of digraph Γ, the *(out) derived coloring* is the result of applying the color change rule until no more changes are possible. An *(out)*

derived set is the set of black vertices in an (out) derived coloring.

- When in the process of obtaining the derived coloring we apply the color change rule to u to change the color of w, we say u *forces* w.

- An *(out) zero forcing set* for a digraph Γ is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of Γ is all black.

- The *(out) zero forcing number* $Z_o(\Gamma)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V_\Gamma$.

Note that the sequence of forces used in constructing the derived set of a given zero forcing set is not unique, even though the derived set (of a specific coloring) is unique, since any vertex that turns black under one sequence of applications of the color change rule can always be turned black regardless of the order of color changes. This can be proved by an induction on the number of color changes necessary to turn the vertex black, but since for our purposes the uniqueness of the derived set is not necessary, we do not supply the details.

Just as it is possible for the maximum nullity of a digraph to be zero, it is possible for the empty set to be a zero forcing set for a digraph (note that both of these are impossible for a graph).

EXAMPLE 4.12. The digraph P shown in Figure 4.2 has the empty set as a zero forcing set (since vertex 1 has out-degree one, 1 forces vertex 2; likewise vertex 2 forces vertex 1).

FIG. 4.2. *A digraph P having the empty set as a zero forcing set*

The proof given in [1] that for a graph G, $M^F(G) \leq Z(G)$ can be extended in a natural way to show $M^F(\Gamma) \leq Z_o(\Gamma)$, but here we give an alternate proof based on the relationship with the triangle number.

THEOREM 4.13. *For any digraph* Γ, $\mathrm{tri}(\Gamma) + Z_o(\Gamma) = |\Gamma|$ *and* $Z_o(\Gamma) = \mathrm{ED}(\Gamma)$.

Proof. Let Z be a zero forcing set that has $Z_o(\Gamma)$ elements. Let Y be the pattern obtained from $\mathcal{Y}(\Gamma)$ by deleting the columns whose indices are in Z. Vertex v forcing vertex w implies that the v, w-entry of Y is nonzero, and it is the only nonzero entry in row v of Y. By Proposition 4.2, $\mathrm{tri}(Y) = \mathrm{tri}(Y(v|w)) + 1$. Proceeding in in this manner, since Z is a a zero forcing set, we see that $|\Gamma| - Z_o(\Gamma) = \mathrm{tri}(Y) \leq \mathrm{tri}(\mathcal{Y}(\Gamma))$.

Now suppose $\mathcal{Y}(\Gamma)$ has a t-triangle. Then vertices of Γ corresponding to the columns not in the t triangle constitute a zero forcing set. So $Z_o(\Gamma) \leq |\Gamma| - \mathrm{tri}(\Gamma)$. □

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COROLLARY 4.14. *For any digraph Γ and any field F, $\mathrm{M}^F(\Gamma) \leq \mathrm{Z}_o(\Gamma)$.*

REMARK 4.15. *For a pattern Y, columns can be colored black or white, analogous definitions can be given for color change rule, derived coloring, zero forcing set, and zero forcing number, and Theorem 4.13 and Corollary 4.14 remain valid for Y with $|\Gamma|$ replaced by the number of columns of Y.*

DEFINITION 4.16. Let Z be a zero forcing set of a digraph Γ. Construct the derived set, recording the forces.

- A *forcing chain* (for this particular choice of forces) is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, v_i forces v_{i+1}.
- The *forcing chain digraph* of the forcing chain $C = (v_1, v_2, \ldots, v_k)$ is the digraph $\Gamma_C = (V_{\Gamma_C}, E_{\Gamma_C})$ where $V_{\Gamma_C} = \{v_1, v_2, \ldots, v_k\}$ and $E_{\Gamma_C} = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$.
- A *maximal forcing chain* is a forcing chain that is not a proper subsequence of another forcing chain.
- A *maximal forcing chain digraph* is the forcing chain digraph of a maximal forcing chain.

The order of the vertices in a forcing chain need not be the order in which the forces happen, as in the next example.
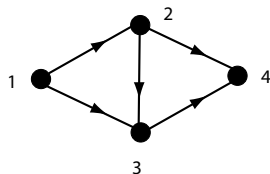

FIG. 4.3. *The digraph for Examples 4.17 and 4.23.*

EXAMPLE 4.17. For the digraph shown in Figure 4.3, $\{1\}$ is a zero forcing set, with the following list of forces: 3 forces 4, 2 forces 3, 1 forces 2. Note that 1 cannot force 2 until after 2 has forced 3, but the maximal forcing chain is $(1, 2, 3, 4)$.

LEMMA 4.18. *Any forcing chain digraph is a path or a cycle. Given a zero forcing set Z and a particular set of forces that produces the derived set (of all vertices), the maximal forcing chain digraphs are disjoint and the elements of the set Z are in one-to-one correspondence with the paths.*

Proof. To see that a forcing chain is a path or a cycle, it suffices to note from the definition of forcing that a vertex can force at most one vertex and can be forced by at most one vertex, i.e., the out-degree and the in-degree are each at most one. Thus a forcing chain does not contain any repeated vertex, except that possibly the

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first and the last vertices are identical, and two maximal forcing chain digraphs have disjoint vertices.

The vertices in Z are exactly the vertices that are never forced by any other vertex, i.e., exactly the initial vertices of the paths. $\square$

4.4. Path Cover Number. The next definition extends the definition of path cover number to digraphs in a more useful way (than the obvious one mentioned in Section 1).

DEFINITION 4.19. Let Γ be a digraph. The *path cover number* $\mathcal{P}(\Gamma)$ of Γ is the minimum number of vertex disjoint paths whose deletion leaves a digraph that requires nonsingularity (or the empty set).

THEOREM 4.20. *For any digraph* Γ, $\mathcal{P}(\Gamma) \leq Z_o(\Gamma)$.

Proof. Let Z be a zero forcing set of order $Z_o(\Gamma)$. Construct the derived set recording the forces and for this set of forces, construct the maximal forcing chain digraphs. Let P be the set of all maximal forcing chain digraphs that are paths. Delete those paths from Γ and the rest of the digraph can force itself, so $\Gamma - V_P$ requires nonsingularity. $\square$

It is possible to construct examples that have the path cover number strictly less than the zero forcing number, as the next example shows.

EXAMPLE 4.21. Consider the complete digraph on $n \geq 3$ vertices $\mathcal{K}_n = (V, V \times V)$ with $|V| = n$.

$$\mathcal{P}(\mathcal{K}_n) = 1 < n - 1 = \mathrm{M}(\mathcal{K}_n) = Z_o(\mathcal{K}_n).$$

To better understand if there is any sort of a relationship between $\mathcal{P}(\Gamma)$ and $\mathrm{M}(\Gamma)$, we pose the following question for investigation. A negative answer would yield Theorem 5.8 below as a corollary to Theorem 5.1.

QUESTION 4.22. *Does there exist a digraph* Γ *for which* $\mathcal{P}(\Gamma) > \mathrm{M}(\Gamma)$?

Note that in [9], the definition of path cover number states that the paths occur as *induced* paths, and this definition has been adopted by many subsequent papers (see [7] and the references therein). However, this distinction is irrelevant for trees or ditrees, so Definition 4.19 is a valid generalization of $P(T)$. Theorem 4.20 would be false if the definition of path cover number required the paths to be induced subdigraphs of Γ, as the next example shows.

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EXAMPLE 4.23. Let Γ be the digraph shown in Figure 4.3. $Z_o(\Gamma) = 1$ because $\{1\}$ is a zero forcing set. Since Γ has no cycles, all vertices must be covered by the paths in a path cover. Since Γ is not a path, at least two paths must be used to cover Γ by induced paths.

5. Directed Trees (Ditrees).

THEOREM 5.1. *For any ditree T, $\mathrm{ED}(T) \leq \mathcal{P}(T)$.*

Proof. Let $P = \{P_1, \ldots, P_k\}$ be a set of vertex-disjoint paths such that $T - V_P$ requires nonsingularity (where $V_P = \cup_{i=1}^k V_{P_i}$). Let v_i be the first vertex and w_i the last vertex of P_i. Edit row w_i (i.e., edit the out-neighborhood of w_i) so that the only out-neighbor of w_i is v_i. This involves at most k row edits and produces a digraph Γ. We show that Γ requires nonsingularity, which implies $\mathrm{ED}(T) \leq k = \mathcal{P}(T)$.

Since $T - V_P$ requires nonsingularity, it has a unique permutation digraph H, and H together with the union of the disjoint cycles $P_i \cup (w_i, v_i), i = 1 \ldots, k$ is a permutation digraph of Γ. We show that this is the only permutation digraph of Γ. A permutation digraph must include w_i in a cycle, and the only arc out of w_i is (w_i, v_i). If the arc (w_i, v_i) were included in a cycle other than $P_i \cup (w_i, v_i)$, there would be a path from v_i to w_i in T that is different from P_i, and so T would not be a ditree. So the only cycle of Γ that includes w_i is $P_i \cup (w_i, v_i)$. Once all the cycles $P_i \cup (w_i, v_i), i = 1 \ldots, k$ are removed, H is the only permutation digraph of $\Gamma - V_P = T - V_P$. Since Γ has a unique permutation digraph, Γ requires nonsingularity. $\square$

Using Theorems 4.20, 5.1, and 4.13 we have the following corollary.

COROLLARY 5.2. *If T is a ditree, then*

$$\mathcal{P}(T) = \mathrm{ED}(T) = Z_o(T).$$

Observe that Theorem 5.1 is false if *ditree* is replaced by *digraph having no cycles of length greater than one.*

EXAMPLE 5.3. Let Γ be the digraph in Figure 5.1, whose only cycle is the loop at 2. Since $\mathcal{Y}(\Gamma) = \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{bmatrix}$, $\mathrm{tri}(\Gamma) = \mathrm{mr}^F(\Gamma) = 1$ and $Z_o(\Gamma) = \mathrm{ED}(\Gamma) = 2$. But $\mathcal{P}(\Gamma) = 1$, because deletion of the path $(1, 2, 3)$ leaves the empty set.

A *loop tree* is graph allowing loops whose associated simple graph (the one obtained by deleting any loops) is a tree, with the presence or absence of the loop at v constraining the v, v-diagonal entry of a symmetric matrix associated with T to be
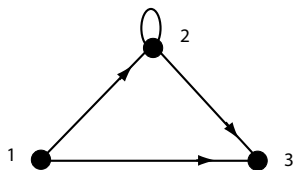
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FIG. 5.1. *A digraph Γ having no cycles of length greater than one such that $\mathcal{P}(\Gamma) < \mathrm{Z}_o(\Gamma)$*

nonzero or zero. In [9], for a simple tree T the parameter $\Delta(T)$ was defined to be the maximum of $p - q$ such that there is a set of q vertices whose deletion leaves p paths, and it was shown that $\mathrm{M}(T) = \Delta(T) = P(T)$. In [5] the definition of Δ was extended to the parameter $\mathcal{C}_0(T)$ defined for a loop tree T, and it was shown that $\mathcal{C}_0(T) = \mathrm{M}(T)$. In [10] Mikkelson extended the applicability of this result by showing that $\mathrm{M}^F(T) = \mathrm{M}(T)$ for every field F of order greater than two.

A loop tree can be viewed as a symmetric ditree, since for computing the minimum rank of trees, symmetry is not an issue (cf. [5]). For convenience and completeness, we reproduce and translate the necessary terminology and the algorithm into the language of ditrees and minimum rank (in [5] it is stated more generally to include sign patterns and nonzero eigenvalues).

Let T be a symmetric ditree. For $Q \subseteq V_T$, define $c_0(Q)$ to be the number of components of $T - Q$ that allow singularity. Then

$$\mathcal{C}_0(T) = \max\{c_0(Q) - |Q| \; : Q \subseteq V_T\}.$$

A *symmetric path* is a symmetric ditree whose underlying graph is a path. A *high degree* vertex of T is a vertex v that has at least three neighbors other than v. Clearly a symmetric ditree is a symmetric path if and only if it does not have any high degree vertices. For $H \subseteq V_T$, an *H-vertex* is a vertex in H. For $R \subseteq V_T$, a component of $T - R$ is *H-free* if it does not contain any H-vertex.

Any digraph Γ can be tested to determine whether it allows singularity by determining the number of permutation digraphs. Alternatively, Γ can be tested by evaluating the determinant of a pattern matrix of variables (Γ allows singularity if and only if there is not exactly one term in the determinant). A symmetric path has maximum nullity 0 or 1, which is distinguished by testing whether it allows singularity.

ALGORITHM 5.4. *Let T be a symmetric ditree that has at least one high degree vertex. This algorithm produces a set $Q \subseteq V_T$ such that $c_0(Q) - |Q| = \mathcal{C}_0(T) = \mathrm{M}(T)$. Initialize: Set $H_1 =$ the set of all high degree vertices of T, $Q = \emptyset$, and $i = 1$. While $H_i \neq \emptyset$:*

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1. *Set T_i = the unique component of $T - Q$ that contains an H_i-vertex.*
2. *Set $Q_i = \emptyset$.*
3. *Set $W_i = \{w \in H_i$: at most one component of $T_i - w$ is not H_i-free\}.*
4. *For each vertex $w \in W_i$,*
   if there are at least two H_i-free components of $T_i - w$ that allow singularity,
   then $Q_i = Q_i \cup \{w\}$.
5. *$Q = Q \cup Q_i$.*
6. *$H_{i+1} = H_i \backslash W_i$.*
7. *For each $v \in H_{i+1}$,*
   if v is not a high degree vertex in $T_i - Q$, remove v from H_{i+1}.
8. *$i = i + 1$.*

LEMMA 5.5. *Let T be a symmetric ditree and $v \in V_T$. Suppose that:*

- *S is a component of $T - v$.*
- *S allows singularity.*
- *If $x \in V_S$, then $T - x$ has at most one component that is a subgraph of S and allows singularity.*

Then there is a path P from v to a vertex $u \in S$ such that every component of $T - V_P$ that is a subgraph of S requires nonsingularity.

Proof. Let w be the neighbor of v in S. Start with path (v, w) and continue adding adjacent vertices one at a time until every component of $T - V_P$ that is a subgraph of S requires nonsingularity. After vertex x is added to the path, if it is not yet the case that every component of $T - V_P$ that is a subgraph of S requires nonsingularity, the next vertex to add to the path is the neighbor of x in the component that allows singularity. □

THEOREM 5.6. *If T is a symmetric ditree and F is a field of order greater than 2, then*

$$\mathrm{M}^F(T) = \mathcal{P}(T) = \mathrm{Z}_o(T) \qquad and \qquad \mathrm{mr}^F(T) = \mathrm{tri}(T).$$

Proof. We show $\mathcal{P}(T) \leq \mathrm{M}(T)$ by induction on $\mathrm{M}(T)$, and the result over $\mathbb{R}$ then follows from Corollaries 4.14 and 5.2 and Theorem 4.13. The extension to other fields follows from [10, Theorem 3.2]. If $\mathrm{M}(T) = 0$, then T requires nonsingularity and $\mathcal{P}(T) = 0$.

Now suppose the result is established for all symmetric ditrees such that $\mathrm{M}(T) < k$ and let $\mathrm{M}(T) = k \geq 1$. Note first that if T is a path, then $\mathrm{M}(T) = 1$ and the deletion of all vertices shows that $\mathcal{P}(T) = 1$. Therefore we may assume T has at least one high degree vertex. Apply Algorithm 5.4, and use the notation from that algorithm

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for T_i, H_i, W_i, Q_i, and Q. If $w \in W_i$ and an H_i-free component S of $T_i - w$ allows singularity, then S satisfies the hypotheses of Lemma 5.5, because any vertex x of S that violated the third hypothesis would have been added to Q (and thus deleted) at an earlier stage of the algorithm.

If $Q = \emptyset$, then we exhibit a path whose deletion leaves a digraph that requires nonsingularity, establishing $\mathcal{P}(T) = 1 \leq \mathrm{M}(T)$. Select any high degree vertex v. If no component of $T - v$ allows singularity, then v itself is the required path. If one component allows singularity, apply Lemma 5.5 to obtain the required path.

So suppose that $Q \neq \emptyset$. Let w be a vertex in Q_m where m is the least index such that $Q_m \neq \emptyset$; note that $T_m = T$. Let $S_i, i = 1, \ldots, \ell$ be the components of $T - w$ that are H_m-free and allow singularity. Note that $\ell \geq 2$. Apply Lemma 5.5 to find paths P_i from w to $u_i \in S_i$ such that the components of $T - V_{P_i}$ in S_i require nonsingularity. Since T is symmetric, we can reverse path $P_{\ell-1}$ and join it to P_ℓ at w to form $P'_{\ell-1}$, and let $P'_i = P_i - w$, for $i = 1, \ldots, \ell - 2$. Let $V_S = \cup_{i=1}^{\ell} V_{S_i}$, $V_P = \cup_{i=1}^{\ell-1} V_{P'_i}$ and let T^o be the component of $T - V_P$ that allows singularity (if there is such; if not $T^o = \emptyset$ and $\mathrm{M}(T^o) = 0$). Note that T^o is playing a role analogous to T_{m+1}, except that the only vertex in Q_m that is deleted is w. Thus $\mathcal{C}_0(T) = \mathcal{C}_0(T^o) + \ell - 1$. By the induction hypothesis, $\mathrm{M}(T^o) = \mathcal{P}(T^o)$, so we can find paths $P''_1, \ldots, P''_{M(T^o)}$ whose deletion from T^o leaves a digraph that requires nonsingularity. Thus the deletion from T of the paths $P''_1, \ldots, P''_{M(T^o)}, P'_1, \ldots, P'_{\ell-1}$ leaves a digraph that requires nonsingularity (note it is possible that $M(T^o) = 0$ and the only paths deleted are $P'_1, \ldots, P'_{\ell-1}$). Thus

$$\mathcal{P}(T) \leq \mathrm{M}(T^o) + \ell - 1 = \mathcal{C}_0(T^o) + \ell - 1 = \mathcal{C}_0(T) = \mathrm{M}(T). \ \square$$

The following lemma will be used to establish Theorem 5.8 below.

LEMMA 5.7. *Let F be any field and let Y be a pattern of the form*

$$Y = \begin{bmatrix} X & O \\ U & W \end{bmatrix},$$

where U is $k \times m$. Let X' be obtained from X by replacing the last column of X by 0s and W' be obtained from W by replacing the first row of W by 0s. If $\mathrm{mr}^F(X) = \mathrm{tri}(X)$, $\mathrm{mr}^F(W) = \mathrm{tri}(W)$, $\mathrm{mr}^F(X') = \mathrm{tri}(X')$, $\mathrm{mr}^F(W') = \mathrm{tri}(W')$, and U has exactly one nonzero entry in the $1, m$ position, then $\mathrm{mr}^F(Y) = \mathrm{tri}(Y)$.

Proof. Observe that

$$\mathrm{mr}^F(X) + \mathrm{mr}^F(W) \leq \mathrm{mr}^F(Y) \leq \mathrm{mr}^F(X) + \mathrm{mr}^F(W) + 1.$$

If $\mathrm{mr}^F(Y) = \mathrm{mr}^F(X) + \mathrm{mr}^F(W)$, then Y has a triangle of order $\mathrm{mr}^F(Y)$, because X has a triangle of order $\mathrm{mr}^F(X)$ and W has a triangle of order $\mathrm{mr}^F(W)$.

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So henceforth we consider the more difficult case:

$$\mathrm{mr}^F(Y) = \mathrm{mr}^F(X) + \mathrm{mr}^F(W) + 1.$$

For any matrix having pattern Y, without loss of generality, we may assume the nonzero entry associated with U is 1, i.e., if M is a matrix such that $\mathcal{Y}(M) = Y$, then M has the form

$$\begin{bmatrix} A & O \\ E_{1m} & B \end{bmatrix}$$

where $\mathcal{Y}(A) = X$ and $\mathcal{Y}(B) = W$. If $\mathrm{rank}(A) = \mathrm{mr}^F(X)$ and $\mathrm{rank}(B) = \mathrm{mr}^F(W)$, then $\mathrm{rank}(M) = \mathrm{mr}^F(X) + \mathrm{mr}^F(W) + 1$. If e_m^T is in the row space $\mathrm{RS}(A)$ or e_1 is in the column space $\mathrm{CS}(B)$, then we have the contradiction that $\mathrm{rank}(M) = \mathrm{rank}(A) + \mathrm{rank}(B) < \mathrm{mr}^F(Y)$. Thus $e_m^T \notin \mathrm{RS}(A)$ and $e_1 \notin \mathrm{CS}(B)$. This implies that the last column of A is in the span of the remaining columns of A, and similarly the first row of B is in the span of the remaining rows of B.

We claim that $\mathrm{mr}^F(X[: \,|\, \{m\})) = \mathrm{mr}^F(X)$. If not, we can construct a matrix A of rank $\mathrm{mr}^F(X)$ by starting with a minimum rank realization of $X[: \,|\, \{m\})$ and appending a (necessarily) independent column whose pattern is that of the last column of X. Such an A would have rank $\mathrm{mr}^F(X)$, and yet its last column would not be in the span of the remaining columns of A. Similarly, $\mathrm{mr}^F(W(\{1\} \,|\, :]) = \mathrm{mr}^F(W)$.

Now replace the last column of X by 0's and the first row of W by 0's to get the patterns X' and W'. By hypothesis, X' has a triangle T_1 of order $\mathrm{tri}(X') = \mathrm{mr}^F(X') = \mathrm{mr}^F(X)$ and W' has a triangle T_2 of order $\mathrm{tri}(W') = \mathrm{mr}^F(W') = \mathrm{mr}^F(W)$. Thus, after rearranging rows and columns, Y has a subpattern of the form

$$\begin{bmatrix} T_1 & ? & O \\ O & 1 & ? \\ O & O & T_2 \end{bmatrix},$$

which is a triangle of order $\mathrm{mr}^F(X) + \mathrm{mr}^F(W) + 1 = \mathrm{mr}^F(Y)$. □

A *forest* is a simple acyclic graph and a *directed forest* or *diforest* is a digraph whose underlying simple graph is a forest.

THEOREM 5.8. *If T is a ditree and F is a field of order greater than 2, then*

$$\mathrm{M}^F(T) = \mathrm{Z}_o(T) = \mathcal{P}(T) = \mathrm{ED}(T) \qquad and \qquad \mathrm{mr}^F(T) = \mathrm{tri}(T).$$

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Proof. We prove $\mathrm{M}^F(T) = \mathrm{Z}_o(T)$ for diforests. Note first that the theorem is true for any symmetric diforest by Theorem 5.6, and for any diforest of order at most 2 by direct examination of cases. Assume it is true for every diforest of order less than $|T|$. If T is symmetric we are done; if not T has two vertices x, w such that (w, x) is an arc and (x, w) is not. Let X be the induced subdigraph containing x in $T - w$ and let W be the induced subdigraph containing w in $T - x$. Let X' be the diforest obtained from X by deleting all in-neighbors of x, and let W' be obtained from W by deleting all out-neighbors of w. Note that $|X|, |W| < |T|$, so by the induction hypothesis $\mathrm{mr}^F(\mathcal{Y}(X)) = \mathrm{tri}(\mathcal{Y}(X)), \ \mathrm{mr}^F(\mathcal{Y}(W)) = \mathrm{tri}(\mathcal{Y}(W))$ and $\mathrm{mr}^F(\mathcal{Y}(X')) = \mathrm{tri}(\mathcal{Y}(X')), \ \mathrm{mr}^F(\mathcal{Y}(W')) = \mathrm{tri}(\mathcal{Y}(W'))$. Apply Lemma 5.7 to $\mathcal{Y}(T)$ to conclude $\mathrm{tri}(T) = \mathrm{mr}^F(T)$. $\square$

By Theorem 5.8, computing $\mathrm{Z}_o(T)$ determines $\mathrm{M}^F(T)$ and thus $\mathrm{mr}^F(T)$ for a ditree T and any field $F \neq \mathbb{Z}_2$. A program for the computation of $\mathrm{Z}_o(\Gamma)$ is available [6] using the free open-source computer mathematics software system *Sage* [11].

The hypotheses about X' and W' are necessary for Lemma 5.7, as the next example shows.

EXAMPLE 5.9. Let

$$
Y = \begin{bmatrix}
* & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & * & 0 & * & 0 & * & * & 0 & 0 \\
0 & 0 & * & * & * & 0 & * & 0 & 0 \\
0 & * & * & 0 & * & * & 0 & 0 & 0 \\
* & 0 & * & * & 0 & * & 0 & 0 & 0 \\
* & * & 0 & * & * & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & * & * & 0 \\
* & * & * & 0 & 0 & 0 & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0
\end{bmatrix}.
$$

Note that Y is of the form $\begin{bmatrix} X & 0 \\ U & W \end{bmatrix}$ with $X = Y(9), W = [0]$, and $U = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$. The pattern $Y(\{8,9\}) = X(8)$ is the pattern X_F in Example 4.3, so $\mathrm{tri}(X(8)) = 3$ and $\mathrm{mr}(X(8)) = 4$. Note that row 8 duplicates row 7, so $\mathrm{tri}(X) = \mathrm{tri}(X(\{8\} \mid :))$ and $\mathrm{mr}(X) = \mathrm{mr}(X(\{8\} \mid :))$. Since $\mathrm{tri}(X(\{7,8\} \mid \{8\})) = 3 = \mathrm{mr}(X(\{7,8\} \mid \{8\}))$, by elimination (Proposition 4.2), $\mathrm{tri}(X(\{8\} \mid :)) = 4 = \mathrm{mr}(X(\{8\} \mid :))$. Thus $\mathrm{tri}(X) = 4 = \mathrm{mr}(X)$. Clearly $\mathrm{tri}(W) = 0 = \mathrm{mr}(W)$. Thus Y satisfies the all hypotheses of Lemma 5.7 except the hypothesis $\mathrm{tri}(X') = \mathrm{mr}(X')$. By elimination and the fact that row 8 duplicates row 7, $\mathrm{tri}(Y) = 1 + \mathrm{tri}(Y(\{8,9\})) = 4$ and $\mathrm{mr}(Y) = 1 + \mathrm{mr}(Y(\{8,9\})) = 5$.

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