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AN ANGLE METRIC THROUGH THE NOTION OF GRASSMANN REPRESENTATIVE*

GRIGORIS I. KALOGEROPOULOS†, ATHANASIOS D. KARAGEORGOS†, AND ATHANASIOS A. PANTELOUS†

Abstract. The present paper has two main goals. Firstly, to introduce different metric topologies on the pencils (F, G) associated with autonomous singular (or regular) linear differential or difference systems. Secondly, to establish a new angle metric which is described by decomposable multi-vectors called Grassmann representatives (or Plücker coordinates) of the corresponding subspaces. A unified framework is provided by connecting the new results to known ones, thus aiding in the deeper understanding of various structural aspects of matrix pencils in system theory.

Key words. Angle metric, Grassmann manifold, Grassmann representative, Plücker coordinates, Exterior algebra.

AMS subject classifications. 15A75, 15A72, 32F45, 14M15.

1. Introduction - Metric Topologies on the Relevant Pencil (F, G). The perturbation analysis of rectangular (or square) matrix pencils is quite challenging due to the fact that arbitrarily small perturbations of a pencil can result in eigenvalues and eigenvectors vanishing. However, there are applications in which properties of a pencil ensure the existence of certain eigenpairs. Motivated by this situation, in this paper we consider the generalized eigenvalue problem (both for rectangular and for square constant coefficient matrices) associated with autonomous singular (or regular) linear differential systems

$$F\underline{x}'(t) = G\underline{x}(t),$$

or, in the discrete analogue, with autonomous singular (or regular) linear difference systems

$$F\underline{x}_{k+1} = G\underline{x}_k,$$

where $\underline{x} \in \mathbb{C}^n$ is the state control vector and $F, G \in \mathbb{C}^{m\times n}$ (or $F, G \in \mathbb{C}^{n\times n}$). In the last few decades the perturbation analysis of such systems has gained importance, not only in the context of matrix theory, but also in numerical linear algebra and control theory; see [6], [8], [9], [11], [13].

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†Department of Mathematics, University of Athens, Panepistimiopolis, GR-15784 Athens, Greece (gkaloger@math.uoa.gr, athkar@math.uoa.gr, apantelous@math.uoa.gr).

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The present paper is divided into two main parts. The first part deals with the topological aspects, and the second part with the "relativistic" aspects of matrix pencils. The topological results have a preliminary nature. By introducing different metric topologies on the space of pencils (F, G), a unified framework is provided aiding in the deeper understanding of perturbation aspects. In addition, our work connects the new angle metric (which in the authors' view is more natural) to some already known results. In particular, the angle metric is connected to the results in [8]-[11] and the notion of deflating subspaces of Stewart [6], [8], and it is shown to be equivalent to the notion of e.d. subspace.

Some useful definitions and notations are required.

DEFINITION 1.1. For a given pair

$$L = (F, G) \in \mathcal{L}_{m,n} \stackrel{\Delta}{=} \left\{ L : L = (F, G)\,; F, G \ \in \mathbb{C}^{m\times n} \right\},$$

we define:
a) the *flat matrix representation* of L, $[L]_f = \left[\begin{array}{cc} F & -G \end{array} \right] \in \mathbb{C}^{m\times 2n}$, and
b) the *sharp matrix representation* of L, $[L]_s = \left[\begin{array}{c} F \\ -G \end{array} \right] \in \mathbb{C}^{2m\times n}$.

DEFINITION 1.2. If $L = (F, G) \in \mathcal{L}_{m,n}$, then L is called *non-degenerate* when
a) $m \leq n$ and $\operatorname{rank} [L]_f = m$, or
b) $m \geq n$ and $\operatorname{rank} [L]_s = n$.

REMARK 1.3. In the case of the flat description, degeneracy implies zero row minimal indices, whereas, in the case of the sharp description, degeneracy implies zero column minimal indices.

REMARK 1.4. If $m = n$, then non-degeneracy implies that L is a regular pair.

By using these definitions, we can express the notion of strict equivalence as follows.

DEFINITION 1.5. Let $L, L' \in \mathcal{L}_{m,n}$.
a) L and L' are called *strict equivalent* ($L \mathcal{E}_{\mathcal{H}} L'$) if and only if there exist non-singular matrices $Q \in \mathbb{F}^{n\times n}$ and $P \in \mathbb{F}^{m\times m}$ such that

$$[L']_f = P\,[L]_f \left[\begin{array}{cc} Q & \mathbb{O} \\ \mathbb{O} & Q \end{array} \right],$$

or equivalently,

$$[L']_s = \left[\begin{array}{cc} P & \mathbb{O} \\ \mathbb{O} & P \end{array} \right] [L]_s\, Q.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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b) L and L' are called *left strict equivalent* ($L\mathcal{E}_{\mathcal{H}}^{L}L'$) if and only if there exists a non-singular matrix $Q \in \mathbb{F}^{n\times n}$ such that

$$[L']_f = P\,[L]_f\,.$$

Similarly, L and L' are called *right strict equivalent* ($L\mathcal{E}_{\mathcal{H}}^{R}L'$) if and only if there exists a non-singular matrix $Q \in \mathbb{F}^{n\times n}$ such that

$$[L']_s = [L]_s\,Q.$$

For the pair $L = (F, G) \in \mathcal{L}_{m,n}$, we can define four vector spaces as follows:

$$\mathcal{X}_f^r(L) \triangleq rowspan_{\mathbb{F}}\,[L]_f\,, \qquad \mathcal{X}_f^c(L) \triangleq colspan_{\mathbb{F}}\,[L]_f\,,$$

$$\mathcal{X}_s^r(L) \triangleq rowspan_{\mathbb{F}}\,[L]_s\,, \qquad \mathcal{X}_s^c(L) \triangleq colspan_{\mathbb{F}}\,[L]_s\,.$$

It is clear that for every $L, L' \in \mathcal{L}_{m,n}$ with $L\mathcal{E}_{\mathcal{H}}^{L}L'$, we have $\mathcal{X}_f^r(L) = \mathcal{X}_f^r(L')$ and $\mathcal{X}_s^r(L) = \mathcal{X}_s^r(L')$. Moreover, we observe that $\mathcal{X}_f^r(L)$ and $\mathcal{X}_s^r(L)$ are invariant under $\mathcal{E}_{\mathcal{H}}^{L}$-equivalence, but $\mathcal{X}_f^c(L)$ and $\mathcal{X}_s^c(L)$ are not always invariant under $\mathcal{E}_{\mathcal{H}}^{L}$-equivalence.

Also, for every $L, L' \in \mathcal{L}_{m,n}$ with $L\mathcal{E}_{\mathcal{H}}^{R}L'$, we see that $\mathcal{X}_f^c(L) = \mathcal{X}_f^c(L')$ and $\mathcal{X}_s^c(L) = \mathcal{X}_s^c(L')$. Moreover, we observe that $\mathcal{X}_f^c(L)$ and $\mathcal{X}_s^c(L)$ are invariant under $\mathcal{E}_{\mathcal{H}}^{R}$-equivalence, but $\mathcal{X}_f^r(L)$ and $\mathcal{X}_s^r(L)$ are not always invariant under $\mathcal{E}_{\mathcal{H}}^{R}$-equivalence.

In the next lines, we provide a classification of the pairs considering their dimensions. Moreover, the invariant properties of the four vector subspaces are also investigated. The following classification is a simple consequence of the definitions and observations, so far. The table below demonstrates the type of subspace and the invariant properties.

DEFINITION 1.6. For a non-degenerate pair $L = (F, G) \in \mathcal{L}_{m,n}$, the *normal-characteristic space*, $\mathcal{X}_{\mathcal{N}}(L)$, is defined as follows:
(i) when $m \leq n$, $\mathcal{X}_{\mathcal{N}}(L) \triangleq \mathcal{X}_f^r(L)$, which is $\mathcal{E}_{\mathcal{H}}^{L}$-invariant,
(ii) when $m > n$, $\mathcal{X}_{\mathcal{N}}(L) \triangleq \mathcal{X}_s^c(L)$, which is $\mathcal{E}_{\mathcal{H}}^{R}$-invariant.

A straightforward consequence of (i) is that for a regular pencil $L = (F, G) \in \mathcal{L}_{n,n}$, we have that $\mathcal{X}_{\mathcal{N}}(L) = \mathcal{X}_f^r(L)$,

$$\mathcal{X}_f^r(L) = rowspan_{\mathbb{F}}\,[L]_f = colspan_{\mathbb{F}}\,[L]_f^t$$

and

$$\mathcal{X}_s^c(L) = colspan_{\mathbb{F}}\,[L]_s = rowspan_{\mathbb{F}}\,[L]_s^t\,.$$

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Furthermore, in the case where the pair is non-degenerate and $m \leq n$, we have $rank\,[L]_f = m$, $\mathcal{X}_{\mathcal{N}}(L) = \mathcal{X}_f^r(L) = rowspan_{\mathbb{F}}\,[L]_f$ and $\dim \mathcal{X}_{\mathcal{N}}(L) = m$. Therefore, we can identify $\mathcal{X}_{\mathcal{N}}(L)$ with a point of the Grassmann manifold $G\left(m, \mathbb{F}^{2n}\right)$. Note that if $L_1 \mathcal{E}_{\mathcal{H}}^L L$, then $\mathcal{X}_{\mathcal{N}}(L_1)$ represents the same point on $G\left(m, \mathbb{F}^{2n}\right)$ as $\mathcal{X}_{\mathcal{N}}(L)$.

Dimensions	$\mathcal{X}_f^r(L)$	$\mathcal{X}_s^r(L)$	$\mathcal{X}_f^c(L)$	$\mathcal{X}_s^c(L)$	Invariant spaces
$m \leq n/2$	$\mathcal{E}_{\mathcal{H}}^L$-invariant	$\mathcal{E}_{\mathcal{H}}^L$-invariant	$\mathcal{X}_f^c(L) \equiv \mathbb{F}^{2m}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{X}_s^c(L) \equiv \mathbb{F}^{2m}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{X}_f^r(L)$, $\mathcal{X}_s^r(L)$.
$n/2 < m \leq 2n$	$\mathcal{E}_{\mathcal{H}}^L$-invariant	$\mathcal{X}_s^r(L) \equiv \mathbb{F}^{2m}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{X}_f^c(L) \equiv \mathbb{F}^{2m}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{E}_{\mathcal{H}}^R$-invariant	$\mathcal{X}_f^r(L)$, $\mathcal{X}_s^c(L)$
$n \leq m/2 < 2m$	$\mathcal{X}_f^r(L) \equiv \mathbb{F}^{2n}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{X}_s^r(L) \equiv \mathbb{F}^{n}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{E}_{\mathcal{H}}^R$-invariant	$\mathcal{E}_{\mathcal{H}}^R$-invariant	$\mathcal{X}_f^c(L)$, $\mathcal{X}_s^c(L)$
$n/2 < m \leq 2n$	$\mathcal{E}_{\mathcal{H}}^L$-invariant	$\mathcal{X}_s^r(L) \equiv \mathbb{F}^{n}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{X}_f^c(L) \equiv \mathbb{F}^{m}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{E}_{\mathcal{H}}^R$-invariant	$\mathcal{X}_f^r(L)$, $\mathcal{X}_s^c(L)$
Regular, $m = n$	$\mathcal{E}_{\mathcal{H}}^L$-invariant	$\mathcal{X}_s^r(L) \equiv \mathbb{F}^{n}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{X}_f^c(L) \equiv \mathbb{F}^{m}$, $\mathcal{E}_{\mathcal{H}}$-invariant	$\mathcal{E}_{\mathcal{H}}^R$-invariant	$\mathcal{X}_f^r(L)$, $\mathcal{X}_s^c(L)$

Now, let us consider the set of regular pairs. For a pair $L = (F, G) \in \mathcal{L}_{n,n}$, the equivalent class $\mathcal{E}_{\mathcal{H}}^L = \left\{ L_1 \in \mathcal{L}_{n,n} : [L_1]_f = P\,[L]_f, P \in \mathbb{F}^{n\times n}, \det P \neq 0 \right\}$ can be considered as a representation of a point on $G\left(n, \mathbb{F}^{2n}\right)$. This point is fully identified by $\mathcal{X}_{\mathcal{N}}(L_1)$, where $L_1 \in \mathcal{E}_{\mathcal{H}}^L(L)$. Note that a point of $G\left(n, \mathbb{F}^{2n}\right)$ does not represent a pair, but only the left-equivalent class of pairs. Furthermore, we shall denote by Σ_1^L the set of all distinct generalized eigenvalues (finite and infinite) of the pair $L = (F, G) \in \mathcal{L}_{n,n}$, and by Σ_2^L the set of normalized generalized eigenvectors (Jordan vectors included) defined for every $\lambda \in \Sigma_1^L$.

We define $\mathcal{Q} = \left\{ \left(\Sigma_1^L, \Sigma_2^L\right) : \forall L \in \mathcal{L}_{n,n} \right\}$ and consider the function

$$f : \mathcal{L}_{n,n} \to \mathcal{Q} : L \to f(L) \triangleq \left(\Sigma_1^L, \Sigma_2^L\right) \in \mathcal{Q}.$$

It is obvious that $f^{-1}\left\{ \left(\Sigma_1^L, \Sigma_2^L\right) : \forall L \in \mathcal{L}_{n,n} \right\} = \mathcal{E}_{\mathcal{H}}^L(L)$, which means that f is $\mathcal{E}_{\mathcal{H}}^L$-invariant. Thus, from the eigenvalue-eigenvector point of view, it is reasonable to identify the whole class $\mathcal{E}_{\mathcal{H}}^L(L)$ as a point of the Grassmann manifold. The study of topological properties of ordered regular pairs is intimately related with the generalized eigenvalue problem. The analysis above demonstrates that such properties may

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be studied on the appropriate Grassmann manifold. In the next section, a new angle metric is introduced from the point of view of metric topology on the Grassmann manifold.

2. A New Angle Metric. In this session, we consider the Grassmann manifolds which are associated with elements of $\mathcal{L}_{m,n}$. In the case where $m \leq n$, the manifold $G\left(m, \mathbb{F}^{2n}\right)$ is constructed, and on the other hand, when $m \geq n$ the manifolds $G\left(n, \mathbb{F}^{2m}\right)$ is the subject of investigation. The relationship between the two Grassmann manifolds and the pairs of $\mathcal{L}_{m,n}$ is clarified by the following remark.

REMARK 2.1. (i) For any $\mathcal{V} \in G\left(m, \mathbb{F}^{2n}\right)$, there exists an $\mathcal{E}_{\mathcal{H}}^{L}$-equivalence class from $\mathcal{L}_{m,n}$ $(m \leq n)$, which is represented by a non-degenerate $L = (F, G) \in \mathcal{L}_{m,n}$, such that $\mathcal{V} = rowspan\,[L]_f$.
(ii) For any $\mathcal{V} \in G\left(n, \mathbb{F}^{2m}\right)$, there exists an $\mathcal{E}_{\mathcal{H}}^{R}$-equivalence class from $\mathcal{L}_{m,n}$ $(m \geq n)$, which is represented by a non-degenerate $L = (F, G) \in \mathcal{L}_{m,n}$, such that $\mathcal{V} = colspan\,[L]_s$.

DEFINITION 2.2 ([8]). A real valued function $d(\cdot,\cdot) : G\left(m, \mathbb{R}^{2n}\right) \times G\left(m, \mathbb{R}^{2n}\right) \rightarrow \mathbb{R}_o^+$ $(m \leq n)$ is called an *orthogonal invariant metric* if for every $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in G\left(m, \mathbb{R}^{2n}\right)$ (where $\mathcal{V}_1 = \mathcal{X}_{\mathcal{N}}(L_1)$, $\mathcal{V}_2 = \mathcal{X}_{\mathcal{N}}(L_2)$, $\mathcal{V}_3 = \mathcal{X}_{\mathcal{N}}(L_3)$, and $L_1, L_2, L_3 \in \mathcal{L}_{m,n}$ are non-degenerate), it satisfies the following properties:
(i) $d\left(\mathcal{X}_{\mathcal{N}}(L_1), \mathcal{X}_{\mathcal{N}}(L_2)\right) \geq 0$, and $d\left(\mathcal{X}_{\mathcal{N}}(L_1), \mathcal{X}_{\mathcal{N}}(L_2)\right) = 0$ if and only if $L_1 \mathcal{E}_{\mathcal{H}}^{L} L_2$,
(ii) $d\left(\mathcal{X}_{\mathcal{N}}(L_1), \mathcal{X}_{\mathcal{N}}(L_2)\right) = d\left(\mathcal{X}_{\mathcal{N}}(L_2), \mathcal{X}_{\mathcal{N}}(L_1)\right)$,
(iii) $d\left(\mathcal{X}_{\mathcal{N}}(L_1), \mathcal{X}_{\mathcal{N}}(L_2)\right) \leq d\left(\mathcal{X}_{\mathcal{N}}(L_1), \mathcal{X}_{\mathcal{N}}(L_3)\right) + d\left(\mathcal{X}_{\mathcal{N}}(L_3), \mathcal{X}_{\mathcal{N}}(L_2)\right)$,
(iv) $d\left(colspan\left[[L_1]_f^t U\right], colspan\left[[L_2]_f^t U\right]\right) = d\left(colspan\,[L_1]_f^t, colspan\,[L_2]_f^t\right)$ for any orthogonal matrix $U \in \mathbb{R}^{m \times m}$.

Similarly, we can define the orthogonal invariant metric on $G\left(n, \mathbb{R}^{2m}\right)$. Furthermore, we summarize some already known results for metrics on $G\left(m, \mathbb{R}^{2n}\right)$, $m \leq n$, and afterwards, we introduce the new angle metric.

THEOREM 2.3 ([12]). *For any two points $\mathcal{V}_{L_1}, \mathcal{V}_{L_2} \in G\left(m, \mathbb{R}^{2n}\right)$ such that*

$$\mathcal{V}_{L_1} = \mathcal{X}_f^r(L_1) = colspan_{\mathbb{R}}\,[L_1]_f^t \quad and \quad \mathcal{V}_{L_2} = \mathcal{X}_f^r(L_2) = colspan_{\mathbb{R}}\,[L_2]_f^t,$$

if we define $X_1 = \left([L_1]_f^t\,[L_1]_f\right)^{-\frac{1}{2}}[L_1]_f^t$ and $X_2 = \left([L_2]_f^t\,[L_2]_f\right)^{-\frac{1}{2}}[L_2]_f^t$, then

$$\mathcal{J}\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right) = \arccos\left[\det\left[X_1 X_2^t X_2 X_1^t\right]\right]^{\frac{1}{2}} \tag{2.1}$$

and

$$d_{\mathcal{J}}\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right) = \sin \mathcal{J}\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right) \tag{2.2}$$

are orthogonal invariant metrics on $G\left(m, \mathbb{R}^{2n}\right)$.

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Note that metrics (2.1) and (2.2) have been used extensively in [8]-[11], on the perturbation analysis of the generalized eigenvalue-eigenvector problem. These metrics have also been related to other metrics on the Grassmann manifold, for instance, the "GAP" between subspaces, $\sin\theta(X,Y)$, etc.; see [8]-[11], [13]-[14].

Now, we denote $\mathcal{L}_{m,n}/\mathcal{E}_{\mathcal{H}}^{L}$ as the quotient $\mathcal{E}_{\mathcal{H}}^{L}$-the collection of equivalence classes. It is easily derived that there is an one-to-one map

$$\varphi : \mathcal{L}_{m,n}/\mathcal{E}_{\mathcal{H}}^{L} \to G\left(m, \mathbb{R}^{2n}\right)$$

such that

$$\mathcal{E}_{\mathcal{H}}^{L}(L) \to \varphi\left(\mathcal{E}_{\mathcal{H}}^{L}(L)\right) \triangleq \mathcal{X}_{N}(L) = \mathcal{X}_{f}^{r}(L_1) \triangleq \mathcal{V}_{L} \in G\left(m, \mathbb{R}^{2n}\right). \tag{2.3}$$

Considering map (2.3), we can always associate an m-dimensional subspace $\mathcal{V}_L$ of $\mathbb{R}^{2n}$ to a one-dimensional subspace of the vector space $\Lambda^m\left(\mathbb{R}^{2n}\right)$ (see [2]-[5]), or equivalently, a point on the Grassmann variety $\Omega(m,2n)$ of the projective space $\mathbb{P}^v(\mathbb{R})$, $v = \begin{pmatrix} 2n \\ m \end{pmatrix} - 1$. This one-dimensional subspace of $\Lambda^m\left(\mathbb{R}^{2n}\right)$ is described by decomposable multi-vectors called *Grassmann representatives* (or *Plücker coordinates*) of the corresponding subspace $\mathcal{V}_L$, which are denoted by $\underline{g}(\mathcal{V}_L)$. It is also known that if $\underline{x}, \underline{x}' \in \Lambda^m\left(\mathbb{R}^{2n}\right)$ are two Grassmann representatives of $\mathcal{V}_L$, then $\underline{x}' = \lambda\underline{x}$ where $\lambda \in \mathbb{R}\backslash\{0\}$.

Definition 2.4. Let $L = (F,G) \in \mathcal{L}_{m,n}$ and $\mathcal{V}_L \in G\left(m, \mathbb{R}^{2n}\right)$ be a linear subspace of $\mathbb{R}^{2n}$ that corresponds to the equivalent class $\mathcal{E}_{\mathcal{H}}^{L}(L)$. If $\underline{x} \in \Lambda^m\left(\mathbb{R}^{2n}\right)$ is a Grassmann representative of $\mathcal{V}_L$ and $\underline{x} = [x_0, x_1, \ldots, x_v]^t$, $v = \begin{pmatrix} 2n \\ m \end{pmatrix} - 1$, then we define the normal Grassmann representative of as the decomposable multi-vector which is given by

$$\widetilde{\underline{x}} = \begin{cases} \frac{sign(x_v)}{\|\underline{x}\|_2}\underline{x}, & \text{if } x_v \neq 0 \\ \frac{sign(x_i)}{\|\underline{x}\|_2}\underline{x}, & \text{if } x_v = 0 \text{ and } x_i \text{ is the first non-zero component of } \underline{x} \end{cases}$$

where $sign(x_i) = \begin{cases} 1, & \text{if } x_i > 0 \\ -1, & \text{if } x_i < 0 \end{cases}$.

Proposition 2.5. *The normal Grassmann representative $\widetilde{\underline{x}}$ of the subspace $\mathcal{V}_L \in G\left(m, \mathbb{R}^{2n}\right)$ is unique and $\|\widetilde{\underline{x}}\|_2 = 1$ ($\|\cdot\|_2$ denotes the Euclidean norm).*

Proof. Let $\widetilde{\underline{x}}' \in \Lambda^m\left(\mathbb{R}^{2n}\right)$ be another Grassmann representative of $\mathcal{V}_L$. It is clear

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that $\underline{x}' = \lambda \underline{x}$ for some $\lambda \in \mathbb{R} \backslash \{0\}$. By Definition 2.4,

$$\begin{aligned}\widetilde{\underline{x}'} &= \begin{cases} \frac{sign(x'_v)}{\|\underline{x}'\|_2}\underline{x}', & \text{if } x'_v = \lambda x_v \neq 0 \\ \frac{sign(x'_i)}{\|\underline{x}'\|_2}\underline{x}', & \text{if } x'_v = \lambda x_v = 0 \text{ and } x'_i = \lambda x_i \end{cases} \\ &= \begin{cases} \frac{\lambda sign(\lambda) sign(x_v)}{|\lambda| \|\underline{x}\|_2}\underline{x}, & \text{if } x_v \neq 0 \\ \frac{\lambda sign(\lambda) sign(x_i)}{|\lambda| \|\underline{x}\|_2}\underline{x}, & \text{if } x_v = 0 \text{ and } x_i \text{ is the first non-zero component of } \underline{x} \end{cases} \\ &= \widetilde{\underline{x}},\end{aligned}$$

keeping in mind that $\frac{\lambda sign(\lambda)}{|\lambda|} = 1$. Thus, $\widetilde{\underline{x}}$ is a unique Grassmann representative of the subspace $\mathcal{V}_L$. The property $\|\widetilde{\underline{x}}\|_2 = 1$ is obvious. □

DEFINITION 2.6 (The new angle metric). For $\mathcal{V}_{L_1}, \mathcal{V}_{L_2} \in G\left(m, \mathbb{R}^{2n}\right)$, we define the angle metric, $\lhd\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right)$, by

$$\lhd\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right) \triangleq \arccos\left|\langle \widetilde{\underline{x}}_1, \widetilde{\underline{x}}_2 \rangle\right|, \tag{2.4}$$

where $\widetilde{\underline{x}}_1$,$\widetilde{\underline{x}}_2$ are the normal Grassmann representatives of $\mathcal{V}_{L_1}, \mathcal{V}_{L_2}$, respectively, and $\langle \cdot, \cdot \rangle$ denotes the inner product.

In what it follows, we denote the set of strictly increasing sequences of l integers $(1 \leq l \leq m)$ chosen from $1, 2, \ldots, m$; for example, $Q_{2,m} = \{(1,2), (1,3), (2,3), \ldots\}$. Clearly, the number of the sequences of $Q_{l,m}$ is equal to $\begin{pmatrix} m \\ l \end{pmatrix}$.

Furthermore, we assume that $A = [a_{ij}] \in \mathcal{M}_{m,n}(\mathbb{R})$, where $\mathcal{M}_{m,n}(\mathbb{R})$ denotes the set of $m \times n$ matrices over $\mathbb{R}$. Let μ, ν be positive integers satisfying $1 \leq \mu \leq m$, $1 \leq \nu \leq n$, $\alpha = (i_1, \ldots, i_\mu) \in Q_{\mu,m}$ and $\beta = (j_1, \ldots j_\nu) \in Q_{\nu,n}$. Then $A[\alpha|\beta] \in \mathcal{M}_{\mu,\nu}(\mathbb{R})$ denotes the submatrix of A formed by the rows $i_1, i_2, \ldots, i_\mu$ and the columns $j_1, j_2, \ldots, j_\nu$.

Finally, for $1 \leq l \leq \min\{m, n\}$ the *l-compound matrix* (or *l-adjugate*) of A is the $\begin{pmatrix} m \\ l \end{pmatrix} \times \begin{pmatrix} n \\ l \end{pmatrix}$ matrix whose entries are $\det\{A[\alpha|\beta]\}$, $\alpha \in Q_{l,m}$, $\beta \in Q_{l,n}$, arranged lexicographically in α and β. This matrix is denoted by $C_l(A)$; see [2]-[5] for more details about this interesting matrix. Moreover, it should be stressed out that a compound matrix is in fact a vector corresponding to the Grassmann representative (or Plücker coordinates) of a space.

PROPOSITION 2.7. *The angle metric (2.4) is an orthogonal invariant metric on the Grassmann manifold* $G\left(m, \mathbb{R}^{2n}\right)$.

Proof. It is straightforward to verify conditions (i) and (ii) of Definition 2.2. Condition (iii) is derived by the corresponding inequality of angles between vectors

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 108-116, February 2009

which is a consequence of the Δ-inequality on the unit sphere. Thus, we need to prove (iv). Let $\mathcal{V}_{L_i} = colspan_{\mathbb{R}}\left[L_i\right]_f^t$, $i = 1, 2$, and U be a $m \times m$ non-singular matrix with $|U| = \lambda$. Then $C_m\left(\left[L_i\right]_f^t U\right) = C_m\left(\left[L_i\right]_f^t\right)|U| = \lambda C_m\left(\left[L_i\right]_f^t\right)$, $i = 1, 2$, and consequently,

$$\sphericalangle\left(\left[L_1\right]_f^t U, \left[L_2\right]_f^t U\right) = \sphericalangle\left(\left[L_1\right]_f^t, \left[L_2\right]_f^t\right).$$

Moreover, if U is an orthogonal matrix, then an orthogonal invariant metric is obtained. □

This angle metric on $G\left(m, \mathbb{R}^{2n}\right)$ is related to the standard metrics (2.1) and (2.2).

THEOREM 2.8. *For every $\mathcal{V}_{L_1}, \mathcal{V}_{L_2} \in G\left(m, \mathbb{R}^{2n}\right)$, we have that*

$$\sphericalangle\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right) = \mathcal{J}\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right)$$

and

$$\sin\left\{\sphericalangle(\mathcal{V}_{L_1}, \mathcal{V}_{L_2})\right\} = d_{\mathcal{J}}\left(\mathcal{V}_1, \mathcal{V}_2\right) = \sin\left\{\mathcal{J}\left(\mathcal{V}_1, \mathcal{V}_2\right)\right\}.$$

Proof. By using the Binet-Cauchy theorem [3], we obtain that

$$\begin{aligned}\det\left[X_1 X_2^t X_2 X_1^t\right] &= C_m\left(X_1\right) C_m\left(X_2^t\right) C_m\left(X_2\right) C_m\left(X_1^t\right) \\ &= C_m\left(X_1\right) C_m^t\left(X_2\right) C_m\left(X_2\right) C_m^t\left(X_1\right).\end{aligned}$$

By Theorem 2.3, we also have

$$C_m\left(X_i\right) = \left[C_m^t\left[L_i\right]_f C_m\left(L_i\right)_f\right]^{-\frac{1}{2}} C_m^t\left(\left[L_i\right]_f\right) = \frac{\underline{g}\left(\mathcal{V}_{L_i}\right)}{\left\|\underline{g}\left(\mathcal{V}_{L_i}\right)\right\|_2} = \varepsilon\widetilde{\underline{x}}_i^t,$$

where $\varepsilon = \pm 1$, $i = 1, 2$.
Thus, since

$$\det\left[X_1 X_2^t X_2 X_1^t\right] = \widetilde{\underline{x}}_1^t \widetilde{\underline{x}}_2 \widetilde{\underline{x}}_2^t \widetilde{\underline{x}}_1 = \left\{\langle\widetilde{\underline{x}}_1, \widetilde{\underline{x}}_2\rangle\right\}^2,$$

it follows that

$$\left(\det\left[X_1 X_2^t X_2 X_1^t\right]\right)^{\frac{1}{2}} = \left|\langle\widetilde{\underline{x}}_1, \underline{x}_2\rangle\right|.$$

Then

$$\arccos\left(\det\left[X_1 X_2^t X_2 X_1^t\right]\right)^{\frac{1}{2}} = \arccos\left|\langle\widetilde{\underline{x}}_1, \widetilde{\underline{x}}_2\rangle\right| \Leftrightarrow \mathcal{J}\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right) = \sphericalangle\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right),$$

and

$$\sin\left\{\sphericalangle(\mathcal{V}_{L_1}, \mathcal{V}_{L_2})\right\} = \sin\left\{\mathcal{J}\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right)\right\} = d_{\mathcal{J}}\left(\mathcal{V}_{L_1}, \mathcal{V}_{L_2}\right). \quad \square$$

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REMARK 2.9. When $m = n$, this angle metric is suitable for the perturbation analysis for the generalized eigenvalue-eigenvector problem.

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