POTENTIALLY STABLE AND 5-BY-5 SPECTRALLY ARBITRARY TREE SIGN PATTERN MATRICES WITH ALL EDGES NEGATIVE[*]

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Abstract. Characterization of potentially stable sign pattern matrices has been a long-standing open problem. In this paper, we give some sufficient conditions for tree sign pattern matrices with all edges negative to allow a properly signed nest. We also characterize potentially stable star and path sign pattern matrices with all edges negative. We give a conjecture on characterizing potentially stable tree sign pattern matrices with all edges negative in terms of allowing a properly signed nest which is verified to be true for sign pattern matrices up to order 6. Finally, we characterize all 5-by-5 spectrally arbitrary tree sign pattern matrices with all edges negative.

1. Introduction. A sign pattern matrix is a matrix with entries from $\{+, -, 0\}$. If $M_{m \times n}(\mathbb{R})$ is the set of all $m \times n$ real matrices, then the qualitative class of an $m \times n$ sign pattern matrix A is denoted by $Q(A)$ and is defined by:

$$Q(A) = \{C \in M_{m \times n}(\mathbb{R}) : \text{sign}(c_{ij}) = a_{ij} \text{ for all } i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}\}.$$

A sign pattern matrix A allows a property P if at least one matrix in $Q(A)$ has the property P and requires a property P if all matrices in $Q(A)$ have the property P. A square sign pattern matrix $A = [a_{ij}]$ is said to be combinatorially symmetric if for each i, j, either both a_{ij}, a_{ji} are zero or both a_{ij}, a_{ji} are nonzero. Let us recall some definitions from [12].

A subpattern $\tilde{A}$ of a sign pattern matrix A is a sign pattern matrix obtained from A by replacing some (possibly none) of the nonzero entries of A with 0. In this case, A is said to be a super-pattern of $\tilde{A}$.

A diagonal sign pattern matrix is a square sign pattern matrix with all off-diagonal entries 0. A signature sign pattern matrix is a diagonal sign pattern matrix with each diagonal entry $+$ or $-$. A signature similarity of a square sign pattern matrix A is a product SAS, where S is a signature sign pattern matrix.

A permutation sign pattern matrix is a square sign pattern matrix with entries from $\{0, +\}$, where the entry $+$ occurs precisely once in each row and each column. A permutation similarity of a square sign pattern matrix A is a product $P^T A P$, where P is a permutation sign pattern matrix.

A sign pattern matrix B is equivalent to another sign pattern matrix A if B is obtained from A by negation and/or permutation similarity and/or signature similarity.

For a simple undirected graph G, we denote the edge between two vertices i, j by $[i, j]$. Let us recall from [13] that the graph of a matrix A of order n, denoted by $G(A)$, is defined to be a simple undirected graph with vertices $1, 2, \ldots, n$ and for $i \neq j$ it has the edge $[i, j]$ if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$. A sign

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pattern matrix A is a tree sign pattern matrix if A is combinatorially symmetric and $G(A)$ is a tree. In particular, if $G(A)$ is a star or a path, then A is called a star sign pattern matrix or a path sign pattern matrix, respectively.

The eigenvalues of matrices in the qualitative class of a tree sign pattern matrix A depend on the signs a_{ii} and $a_{ij}a_{ji}$. Therefore each tree sign pattern matrix can be identified by a signed tree whose vertices may be signed $+, -, 0$ and whose edges may be signed $+, -$. Vertex i has the sign a_{ii}, and for $i \neq j$, if $a_{ij}a_{ji} \neq 0$, then the edge between vertices i and j has the sign $a_{ij}a_{ji}$. For example, a sign pattern matrix with its signed tree is as follows:

$$\begin{bmatrix} 0 & - & 0 & + & 0 \\ - & 0 & + & 0 & - \\ 0 & - & + & 0 & 0 \\ - & 0 & 0 & + & 0 \\ 0 & - & 0 & 0 & 0 \end{bmatrix}$$
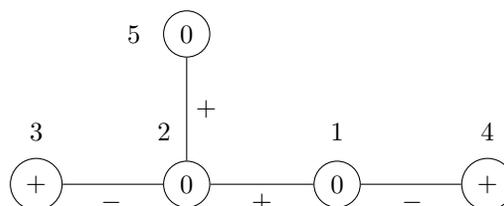
Two signed trees are said to be equivalent if one is obtained from the other after relabeling its vertices. A tree sign pattern matrix is symmetric if each edge of its signed tree is $+$.

Let C be a square matrix of order n. If $\alpha \subseteq \{1, 2, \ldots, n\}$, then $C[\alpha]$ is the principal submatrix of C having rows and columns corresponding to the indices in α. The inertia of C is the triple $i(C) = (i_+(C), i_-(C), i_0(C))$, where $i_+(C)$ is the number of eigenvalues of C with positive real part, $i_-(C)$ is the number of eigenvalues of C with negative real part and $i_0(C)$ is the number of eigenvalues of C with zero real part. A sign pattern matrix A of order n is said to be potentially stable if there exists a $C \in Q(A)$ such that $i(C) = (0, n, 0)$.

For a real number a, the function sgn is defined by:

$$\mathrm{sgn}(a) = \begin{cases} 1, & \text{if } a > 0; \\ 0, & \text{if } a = 0; \\ -1, & \text{if } a < 0. \end{cases}$$

We recall from [14] that a sign pattern matrix A of order n allows a properly signed nest if there exists a $C \in Q(A)$ and a permutation matrix P of order n such that

$$\mathrm{sgn} \det((PCP^T)[\{1, 2, \ldots, k\}]) = (-1)^k \text{ for } k = 1, 2, \ldots, n.$$

The following result was given by Johnson et al. [14] in an attempt to obtain a sufficient condition for a tree sign pattern matrix with negative edges to allow a properly signed nest.

[14, Corollary 3.7]. If A is a tree sign pattern matrix in which at least one diagonal entry is negative and every edge is negative (except possibly those with both end vertices negative), and A allows a nonzero determinant, then A allows a properly signed nest.

However, the following example shows that the above result is not true in general.

EXAMPLE 1.1. Let us consider the tree sign pattern matrix:

$$A = \begin{bmatrix} + & + & 0 \\ - & - & + \\ 0 & - & + \end{bmatrix}.$$

Let $C \in Q(A)$. Then without loss of generality we may assume that

$$C = \begin{bmatrix} a & 1 & 0 \\ -d & -b & 1 \\ 0 & -e & c \end{bmatrix},$$

where $a, b, c, d, e > 0$. If A allows a properly signed nest, then we must have

$$-b < 0, \quad e - bc > 0 \text{ or } d - ab > 0 \quad \text{and} \quad ae + cd - abc < 0,$$

for some $a, b, c, d, e > 0$, which is not possible. So A does not allow a properly signed nest.

REMARK 1.2. By [15, Fig. 2], A in the above example is not potentially stable.

In Section 2, we give some sufficient conditions for tree sign pattern matrices with all edges negative to allow a properly signed nest.

Johnson et al. [14] proved that if a sign pattern matrix allows a properly signed nest, then it is potentially stable. They also mention that the converse is not true, even for tree sign pattern matrices. In Section 3, we discuss the cases for which the converse is also true and characterize potentially stable star and path sign pattern matrices with all edges negative.

Drew et al. [6] introduced the concept of spectrally arbitrary patterns. A sign pattern matrix A of order n is said to be a spectrally arbitrary pattern if, for any given real monic polynomial $r(x)$ of degree n, there is a matrix in $Q(A)$ with characteristic polynomial $r(x)$. In Section 4, we describe all 5-by-5 spectrally arbitrary tree sign pattern matrices with all edges negative.

2. Tree sign pattern matrices with all edges negative that allow properly signed nests. Olesky et al. [17] characterized sign pattern matrices allowing a properly signed nest in terms of another allow problem. Cavalcanti [4] characterized sign pattern matrices allowing a properly signed nest through a relation between two subsequent leading principal minors (after permutation, if required). Using matchings of a graph, we give some sufficient conditions for a sign pattern matrix with all edges negative to allow a properly signed nest. Let us recall the following definitions from [10].

If G is a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$, then the subgraph of G induced by $S \subseteq V(G)$ is the graph with vertex set S and edge set $\{[i, j] \in E(G) : i, j \in S\}$. We denote this induced subgraph by $G(S)$. If $S \subseteq V(G)$, then $G - S$ is the subgraph of G obtained from G by deleting the vertices in S and the edges incident to them. A path between two vertices u, v of G is a sequence $\{v_0, v_1, v_2, \ldots, v_{k-1}, v_k\}$ of distinct vertices such that $v_0 = u, v_k = v$, and $[v_{i-1}, v_i] \in E(G)$ for all $i \in \{1, 2, \ldots, k\}$. A set of edges in an undirected graph G is called a matching if no two edges in that set have a common end vertex. A perfect matching of a graph G is a matching such that each vertex of G is an end vertex of an edge in that matching. The degree of a vertex u, denoted by $\deg(u)$, is the number of distinct vertices of G adjacent to u. If $\deg(u) = 1$, then u is called pendant. Throughout this paper, $\langle n \rangle$ denotes the set $\{1, 2, \ldots, n\}$.

LEMMA 2.1. *Let T be a tree with a vertex v such that $T - \{v\}$ has a perfect matching. Then for each vertex u adjacent to v, there is a path $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}, v_{i_k}\}$ such that $\deg(v_{i_k}) = 1$ and $\deg(v_{i_{k-1}}) = 2$, where $v_{i_1} = v$ and $v_{i_2} = u$.*

Proof. Let u be a vertex adjacent to v. Since $T - \{v\}$ has a perfect matching, u is not pendant. We consider a longest path $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}, v_{i_k}\}$ in T, where $v_{i_1} = v$ and $v_{i_2} = u$. Then, v_{i_k} must be a pendant vertex. If $\deg(v_{i_{k-1}}) > 2$, then $v_{i_{k-1}}$ must be adjacent to another pendant vertex, say w. So we have two pendant vertices w, v_{i_k} distinct from v such that $v_{i_{k-1}}$ is adjacent to both w, v_{i_k}. Then, $T - \{v\}$ cannot have a perfect matching, a contradiction. So $\deg(v_{i_{k-1}}) = 2$.   □

LEMMA 2.2. *Let T be a tree with $2n + 1$ vertices and let v be a vertex of T. If $T - \{v\}$ has a perfect matching, then we can label the vertices with the numbers from $\langle 2n + 1 \rangle$ such that the vertex v is labeled as 1 and for each $s \in \langle 2n + 1 \rangle$, $G(\langle s \rangle)$ has a perfect matching when s is even, and $G(\langle s \rangle) - \{1\}$ has a perfect matching when s is odd.*

Proof. We prove this by induction on n. For $n = 1$, T is a tree with three vertices. Since $T - \{v\}$ has a perfect matching, v must be a pendant vertex. If we label v as 1, the vertex adjacent to v as 2, and the other vertex as 3, we get the desired result.

Suppose the required result is true for any tree with $2n - 1$ vertices. Let T be a tree with $2n + 1$ vertices, and let T has a vertex v such that $T - \{v\}$ has a perfect matching. Then by Lemma 2.1, there is a path $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}, v_{i_k}\}$ such that $\deg(v_{i_k}) = 1$ and $\deg(v_{i_{k-1}}) = 2$, where $v_{i_1} = v$ and $v_{i_{k-1}}, v_{i_k} \neq v$. So $\tilde{T} = T - \{v_{i_{k-1}}, v_{i_k}\}$ is a tree such that $\tilde{T} - \{v\}$ has a perfect matching. Further, $\tilde{T}$ is a tree with $2n - 1$ vertices. So by induction hypothesis, we can label $\tilde{T}$ such that the vertex v is labeled as 1; and for each $s \in \langle 2n - 1 \rangle$, $G(\langle s \rangle)$ has a perfect matching when s is even, and $G(\langle s \rangle) - \{1\}$ has a perfect matching when s is odd. If $v_{i_{k-2}}$ is labeled as m in the above labeling, then label $v_{i_{k-1}}$ as $m + 1$, v_{i_k} as $m + 2$, and relabel all vertices $j(> m)$ of $\tilde{T}$ as $j + 2$.

If m is even, then by induction hypothesis, both $G(\langle m \rangle)$ and $G(\langle m - 1 \rangle) - \{1\}$ have perfect matchings in $\tilde{T}$. So $G(\langle m + 1 \rangle) - \{1\}$ has a perfect matching, namely the perfect matching for $G(\langle m - 1 \rangle) - \{1\}$ in $\tilde{T}$ together with the edge $[m, m + 1]$; and $G(\langle m + 2 \rangle)$ has a perfect matching, namely the perfect matching for $G(\langle m \rangle)$ in $\tilde{T}$ together with the edge $[m + 1, m + 2]$.

If m is odd, then by induction hypothesis, both $G(\langle m \rangle) - \{1\}$ and $G(\langle m - 1 \rangle)$ have perfect matchings in $\tilde{T}$. So $G(\langle m + 1 \rangle)$ has a perfect matching, namely the perfect matching for $G(\langle m - 1 \rangle)$ in $\tilde{T}$ together with the edge $[m, m + 1]$; and $G(\langle m + 2 \rangle) - \{1\}$ has a perfect matching, namely the perfect matching for $G(\langle m \rangle) - \{1\}$ in $\tilde{T}$ together with the edge $[m + 1, m + 2]$.

For each odd $s > m + 2$, $G(\langle s \rangle) - \{1\}$ has a perfect matching, namely the perfect matching for $G(\langle s - 2 \rangle) - \{1\}$ in $\tilde{T}$ together with the edge $[m + 1, m + 2]$. For each even $s > m + 2$, $G(\langle s \rangle)$ has a perfect matching, namely the perfect matching for $G(\langle s - 2 \rangle)$ in $\tilde{T}$ together with the edge $[m + 1, m + 2]$.

For each odd $s \leq m$, $G(\langle s \rangle) - \{1\}$ has a perfect matching, namely the perfect matching for $G(\langle s \rangle) - \{1\}$ in $\tilde{T}$. For each even $s \leq m$, $G(\langle s \rangle)$ has a perfect matching, namely the perfect matching for $G(\langle s \rangle)$ in $\tilde{T}$.

Therefore for each $s \in \langle 2n + 1 \rangle$, $G(\langle s \rangle)$ has a perfect matching when s is even, and $G(\langle s \rangle) - \{1\}$ has a perfect matching when s is odd.   □

Corollary 3.8 of [14] can be rephrased as follows.

LEMMA 2.3 ([14]). *If a sign pattern matrix A allows a properly signed nest, then every super-pattern of A also allows a properly signed nest.*

The following theorem gives a sufficient condition for a tree sign pattern matrix of odd order with negative edges to allow a properly signed nest.

THEOREM 2.4. *Let A be a tree sign pattern matrix of odd order n with all edges negative. If $G(A)$ contains a vertex v such that $G(A) - \{v\}$ has a perfect matching and $a_{vv} = -$, then A allows a properly signed nest.*

Proof. Let B be a subpattern of A such that B is a tree sign pattern matrix, $b_{vv} = -$ and $b_{uu} = 0$ for all $u \neq v$. By Lemma 2.2, we can label $G(B)$ such that v is labeled as 1 so that for each $s \in \langle n \rangle$, the induced subgraph $G(\langle s \rangle)$ has a perfect matching when s is even, and $G(\langle s \rangle) - \{1\}$ has a perfect matching when s is odd. Let $C \in Q(B)$ be such that $c_{ij} = 1, 0, -1$ accordingly as $b_{ij} = +, 0, -$. So there exists a permutation matrix P such that $\det((PCP^T)[\langle s \rangle]) = (-1)^s$ for all $s \in \langle n \rangle$. Thus, B allows a properly signed nest, and therefore by Lemma 2.3, A allows a properly signed nest. □

The converse of the above result is not true.

EXAMPLE 2.5. Let us consider a tree sign pattern matrix A with its signed tree as follows:

$$A = \begin{bmatrix} - & + & 0 & + & 0 \\ - & 0 & + & 0 & + \\ 0 & - & + & 0 & 0 \\ - & 0 & 0 & + & 0 \\ 0 & - & 0 & 0 & 0 \end{bmatrix}$$
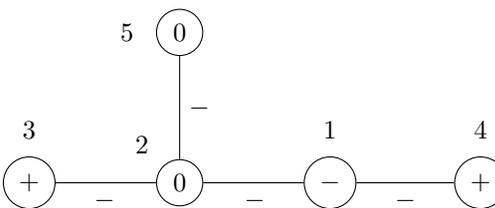
$$C = \begin{bmatrix} -3 & 1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \in Q(A).$$

The leading principal minors of C are -3, 2, -1, 1, and -1. So A allows a properly signed nest. However, $G(A) - \{1\}$ does not have a perfect matching.

LEMMA 2.6. *If two sign pattern matrices A_1 and A_2 allow properly signed nests, then $A_1 \oplus A_2$ allows a properly signed nest.*

Proof. Let the orders of A_1 A_2 be r and s respectively. Since both A_1 and A_2 allow properly signed nests, there exist permutation matrices P_1 and P_2 and $C_1 \in Q(A_1)$ and $C_2 \in Q(A_2)$ such that

$$\operatorname{sgn} \det(P_1 C_1 P_1^T)[\langle t \rangle] = (-1)^t \text{ for } t = 1, 2, \ldots, r,$$

and

$$\operatorname{sgn} \det(P_2 C_2 P_2^T)[\langle t \rangle] = (-1)^t \text{ for } t = 1, 2, \ldots, s.$$

Then, $\operatorname{sgn} \det[(P_1 \oplus P_2)(C_1 \oplus C_2)(P_1 \oplus P_2)^T][\langle t \rangle] = (-1)^t$ for $t = 1, 2, \ldots, r + s$. Since $C_1 \oplus C_2 \in Q(A_1 \oplus A_2)$, $A_1 \oplus A_2$ allows a properly signed nest. □

The following corollary is a generalization of Theorem 2.4.

COROLLARY 2.7. *Let A be a tree sign pattern matrix with all edges negative. If there exist k vertices $v_1, \ldots, v_k$ such that $G(A) - \{v_1, \ldots, v_k\}$ has a perfect matching and $a_{v_i v_i} = -$ for $i = 1, 2, \ldots, k$, then A allows a properly signed nest.*

Proof. Suppose there exist vertices $v_1, \ldots, v_k$ such that $G(A) - \{v_1, \ldots, v_k\}$ has a perfect matching. Let M be such a perfect matching. If two vertices, say v_{k-1}, v_k are adjacent, then $G(A) - \{v_1, \ldots, v_{k-2}\}$ has a perfect matching $M \cup \{[v_{k-1}, v_k]\}$. Without loss of generality, we may assume that no two vertices from $\{v_1, \ldots, v_k\}$ are adjacent.

We prove this by induction on k. For $k = 1$, the result is true by Theorem 2.4. Suppose the result is true for $k = m$ and we consider a tree sign pattern matrix A with $k = m + 1$.

Let $\mathcal{P}$ be the path in $G(A)$ from v_1 to v_2. Then the edge e in $\mathcal{P}$ with an end vertex v_1 does not belong to M. If we delete the edge e, then we will have two tree components. Let $\tilde{A}$ be the sign pattern matrix obtained from A by converting the two entries corresponding to the edge e to zero. So there exists a permutation sign pattern matrix P such that $P\tilde{A}P^T = A_1 \oplus A_2$, where both A_1 and A_2 allow properly signed nests, by induction hypothesis. Therefore, by Lemma 2.6, $A_1 \oplus A_2$ allows a properly signed nest, and thus by Lemma 2.3, A allows a properly signed nest. □

The following theorem gives a sufficient condition for a tree sign pattern matrix of even order with negative edges to allow a properly signed nest.

THEOREM 2.8. *Let A be a tree sign pattern matrix of even order with all edges negative. If A has at least one negative diagonal entry and $G(A)$ has a perfect matching, then A allows a properly signed nest.*

Proof. Let v be a vertex such that $a_{vv} = -$. If B is a subpattern of A such that B is a tree sign pattern matrix, $b_{vv} = -$ and $b_{uu} = 0$ for all $u \neq v$, then $G(B)$ also has a perfect matching. Let M be a perfect matching of $G(B)$. Consider a path $\{v_1, v_2, \ldots, v_{2k}\}$ in $G(B)$ such that $v_1 = v, \deg(v_{2k}) = 1$ and

$$S = \{[v_{2i-1}, v_{2i}] : 1 \leq i \leq k\} \subseteq M.$$

Let $T = G(B) - \{v_{2k}\}$. Then $(M \setminus S) \cup \{[v_{2i}, v_{2i+1}] : 1 \leq i \leq k - 1\}$ is a perfect matching for $T - \{v\}$. Therefore, by Theorem 2.4, $B(\{v_{2k}\})$ allows a properly signed nest and $\operatorname{sign} \det B(\{v_{2k}\}) = -$. Again, $\operatorname{sign} \det(B) = +$. Therefore, B allows a properly signed nest. Hence, by Lemma 2.3, A allows a properly signed nest. □

The converse of the above result is not true.

EXAMPLE 2.9. Let us consider a tree sign pattern matrix A with its signed tree as follows:

$$A = \begin{bmatrix} 0 & - & + & + \\ + & 0 & 0 & 0 \\ - & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}$$

By Corollary 2.7, A allows a properly signed nest. However, $G(A)$ does not have a perfect matching.

3. Potentially stable tree sign pattern matrices with all edges negative. Characterization of potentially stable sign pattern matrices has been a long-standing open problem. Gao and Li [7] gave a necessary and sufficient condition for potentially stable star sign pattern matrices. Few methods to construct higher-order potentially stable sign pattern matrices from lower-order potentially stable sign pattern matrices are given in [2, 8]. Cavers [5] gave few methods to identify some sign pattern matrices which are not potentially stable.

For a tree sign pattern matrix A, the symmetric factorization of A is $A = \tilde{S}\tilde{A}$, where $\tilde{S}$ is a signature sign pattern matrix with the $(1,1)$ entry $+$ and $\tilde{A}$ is a symmetric tree sign pattern matrix. Let $i_+(S)$ denotes the number of $+$ of a signature sign pattern matrix S. The following result by Jeffries and Johnson [13] is very useful in identifying some tree sign pattern matrices which are not potentially stable.

LEMMA 3.1 ([13]). *Let $A = \tilde{S}\tilde{A}$ be the symmetric factorization of an $n \times n$ tree sign pattern matrix A. If A is potentially stable, then there is a symmetric matrix $\tilde{C} \in Q(\tilde{A})$ such that $i_+(\tilde{C}) = n - i_+(\tilde{S})$.*

The following result is given by Johnson et al. [14].

LEMMA 3.2 ([14]). *If a sign pattern matrix A allows a properly signed nest, then A is potentially stable.*

As we discussed earlier, the converse is not true, even for tree sign pattern matrices. Now we discuss the cases for which the converse is also true. Johnson and Summers [15] proved that a super-pattern of a potentially stable sign pattern matrix is also a potentially stable sign pattern matrix.

LEMMA 3.3 ([15]). *Suppose that B is a potentially stable $n \times n$ sign pattern matrix, and suppose that B is a subpattern of the $n \times n$ sign pattern matrix A. Then A is also potentially stable.*

The following result characterizes all potentially stable path sign pattern matrices with all edges negative.

THEOREM 3.4. *If A is a path sign pattern matrix of order n with all edges negative and $G(A)$ is the graph of A with vertices $1, 2, \ldots, n$ such that i is adjacent to $i+1$ for $i = 1, 2, \ldots, n-1$, then the following statements are equivalent.*

1. *A is potentially stable.*
2. *A allows a properly signed nest.*
3. *Exactly one of the following statements is true.*
   (a) *n is even and A has a negative diagonal entry.*
   (b) *n is odd and the i-th diagonal entry of A is negative for some odd i.*

Proof. Theorems 2.4 and 2.8 imply $3 \Rightarrow 2$. By Lemma 3.2, $2 \Rightarrow 1$. Now we show $1 \Rightarrow 3$.

A is potentially stable implies that A has a negative diagonal entry. So $1 \Rightarrow 3$ when n is even. If n is odd, then suppose $a_{ii} \neq -$ for all odd i. Consider a path sign pattern matrix B of order n such that A is a subpattern of B and the $(2i-1)$-th diagonal entry of B is positive for all i with $1 \leq i \leq \frac{n+1}{2}$. Let $B = \tilde{S}\tilde{B}$ be the symmetric factorization of B. Since the number of vertices at even distance from the vertex 1 including itself is $\frac{n+1}{2}$, $i_+(\tilde{S}) = \frac{n+1}{2}$. If B is potentially stable, then by Lemma 3.1, there exists a symmetric matrix $\tilde{C} \in Q(\tilde{B})$ such that $i_+(\tilde{C}) = n - i_+(\tilde{S}) = \frac{n-1}{2}$. But $\tilde{B}$ has a principal submatrix, which is a direct sum of $\frac{n+1}{2}$ $[+]$s. So by Cauchy's interlacing theorem for real symmetric matrices ([11, Theorem 4.3.17]), $i_+(\tilde{X}) \geq \frac{n+1}{2}$ for any symmetric matrix $\tilde{X} \in Q(\tilde{B})$. Thus, B is not potentially stable, and therefore by Lemma 3.3, A is not potentially stable, which is a contradiction. So A is a potentially stable sign pattern matrix of order n implies that the i-th diagonal entry of A is negative for some odd i. Therefore, $1 \Rightarrow 3$. □

The following result characterizes all potentially stable star sign pattern matrices with all edges negative.

THEOREM 3.5. *If A is a star sign pattern matrix of order $n \geq 2$ with all edges negative, then the following statements are equivalent.*

1. *A is potentially stable.*
2. *A allows a properly signed nest.*
3. *A diagonal entry corresponding to at most one pendant vertex of $G(A)$ is nonnegative.*

Proof. For $n = 2$, the potentially stable star sign pattern matrices (up to equivalence) with all edges negative are one of the following sign pattern matrices:

$$
\begin{bmatrix} - & + \\ - & 0 \end{bmatrix}, \quad
\begin{bmatrix} + & + \\ - & - \end{bmatrix}, \quad
\begin{bmatrix} - & + \\ - & - \end{bmatrix}.
$$

Each of these sign pattern matrices satisfies all the above conditions. So the result is true for $n = 2$.

For $n \geq 3$, Corollary 2.7 implies that $3 \Rightarrow 2$. Again by Lemma 3.2, $2 \Rightarrow 1$. Now we show $1 \Rightarrow 3$.

If condition 3 is not true, then at least two diagonal entries corresponding to pendant vertices of $G(A)$ are nonnegative. Let B be a star sign pattern matrix of order n such that A is a subpattern of B and two diagonal entries of B corresponding to pendant vertices of $G(A)$ are $+$. Without loss of generality, we may assume that one of these pendant vertices is labeled as 1. Let $B = \tilde{S}\tilde{B}$ be the symmetric factorization of B. Since the number of vertices at even distance from the vertex 1 including itself is $n-1$, $i_+(\tilde{S}) = n-1$. If B is potentially stable, then by Lemma 3.1, there exists a symmetric matrix $\tilde{C} \in Q(\tilde{B})$ such that $i_+(\tilde{C}) = n - i_+(\tilde{S}) = 1$. But $\tilde{B}$ has a principal submatrix $[+] \oplus [+]$. So by Cauchy's interlacing theorem for real symmetric matrices, $i_+(\tilde{X}) \geq 2$ for any symmetric matrix $\tilde{X} \in Q(\tilde{B})$. Thus, B is not potentially stable. So by Lemma 3.3, A is not potentially stable, which is a contradiction. Therefore, $1 \Rightarrow 3$. □

We get from Lemma 3.2 that if a sign pattern matrix allows a properly signed nest, then it is potentially stable. Since from Theorems 3.4 and 3.5, we get that for path and star sign pattern matrices with all edges negative, potential stability implies the existence of a properly signed nest, we believe that the same is true for all tree sign pattern matrices with all edges negative.

CONJECTURE 3.6. *A tree sign pattern matrix A with all edges negative is potentially stable if and only if it allows a properly signed nest.*

If A has exactly one nonzero diagonal entry, then the conjecture is true by [14, Theorem 4.2]. We do not have a complete answer to the above conjecture as yet; however, the following theorem shows that the above conjecture is true for tree sign pattern matrices up to order 6.

THEOREM 3.7. *Let A be a tree sign pattern matrix of order at most 6 with all edges negative. Then A is potentially stable if and only if A allows a properly signed nest.*

Proof. Let A be a tree sign pattern matrix of order n with all edges negative. By Lemma 3.2, if a sign pattern matrix allows a properly signed nest, then it is potentially stable. So we need to show that if A is potentially stable, then A allows a properly signed nest.

Case I: $n \leq 4$. In this case, $G(A)$ is either a path or a star. So the result is true by Theorems 3.4 and 3.5, respectively.

Case II: $n = 5$. If $G(A)$ is a path or a star, then the result is true by Theorems 3.4 and 3.5, respectively. Suppose $G(A)$ is neither a star nor a path. Then A with its signed tree is as follows:

$$A = \begin{bmatrix} a_1 & + & + & + & 0 \\ - & a_2 & 0 & 0 & 0 \\ - & 0 & a_3 & 0 & 0 \\ - & 0 & 0 & a_4 & + \\ 0 & 0 & 0 & - & a_5 \end{bmatrix}$$
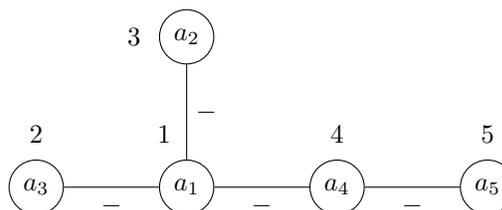
The symmetric factorization of A is

$$A = \begin{bmatrix} a_1 & + & + & + & 0 \\ - & a_2 & 0 & 0 & 0 \\ - & 0 & a_3 & 0 & 0 \\ - & 0 & 0 & a_4 & + \\ 0 & 0 & 0 & - & a_5 \end{bmatrix} = \begin{bmatrix} + & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} a_1 & + & + & + & 0 \\ + & -a_2 & 0 & 0 & 0 \\ + & 0 & -a_3 & 0 & 0 \\ + & 0 & 0 & -a_4 & - \\ 0 & 0 & 0 & - & a_5 \end{bmatrix} = \tilde{S}\tilde{A}.$$

If A is potentially stable, then by Lemma 3.1, there exists a symmetric matrix $\tilde{C} \in Q(\tilde{A})$ such that $i_+(\tilde{C}) = 5 - i_+(\tilde{S}) = 3$ and thus $i_-(\tilde{C}) \leq 2$.

Case II(1). If $a_2, a_3, a_4 = +$, then the eigenvalues of $\tilde{X}[\{2,3,4\}]$ are negative for all $\tilde{X} \in Q(\tilde{A})$. So by Cauchy's interlacing theorem, $i_-(\tilde{X}) \geq 3$ for all symmetric matrix $\tilde{X} \in Q(\tilde{A})$, a contradiction. Therefore, A is not potentially stable. Hence, by Lemma 3.3, A is not potentially stable if, $a_2, a_3, a_4 \in \{0, +\}$.

Case II(2). If $a_2, a_3 = +, a_5 = -$, then the eigenvalues of $\tilde{X}[\{2,3,5\}]$ are negative for all $\tilde{X} \in Q(\tilde{A})$. So by Cauchy's interlacing theorem, $i_-(\tilde{X}) \geq 3$ for all symmetric matrix $\tilde{X} \in Q(\tilde{A})$, a contradiction. Therefore, A is not potentially stable. Hence, by Lemma 3.3, A is not potentially stable if, $a_2, a_3 \in \{0, +\}$ and $a_5 \in \{0, -\}$.

Now a_2, a_3 can be interchanged by a permutation similarity, and the potential stability of a sign pattern matrix is not affected by permutation similarity. Suppose A is potentially stable. Then by **Case II(1)**, at least one of a_2, a_3, a_4 is $-$. If $a_2 = -$ or $a_3 = -$, then A is permutation similar to some super-pattern of
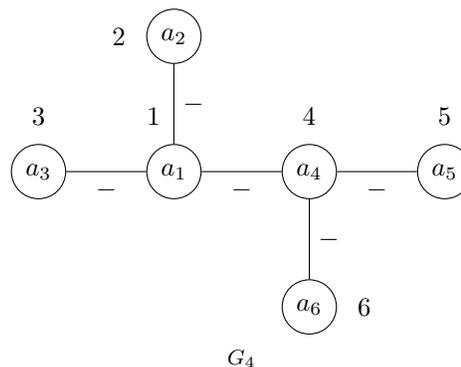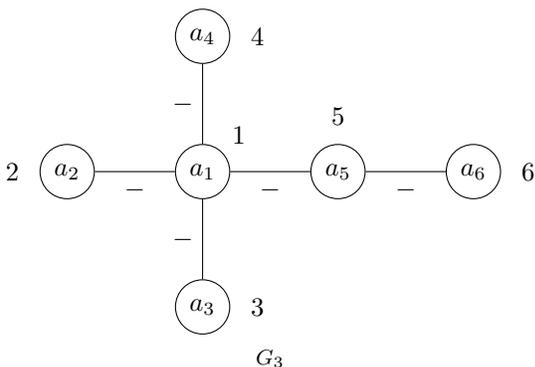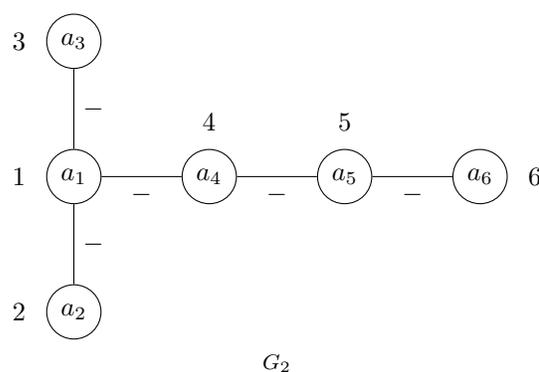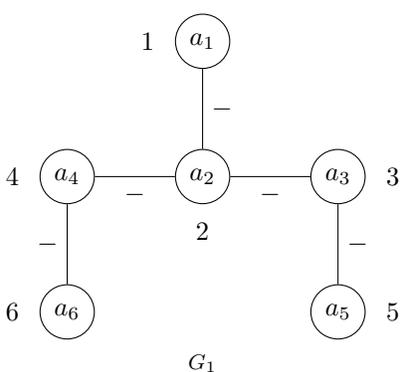
$$A_1 = \begin{bmatrix} 0 & + & + & + & 0 \\ - & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

If $a_4 = -$ and A is not a super-pattern of A_1, then $a_2, a_3 \in \{0, +\}$ and thus by **Case II(2)**, $a_5 = +$. Further, A is potentially stable implies A allows nonsingularity and thus at least one of a_2 and a_3 is $+$. Therefore, A is permutation similar to some super-pattern of

$$A_2 = \begin{bmatrix} 0 & + & + & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & + \\ 0 & 0 & 0 & - & + \end{bmatrix}.$$

Now A_1 allows a properly signed nest by Theorem 2.4. Again A_2 is permutation similar to the sign pattern matrix in Example 2.5, and thus A_2 allows a properly signed nest. Therefore, by Lemma 2.3, A is potentially stable implies A allows a properly signed nest.

Case III: $n = 6$. If $G(A)$ is a path or a star, then the result is true by Theorems 3.4 and 3.5, respectively. Suppose $G(A)$ is neither a star nor a path. There are four nonequivalent such signed trees, which are listed below:

G_1

G_2

G_3

G_4

1. Let $G(A)$ be equivalent to G_1. If A is potentially stable, then A has a negative diagonal entry. So by Theorem 2.8, A is potentially stable implies A allows a properly signed nest.
2. If $G(A)$ is equivalent to G_2, then we can consider

$$
A = \begin{bmatrix}
a_1 & + & + & + & 0 & 0 \\
- & a_2 & 0 & 0 & 0 & 0 \\
- & 0 & a_3 & 0 & 0 & 0 \\
- & 0 & 0 & a_4 & + & 0 \\
0 & 0 & 0 & - & a_5 & + \\
0 & 0 & 0 & 0 & - & a_6
\end{bmatrix}.
$$

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The symmetric factorization of A is

$$
A = \begin{bmatrix} + & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & - \end{bmatrix} \begin{bmatrix} a_1 & + & + & + & 0 & 0 \\ + & -a_2 & 0 & 0 & 0 & 0 \\ + & 0 & -a_3 & 0 & 0 & 0 \\ + & 0 & 0 & -a_4 & - & 0 \\ 0 & 0 & 0 & - & a_5 & + \\ 0 & 0 & 0 & 0 & + & -a_6 \end{bmatrix} = \tilde{S}\tilde{A}.
$$

If A is potentially stable, then by Lemma 3.1, there exists a symmetric matrix $\tilde{C} \in Q(\tilde{A})$ such that $i_+(\tilde{C}) = 6 - i_+(\tilde{S}) = 4$ and thus $i_-(\tilde{C}) \le 2$.

Case III(2.1) If any three of a_2, a_3, a_4, and a_6 are $+$, then by Cauchy's interlacing theorem, $i_-(\tilde{X}) \ge 3$ for all symmetric matrix $\tilde{X} \in Q(\tilde{A})$, a contradiction. Therefore, A is not potentially stable. Hence, by Lemma 3.3, A is not potentially stable if, any three of a_2, a_3, a_4, and a_6 are in $\{0, +\}$.

Case III(2.2) If $a_2, a_3 = +, a_5 = -$, then by Cauchy's interlacing theorem, $i_-(\tilde{X}) \ge 3$ for all symmetric matrix $\tilde{X} \in Q(\tilde{A})$, a contradiction. Therefore, A is not potentially stable. Hence, by Lemma 3.3, A is not potentially stable if, $a_2, a_3 \in \{0, +\}$ and $a_5 \in \{0, -\}$.

Now a_2 and a_3 can be interchanged by a permutation similarity, and the potential stability of a sign pattern matrix is not affected by permutation similarity. Suppose A is potentially stable. Then from Case III(2.1), at least one of $(a_2, a_4), (a_2, a_6), (a_3, a_4), (a_3, a_6), (a_2, a_3), (a_4, a_6)$ is $(-, -)$. If at least one of $(a_2, a_4), (a_2, a_6), (a_3, a_4), (a_3, a_6), (a_2, a_3)$ is $(-, -)$, then A is permutation similar to some super-pattern of one among

$$
A_{21} = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ - & - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & + & 0 \\ 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & - & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ - & - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & - & - \end{bmatrix},
$$

$$
A_{23} = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ - & - & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & - & 0 \end{bmatrix}.
$$

If $(a_4, a_6) = (-, -)$ and A is not a super-pattern of one among A_{21}, A_{22}, A_{23}, then $a_2, a_3 \in \{0, +\}$ and thus by Case III(2.2), $a_5 = +$. Further, A is potentially stable implies A allows nonsingularity and thus at least one of a_2, a_3 is $+$. Therefore A is permutation similar to some super-pattern of

$$
A_{24} = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ - & + & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & + & 0 \\ 0 & 0 & 0 & - & + & + \\ 0 & 0 & 0 & 0 & - & - \end{bmatrix}.
$$

Now by Corollary 2.7, A_{21}, A_{22}, A_{23} allow properly signed nests. Further, $A_{24}(\{6\})$ allows a properly signed nest, since it is permutation similar to the sign pattern matrix in Example 2.5. Thus by Lemmas 2.3 and 2.6, A_{24} allows a properly signed nest. Therefore by Lemma 2.3, A is potentially stable implies A allows a properly signed nest.

3. If $G(A)$ is equivalent to G_3, then we can consider

$$A = \begin{bmatrix} a_1 & + & + & + & + & 0 \\ - & a_2 & 0 & 0 & 0 & 0 \\ - & 0 & a_3 & 0 & 0 & 0 \\ - & 0 & 0 & a_4 & 0 & 0 \\ - & 0 & 0 & 0 & a_5 & + \\ 0 & 0 & 0 & 0 & - & a_6 \end{bmatrix}.$$

The symmetric factorization of A is

$$A = \begin{bmatrix} + & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} a_1 & + & + & + & + & 0 \\ + & -a_2 & 0 & 0 & 0 & 0 \\ + & 0 & -a_3 & 0 & 0 & 0 \\ + & 0 & 0 & -a_4 & 0 & 0 \\ + & 0 & 0 & 0 & -a_5 & - \\ 0 & 0 & 0 & 0 & - & a_6 \end{bmatrix} = \tilde{S}\tilde{A}.$$

If A is potentially stable, then by Lemma 3.1, there exists a symmetric matrix $\tilde{C} \in Q(\tilde{A})$ such that $i_+(\tilde{C}) = 6 - i_+(\tilde{S}) = 4$ and thus $i_-(\tilde{C}) \leq 2$.

Case III(3.1) If any three of a_2, a_3, a_4, a_5 are $+$, then by Cauchy's interlacing theorem, $i_-(\tilde{X}) \geq 3$ for all symmetric matrix $\tilde{X} \in Q(\tilde{A})$, a contradiction. Therefore A is not potentially stable. Hence by Lemma 3.3, A is not potentially stable if, any three of a_2, a_3, a_4, a_5 are in $\{0, +\}$.

Case III(3.2) If any two of a_2, a_3, a_4 are $+$ and $a_6 = -$, then by Cauchy's interlacing theorem, $i_-(\tilde{X}) \geq 3$ for all symmetric matrix $\tilde{X} \in Q(\tilde{A})$, a contradiction. Therefore A is not potentially stable. Hence by Lemma 3.3, A is not potentially stable if, any two of a_2, a_3, a_4 are in $\{0, +\}$ and $a_6 \in \{0, -\}$.

Now a_2, a_3, a_4 can be permuted by a permutation similarity, and the potential stability of a sign pattern matrix is not affected by permutation similarity. Suppose A is potentially stable. Then by **Case III(3.1)**, at least one of $(a_2, a_3), (a_2, a_4), (a_3, a_4), (a_2, a_5), (a_3, a_5), (a_4, a_5)$ is $(-, -)$. If at least one of $(a_2, a_3), (a_2, a_4), (a_3, a_4)$ is $(-, -)$, then A is permutation similar to some super-pattern of

$$A_{31} = \begin{bmatrix} 0 & + & + & + & + & 0 \\ - & - & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

If at least one of $(a_2, a_5), (a_3, a_5), (a_4, a_5)$ is $(-, -)$ and A is not a super-pattern of A_{31}, then exactly one of a_2, a_3, a_4 is $-$ and thus by **Case III(3.2)**, $a_6 = +$. Further, A is potentially stable implies

A allows nonsingularity and thus at least one of a_2, a_3, a_4 is $+$. Therefore A is permutation similar to some super-pattern of

$$A_{32} = \begin{bmatrix} 0 & + & + & + & + & 0 \\ - & + & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & - & + \\ 0 & 0 & 0 & 0 & - & + \end{bmatrix}.$$

Now by Corollary 2.7, A_{31} allows a properly signed nest. Further, $A_{32}(\{3\})$ allows a properly signed nest, since it is permutation similar to the sign pattern matrix in Example 2.5. Thus by Lemmas 2.3 and 2.6, A_{32} allows a properly signed nest. Therefore by Lemma 2.3, A is potentially stable implies A allows a properly signed nest.

4. If $G(A)$ is equivalent to G_4, then we can consider

$$A = \begin{bmatrix} a_1 & + & + & + & 0 & 0 \\ - & a_2 & 0 & 0 & 0 & 0 \\ - & 0 & a_3 & 0 & 0 & 0 \\ - & 0 & 0 & a_4 & + & + \\ 0 & 0 & 0 & - & a_5 & 0 \\ 0 & 0 & 0 & - & 0 & a_6 \end{bmatrix}.$$

The symmetric factorization of A is

$$A = \begin{bmatrix} + & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} a_1 & + & + & + & 0 & 0 \\ + & -a_2 & 0 & 0 & 0 & 0 \\ + & 0 & -a_3 & 0 & 0 & 0 \\ + & 0 & 0 & -a_4 & - & - \\ 0 & 0 & 0 & - & a_5 & 0 \\ 0 & 0 & 0 & - & 0 & a_6 \end{bmatrix} = \tilde{S}\tilde{A}.$$

If A is potentially stable, then by Lemma 3.1, there exists a symmetric matrix $\tilde{C} \in Q(\tilde{A})$ such that $i_+(\tilde{C}) = 6 - i_+(\tilde{S}) = 3$ and thus $i_-(\tilde{C}) \leq 3$.

Case III(4.1) If $a_2, a_3 = +, a_5, a_6 = -$, then by Cauchy's interlacing theorem, $i_-(\tilde{X}) \geq 4$ for all symmetric matrix $\tilde{X} \in Q(\tilde{A})$, a contradiction. Therefore A is not potentially stable. Hence by Lemma 3.3, A is not potentially stable if, $a_2, a_3 \in \{0, +\}$ and $a_5, a_6 \in \{0, -\}$.

Case III(4.2) If $a_2, a_3 = -, a_5, a_6 = +$, then by Cauchy's interlacing theorem, $i_+(\tilde{X}) \geq 4$ for all symmetric matrix $\tilde{X} \in Q(\tilde{A})$, a contradiction. Therefore A is not potentially stable. Hence by Lemma 3.3, A is not potentially stable if, $a_2, a_3 \in \{0, -\}$ and $a_5, a_6 \in \{0, +\}$.

Now a_2, a_3 can be interchanged, a_5, a_6 can be interchanged, and a_1, a_2, a_3 can be interchanged with a_4, a_5, a_6 respectively by a permutation similarity, and the potential stability of a sign pattern matrix is not affected by permutation similarity. Suppose A is potentially stable. Then by Case III(4.1) and Case III(4.2), at least one of $(a_2, a_5), (a_2, a_6), (a_3, a_5)$, and (a_3, a_6) is either $(-, -)$ or $(+, +)$.

If at least one of (a_2, a_5), (a_2, a_6), (a_3, a_5), and (a_3, a_6) is $(-, -)$, then A is permutation similar to some super-pattern of

$$A_{41} = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ - & - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & - & 0 & 0 \end{bmatrix}.$$

If at least one of (a_2, a_5), (a_2, a_6), (a_3, a_5), and (a_3, a_6) is $(+, +)$ and A is not a super-pattern of A_{41}, then A is permutation similar to some super-pattern of one among

$$A_{42} = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ - & + & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & - & + & 0 \\ 0 & 0 & 0 & - & 0 & 0 \end{bmatrix}, \quad A_{43} = \begin{bmatrix} - & + & + & + & 0 & 0 \\ - & + & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & - & + & 0 \\ 0 & 0 & 0 & - & 0 & 0 \end{bmatrix},$$

since A is potentially stable implies A has a $-$ on the diagonal.

Now by Corollary 2.7, A_{41} allows a properly signed nest. Since the leading principal minors of PCP^T are -1, 2, -1, 1, -1, and 1, where

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \in Q(A_{42}) \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

A_{42} allows a properly signed nest. Further, $A_{43}(\{3\})$ allows a properly signed nest, since it is permutation similar to the sign pattern matrix in Example 2.5. Since $\det A_{43} = +$, A_{43} allows a properly signed nest. Therefore, by Lemma 2.3, A is potentially stable implies A allows a properly signed nest. □

4. 5-by-5 spectrally arbitrary tree sign pattern matrices with all edges negative.

A sign pattern matrix A of order n is potentially nilpotent if there exists a $C \in Q(A)$ such that $\det(zI - C) = z^n$, where I is an identity matrix. In this case, C is a nilpotent realization of A. Drew et al. [6] had introduced the following method to find spectrally arbitrary sign pattern matrices.

Nilpotent-Jacobian method. To show that an n-by-n irreducible pattern Y is a spectrally arbitrary pattern, consider $X \in Q(Y)$ with its nonzero entries specified by positive parameters $x_0, x_1, \ldots, x_k$. If X has a nilpotent realization $\hat{X}$ with $(x_0, x_1, \ldots, x_k) = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_k)$, and there are n x_is, say $x_0, x_1, \ldots, x_{n-1}$, such that the Jacobian of the coefficients of $z^{n-1}, z^{n-2}, \ldots, z^1, z^0$ in $\det(zI - X)$ with respect to $x_0, x_1, \ldots, x_{n-1}$ is nonzero when evaluated at $(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_k)$, then any super-pattern of Y is a spectrally arbitrary pattern.

MacGillivray et al. [16] identified all spectrally arbitrary star sign pattern matrices. All spectrally arbitrary sign pattern matrices up to order 3 were determined by Britz et al. [3]. Arav et al. [1] characterized all 4-by-4 spectrally arbitrary tree sign pattern matrices.

In this paper, we identify all 5-by-5 tree sign pattern matrices with all edges negative. We get the following result from [16, Theorem 7.1].

LEMMA 4.1. *There are no 5-by-5 spectrally arbitrary star sign pattern matrices with all edges negative.*

If A is a 5-by-5 tree sign pattern matrix with all edges negative such that $G(A)$ is not a star, then A is equivalent to one among

(4.1)
$$
P = \begin{bmatrix}
p_1 & + & 0 & 0 & 0 \\
- & p_2 & + & 0 & 0 \\
0 & - & p_3 & + & 0 \\
0 & 0 & - & p_4 & + \\
0 & 0 & 0 & - & p_5
\end{bmatrix}, \quad
R = \begin{bmatrix}
r_1 & + & + & + & 0 \\
- & r_2 & 0 & 0 & 0 \\
- & 0 & r_3 & 0 & 0 \\
- & 0 & 0 & r_4 & + \\
0 & 0 & 0 & - & r_5
\end{bmatrix},
$$

where $p_1, p_2, p_3, p_4, p_5, r_1, r_2, r_3, r_4, r_5 \in \{+, -, 0\}$.

Let $n \geq 2$ and a sign pattern matrix of order n be given by:

$$
T_n = \begin{bmatrix}
- & + & 0 & \cdots & \cdots & 0 \\
- & 0 & + & \ddots & & \vdots \\
0 & - & 0 & + & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & & \ddots & - & 0 & + \\
0 & \cdots & \cdots & 0 & - & +
\end{bmatrix}.
$$

LEMMA 4.2 ([9]). *Every super-pattern of T_n is a spectrally arbitrary pattern for all $n \geq 2$.*

If G is a simple undirected graph, then the path $\{u = v_0, v_1, v_2, \ldots, v_{k-1}, v_k = v\}$ between two vertices u and v of G has length k. The distance between u and v, denoted by $d(u, v)$, is the length of the shortest path between them.

LEMMA 4.3. *Let A be a path sign pattern matrix of order n with all edges negative and $G(A)$ be the graph of A with vertices $1, 2, \ldots, n$ such that i is adjacent to $i + 1$ for $i = 1, 2, \ldots, n - 1$. Suppose A has a nonzero diagonal entry and $u_{\min} = \min\{u : a_{uu} \neq 0\}$, $u_{\max} = \max\{u : a_{uu} \neq 0\}$. If*

$$
\max\{d(1, u_{\min}), d(n, u_{\max})\} \geq \left\lfloor \frac{n}{2} \right\rfloor,
$$

then A is not potentially nilpotent.

Proof. Let $C \in Q(A)$.

Let n be even. If $d(1, u_{\min}) \geq \frac{n}{2}$, then $(C^n)_{11} \neq 0$. If $d(n, u_{\max}) \geq \frac{n}{2}$, then $(C^n)_{nn} \neq 0$. So C is not nilpotent.

Let n be odd. Suppose $d(1, u_{\min}) \geq \frac{n-1}{2}$. Let w be the vertex of $G(A)$ such that $w < u_{\min}$ and $d(w, u_{\min}) = \frac{n-1}{2}$. Then $(C^n)_{ww} \neq 0$. So C is not nilpotent. Similarly, if $d(n, u_{\max}) \geq \frac{n-1}{2}$, then C is not nilpotent.

Hence, A is not potentially nilpotent. □

LEMMA 4.4. *The sign pattern matrices*

$$
N_1 = \begin{bmatrix} + & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & - & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \quad
N_2 = \begin{bmatrix} + & + & 0 & 0 & 0 \\ - & + & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & - & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \quad
N_3 = \begin{bmatrix} + & + & 0 & 0 & 0 \\ - & - & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & - & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix},
$$

do not allow a nilpotent matrix.

Proof. Let $b_1, b_2, b_3, b_4, a_1, a_3, a_4 > 0$, $a_2 \in \mathbb{R}$ and

$$
C = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 \\ -b_1 & a_2 & 1 & 0 & 0 \\ 0 & -b_2 & -a_3 & 1 & 0 \\ 0 & 0 & -b_3 & -a_4 & 1 \\ 0 & 0 & 0 & -b_4 & 0 \end{bmatrix}.
$$

The characteristic polynomial of C is $z^5 - c_1 z^4 + c_2 z^3 - c_3 z^2 + c_4 z - c_5$, where

$$c_1 = a_1 + a_2 - a_3 - a_4;$$
$$c_2 = a_1(a_2 - a_3 - a_4) - a_2(a_3 + a_4) + a_3 a_4 + b_1 + b_2 + b_3 + b_4;$$
$$c_3 = a_1(b_2 + b_3 + b_4) + a_2(b_3 + b_4) - a_3(b_1 + b_4) - a_4(b_1 + b_2) - a_1 a_2(a_3 + a_4) + (a_1 + a_2)a_3 a_4;$$
$$c_4 = b_1 b_3 + b_1 b_4 + b_2 b_4 + a_1 a_2(b_3 + b_4) - a_1 a_3 b_4 - a_1 a_4 b_2 - a_2 a_3 b_4 + a_3 a_4 b_1 + a_1 a_2 a_3 a_4;$$
$$c_5 = a_1 b_2 b_4 - a_3 b_1 b_4 - a_1 a_2 a_3 b_4.$$

Suppose C is nilpotent. Then, $c_1 = c_2 = c_3 = c_4 = c_5 = 0$.

Now $c_1 = 0 \Rightarrow a_1 + a_2 = a_3 + a_4$, $c_5 = 0 \Rightarrow a_1 b_2 - a_3 b_1 - a_1 a_2 a_3 = 0$ and thus $b_1 + a_1 a_2 > 0$.

$$
\begin{aligned}
c_3 = 0 &\Rightarrow a_1(b_3 + b_4) + a_2(b_3 + b_4) - a_3 b_4 - a_4(b_1 + b_2) - a_1 a_2 a_4 + (a_1 + a_2)a_3 a_4 = 0 \\
&\Rightarrow (a_1 + a_2)b_3 + b_4(a_1 + a_2 - a_3) - a_4(b_1 + b_2 + a_1 a_2 - a_1 a_3 - a_2 a_3) = 0 \\
&\Rightarrow (a_3 + a_4)b_3 + a_4 b_4 - a_4(b_1 + b_2 + a_1 a_2 - a_1 a_3 - a_2 a_3) = 0 \\
&\Rightarrow b_1 + b_2 + a_1 a_2 - a_1 a_3 - a_2 a_3 > 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{Then } c_4 &= b_1 b_3 + b_1 b_4 + b_2 b_4 + a_1 a_2(b_3 + b_4) - a_1 a_3 b_4 - a_1 a_4 b_2 - a_2 a_3 b_4 + a_3 a_4 b_1 + a_1 a_2 a_3 a_4 \\
&= b_1 b_3 + a_1 a_2 b_3 + b_4(b_1 + b_2 + a_1 a_2 - a_1 a_3 - a_2 a_3) + a_4(a_3 b_1 - a_1 b_2 + a_1 a_2 a_3) \\
&= (b_1 + a_1 a_2)b_3 + b_4(b_1 + b_2 + a_1 a_2 - a_1 a_3 - a_2 a_3) > 0.
\end{aligned}
$$

This is a contradiction. So C is not nilpotent and thus none among N_1, N_2, N_3 allow a nilpotent matrix. □

A sign pattern matrix A is said to allow the inertia (n_1, n_2, n_3) if there exists a $C \in Q(A)$ such that $i(C) = (n_1, n_2, n_3)$.

LEMMA 4.5. *If the path sign pattern matrix P in (4.1) is potentially nilpotent and allows the inertias* $(5, 0, 0), (0, 5, 0)$, *then P is equivalent to some super-pattern of one among*

$$P_1 = \begin{bmatrix} + & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & 0 & + & 0 \\ 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & - & - \end{bmatrix}, \ P_2 = \begin{bmatrix} + & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & - & + \end{bmatrix}, P_3 = \begin{bmatrix} + & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & + & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

Proof. By Theorem 3.4,

$$P = \begin{bmatrix} p_1 & + & 0 & 0 & 0 \\ - & p_2 & + & 0 & 0 \\ 0 & - & p_3 & + & 0 \\ 0 & 0 & - & p_4 & + \\ 0 & 0 & 0 & - & p_5 \end{bmatrix},$$

allows the inertia $(0, 5, 0)$ if and only if at least one of p_1, p_3, p_5 is $-$. Similarly, P allows the inertia $(5, 0, 0)$ if and only if at least one of p_1, p_3, p_5 is $+$. Hence, by Lemma 4.3, the path sign pattern matrices of order 5, which are potentially nilpotent and allow the inertias $(0, 5, 0)$ and $(5, 0, 0)$ can only be equivalent to some super-pattern of one among P_1, P_2, P_3 and

$$P_4 = \begin{bmatrix} + & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & - & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

The only super-patterns of P_4 of the form P, which are super-patterns of none among P_1, P_2, P_3 are N_1, N_2, N_3. By Lemma 4.4, none among N_1, N_2, N_3 allow a nilpotent matrix. Hence, the result follows: □

LEMMA 4.6. *The sign pattern matrix*

$$\tilde{R} = \begin{bmatrix} r_1 & + & + & + & 0 \\ - & r_2 & 0 & 0 & 0 \\ - & 0 & r_3 & 0 & 0 \\ - & 0 & 0 & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix},$$

is not potentially nilpotent, where $r_1, r_2, r_3 \in \{+, -, 0\}$.

Proof. Let

$$C = \begin{bmatrix} a_1 & 1 & 1 & 1 & 0 \\ -b_1 & a_2 & 0 & 0 & 0 \\ -b_2 & 0 & a_3 & 0 & 0 \\ -b_3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -b_4 & 0 \end{bmatrix} \in Q(\tilde{R}),$$

where $a_1, a_2, a_3 \in \mathbb{R}$ and $b_1, b_2, b_3, b_4 > 0$.

The characteristic polynomial of C is $z^5 - c_1 z^4 + c_2 z^3 - c_3 x^2 + c_4 z - c_5$, where

$$c_1 = a_1 + a_2 + a_3,$$
$$c_2 = a_2 a_3 + a_1 a_2 + a_1 a_3 + b_1 + b_2 + b_3 + b_4,$$
$$c_3 = a_1 b_4 + a_2(b_2 + b_3 + b_4) + a_3(b_1 + b_3 + b_4) + a_1 a_2 a_3,$$
$$c_4 = b_4(a_2 a_3 + a_1 a_2 + a_1 a_3 + b_1 + b_2) + a_2 a_3 b_3,$$
$$c_5 = b_4(a_1 a_2 a_3 + a_3 b_1 + a_2 b_2).$$

Suppose C is a nilpotent matrix. Then $c_1 = c_2 = c_3 = c_4 = c_5 = 0$.

Case I: Assume $a_1 \neq 0$.

Then, $c_1 = 0 \Rightarrow a_1 + a_2 + a_3 = 0$ and $c_5 = 0 \Rightarrow a_1 a_2 a_3 + a_3 b_1 + a_2 b_2 = 0$.

Hence, $c_3 = 0 \Rightarrow a_2 + a_3 = 0 \Rightarrow a_1 = 0$, a contradiction.

Case II: Assume $a_1 = 0$.

Then, $c_2 = 0 \Rightarrow b_1 + b_2 + a_2 a_3 < 0$ and $c_1 = 0 \Rightarrow a_2 + a_3 = 0 \Rightarrow a_2 a_3 \leq 0$.

Therefore, $c_4 = b_4(a_2 a_3 + b_1 + b_2) + a_2 a_3 b_3 < 0$, a contradiction.

Hence, C is not a nilpotent matrix and thus $\tilde{R}$ is not potentially nilpotent.   □

LEMMA 4.7. *The sign pattern matrix R in* (4.1) *is spectrally arbitrary if and only if it is equivalent to some super-pattern of one among*

$$R_1 = \begin{bmatrix} 0 & + & + & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 \\ - & 0 & 0 & - & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & + & + & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 \\ - & 0 & 0 & 0 & + \\ 0 & 0 & 0 & - & - \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & + & + & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & + \\ 0 & 0 & 0 & - & + \end{bmatrix}.$$

Proof. We have

$$R = \begin{bmatrix} r_1 & + & + & + & 0 \\ - & r_2 & 0 & 0 & 0 \\ - & 0 & r_3 & 0 & 0 \\ - & 0 & 0 & r_4 & + \\ 0 & 0 & 0 & - & r_5 \end{bmatrix}.$$

Suppose R is spectrally arbitrary. Then, R is potentially nilpotent and allows the inertias $(5,0,0)$ and $(0,5,0)$. Since R allows the inertia $(0,5,0)$, from **Case II** in the proof of Theorem 3.7, R is permutation similar to some super-pattern of one among

$$A_1 = \begin{bmatrix} 0 & + & + & + & 0 \\ - & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & + & + & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & + \\ 0 & 0 & 0 & - & + \end{bmatrix}.$$

Now

$$S(-R)S = \begin{bmatrix} -r_1 & + & + & + & 0 \\ - & -r_2 & 0 & 0 & 0 \\ - & 0 & -r_3 & 0 & 0 \\ - & 0 & 0 & -r_4 & + \\ 0 & 0 & 0 & - & -r_5 \end{bmatrix}, \quad \text{where} \quad S = \begin{bmatrix} + & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & + \end{bmatrix}.$$

So both A_2 and $S(-A_2)S$ allow the inertia $(0, 5, 0)$. Since the inertia of a matrix remains unchanged after a similarity transformation, both A_2 and $-A_2$ allow the inertia $(0, 5, 0)$, that is, A_2 allows the inertias $(0, 5, 0)$ and $(5, 0, 0)$. If R allows the inertia $(0, 5, 0)$ and R is not permutation similar to any super-pattern of A_2, then R must be permutation similar to some super-pattern of A_1. Any super-pattern of A_1, which is not equivalent to any super-pattern of A_2 and allows the inertia $(5, 0, 0)$, must be equivalent to some super-pattern of

$$A_3 = \begin{bmatrix} 0 & + & + & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 \\ - & 0 & 0 & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

Therefore, if R is spectrally arbitrary, then by Lemma 4.6, we can conclude that R must be equivalent to some super-pattern of one among R_1, R_2, R_3.

For the converse part, let $X = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ -x_0 & x_1 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & -x_4 & 1 \\ 0 & 0 & 0 & -x_5 & x_6 \end{bmatrix} \in Q(R_3)$, where $x_0, x_1, x_2, x_3, x_4, x_5, x_6 > 0$.

The characteristic polynomial of X is $z^5 - c_1 z^4 + c_2 z^3 - c_3 z^2 + c_4 z - c_5$, where

$$c_1 = x_1 - x_4 + x_6,$$
$$c_2 = x_0 + x_2 + x_3 + x_5 - x_1 x_4 + x_1 x_6 - x_4 x_6,$$
$$c_3 = x_1 x_2 + x_1 x_3 + x_1 x_5 - x_0 x_4 - x_2 x_4 + x_0 x_6 + x_2 x_6 + x_3 x_6 - x_1 x_4 x_6,$$
$$c_4 = (x_0 + x_2)(x_5 - x_4 x_6) - x_1 x_2 x_4 + x_1 x_6 (x_2 + x_3),$$
$$c_5 = x_1 x_2 (x_5 - x_4 x_6).$$

X has a nilpotent realization at $(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5, \hat{x}_6) = (2, 2, 1, 1, 4, 8, 2)$. The Jacobian of the coefficients of z^4, z^3, z^2, z^1, z^0 with respect to x_0, x_1, x_2, x_3, x_5 at $(2, 2, 1, 1, 4, 8, 2)$ is

$$\begin{vmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & -x_4 + x_6 & 1 & 1 & 1 \\ x_4 - x_6 & -x_2 - x_3 - x_5 + x_4 x_6 & -x_1 + x_4 - x_6 & -x_1 - x_6 & -x_1 \\ x_5 - x_4 x_6 & -x_2 x_4 + x_6(x_2 + x_3) & x_5 - x_4 x_6 - x_1 x_4 + x_1 x_6 & x_1 x_6 & x_0 + x_2 \\ 0 & -x_2(x_5 - x_4 x_6) & -x_1(x_5 - x_4 x_6) & 0 & -x_1 x_2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 1 & 1 \\ 2 & -2 & 0 & -4 & -2 \\ 0 & 0 & -4 & 4 & 3 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & -4 \\ 0 & -4 & 4 \end{vmatrix} = 64 \neq 0.$$

Therefore, by the Nilpotent-Jacobian method, every super-pattern of R_3 is spectrally arbitrary. Similarly, the super-patterns of the sign pattern matrices R_1 and R_2 can be shown to be spectrally arbitrary by the Nilpotent-Jacobian method. Such nilpotent matrices in $Q(R_1)$ and $Q(R_2)$ are respectively,

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 0 \\
-\frac{9+5\sqrt{3}}{3} & 2+\sqrt{3} & 0 & 0 & 0 \\
-\frac{3+\sqrt{3}}{3} & 0 & -1 & 0 & 0 \\
-\frac{2+\sqrt{3}}{2} & 0 & 0 & -1-\sqrt{3} & 1 \\
0 & 0 & 0 & -\frac{2+\sqrt{3}}{2} & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 & 1 & 0 \\
-\frac{9+5\sqrt{3}}{6} & \sqrt{3}+1 & 0 & 0 & 0 \\
-\frac{9-5\sqrt{3}}{6} & 0 & -\sqrt{3}+1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2 & -2
\end{bmatrix}.
$$

Since any sign pattern matrix equivalent to a spectrally arbitrary pattern is also spectrally arbitrary, the converse part follows.   ☐

The following result characterizes all 5-by-5 spectrally arbitrary tree sign pattern matrices with all edges negative.

THEOREM 4.8. *A 5-by-5 tree sign pattern matrix with all edges negative is spectrally arbitrary if and only if it is equivalent to some super-pattern of one among* $P_1, P_2, P_3, R_1, R_2, R_3$.

Proof. Let A be a 5-by-5 spectrally arbitrary tree sign pattern matrix with all edges negative. Then by Lemma 4.1, A is not a star sign pattern matrix. If A is a path sign pattern matrix, then it is equivalent to a matrix of the form P in (4.1). If A is spectrally arbitrary, then A is potentially nilpotent and allows the inertias $(5,0,0)$ and $(0,5,0)$. So by Lemma 4.5, A is equivalent to some super-pattern of one among P_1, P_2, P_3.

If A is neither a star sign pattern matrix nor a path sign pattern matrix, then it is equivalent to a matrix of the form R in (4.1). If A is spectrally arbitrary, then by Lemma 4.7, A must be equivalent to some super-pattern of one among R_1, R_2, R_3.

Now we prove the converse part. By Lemma 4.2, the super-patterns of P_1 are spectrally arbitrary. The super-patterns of the sign pattern matrices P_2 and P_3 can be shown to be spectrally arbitrary by the Nilpotent-Jacobian method, and such nilpotent matrices in $Q(P_2)$ and $Q(P_3)$ are respectively,

$$
\begin{bmatrix}
\sqrt{5}+1 & 1 & 0 & 0 & 0 \\
-4 & 0 & 1 & 0 & 0 \\
0 & -4 & -2\sqrt{5} & 1 & 0 \\
0 & 0 & -4 & 0 & 1 \\
0 & 0 & 0 & -4 & \sqrt{5}-1
\end{bmatrix},
\begin{bmatrix}
6 & 1 & 0 & 0 & 0 \\
-12 & 0 & 1 & 0 & 0 \\
0 & -24 & -12 & 1 & 0 \\
0 & 0 & -54 & 6 & 1 \\
0 & 0 & 0 & -18 & 0
\end{bmatrix}.
$$

The super-patterns of the sign pattern matrices R_1, R_2, and R_3 are spectrally arbitrary by Lemma 4.7. Since any sign pattern matrix equivalent to a spectrally arbitrary pattern is also spectrally arbitrary, the converse part follows.   ☐

REFERENCES

[1] M. Arav, F. Hall, Z. Li, K. Kaphle, and N. Manzagol. Spectrally arbitrary tree sign patterns of order 4. *Electron. J. Linear Algebra*, 20:180–197, 2010.

[2] T. Bone. Positive feedback may sometimes promote stability. *Linear Algebra Appl.*, 51:143–151, 1983.

[3] T. Britz, J.J. McDonald, D.D. Olesky, and P. van den Driessche. Minimal spectrally arbitrary sign patterns. *SIAM J. Matrix Anal. Appl.*, 26(1):257–271, 2004.

[4] J. Cavalcanti. Potentially Hurwitz structures: A characterization of nests. In: *2020 59th IEEE Conference on Decision and Control (CDC)*, 5180–5187, 2020.

[5] M. Cavers. Polynomial stability and potentially stable patterns *Linear Algebra Appl.*, 613:87–114, 2021.

[6] J.H. Drew, C.R. Johnson, D.D. Olesky, and P. van den Driessche. Spectrally arbitrary patterns. *Linear Algebra Appl.*, 308(1–3):121–137, 2000.

[7] Y. Gao and J. Li. On the potential stability of star sign pattern matrices. *Linear Algebra Appl.*, 327(1–3):61–68, 2001.

[8] D.A. Grundy, D.D. Olesky, and P. van den Driessche. Constructions for potentially stable sign patterns. *Linear Algebra Appl.*, 436(12):4473–4488, 2012.

[9] C. Garnett and B.L. Shader. A proof of the T_n conjecture: centralizers, Jacobians and spectrally arbitrary sign patterns. *Linear Algebra Appl.*, 436(12):4451–4458, 2012.

[10] J.M. Harris, J.L. Hirst, and M.J. Mossinghoff. *Combinatorics and Graph Theory*, Second edition. Undergraduate Texts in Mathematics. Springer, New York, 2008.

[11] R.A. Horn and C.R. Johnson. *Matrix Analysis*, Second edition. Cambridge University Press, Cambridge, 2013.

[12] F.J. Hall and Z. Li. In: L. Hogben (editor), *Handbook of Linear Algebra*, Second edition. Discrete Mathematics and Its Applications. CRC Press, Boca Raton, FL, 2014.

[13] C. Jeffries and C.R. Johnson. Some sign patterns that preclude matrix stability. *SIAM J. Matrix Anal. Appl.*, 9(1):19–25, 1988.

[14] C.R. Johnson, J.S. Maybee, D.D. Olesky, and P. van den Driessche. Nested sequences of principal minors and potential stability. *Linear Algebra Appl.*, 262:243–257, 1997.

[15] C.R. Johnson and T.A. Summers. The potentially stable tree sign patterns for dimensions less than five. *Linear Algebra Appl.*, 126:1–13, 1989.

[16] G. MacGillivray, R.M. Tifenbach, and P. van den Driessche. Spectrally arbitrary star sign patterns. *Linear Algebra Appl.*, 400:99–119, 2005.

[17] D.D. Olesky, M.J. Tsatsomeros, and P. van den Driessche. Sign patterns with a nest of positive principal minors. *Linear Algebra Appl.*, 436(12):4392–4399, 2012.