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SIMPLE NECESSARY CONDITIONS FOR HADAMARD FACTORIZABILITY OF HURWITZ POLYNOMIALS*

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Abstract. In this paper, we focus the attention on the Hadamard factorization problem for Hurwitz polynomials. We give a new necessary condition for Hadamard factorizability of Hurwitz stable polynomials of degree $n \geq 4$ and show that for $n = 4$ this condition is also sufficient. The effectiveness of the result is illustrated during construction of examples of stable polynomials that are not Hadamard factorizable.

Key words. Hadamard factorization, Hadamard product of polynomials, Hurwitz stable polynomials.

AMS subject classifications. 26C10, 30C15, 93D99.

1. Introduction.

A problem of the existence of a Hadamard factorization for a given Hurwitz stable polynomial has been taken by many authors. Recall that a Hurwitz stable polynomial of degree $n \geq 1$ has a Hadamard factorization if it is a Hadamard product of two Hurwitz stable polynomials of degree n. It is known that every stable polynomial of degree $n \leq 3$ admits a Hadamard factorization (see Garloff and Shrinivasan [5]) and that for every $n \geq 4$ there exists an n-th degree stable polynomial which is not Hadamard factorizable (see Białas and Góra [2]). Some conditions for the existence of a Hadamard factorization can be found in Loredo–Villalobos and Aguirre–Hernández [8, 9], but these conditions cannot be effectively applied in practice. In turn, some topological properties of the entire family of polynomials admitting a Hadamard factorization can be found in Aguirre–Hernández *et al.* [1].

Note also, that there are some issues in which polynomials having a Hadamard factorization play an important role. In [3], the authors have considered the stability problem for the generalized Hadamard product of polynomials (recall that the generalized Hadamard product $f \bullet g$ of polynomials f of degree m and g of degree $n \geq m$ is a set consisting of polynomials of degree m which are defined as the Hadamard products of the polynomial f and some other $n-m+1$ polynomials formed from the polynomial g). It was shown, among others, that if f and g are Hurwitz stable, then $f \bullet g$ is quasi-stable (i.e. all zeros of every polynomial belonging to $f \bullet g$ have non-positive real parts). If, additionally, f has a Hadamard factorization, then $f \bullet g$ occurs to be Hurwitz stable. This shows that the Hadamard factorization problem has both theoretical and applied significance.

In this work, we develop the idea used in our recent paper [2]. We give a new necessary condition for Hadamard factorizability of a polynomial and show that for $n = 4$ this condition is also sufficient. The effectiveness of the result is illustrated during construction of examples of stable polynomials that are not Hadamard factorizable.

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2. Preliminary results. Let us now introduce the notations and remind some results which will be used in the sequel.

2.1. Basic notations. We use standard notation: $\mathbb{R}$ and $\mathbb{R}^{n\times n}$ stand for the set of real numbers and for the set of real matrices of order $n \times n$, respectively; π_n^+ denotes the family of n–th degree polynomials with positive coefficients.

2.2. Stable polynomials. A polynomial $f \in \pi_n^+$ $(n \geq 1)$,

$$f(s) = a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + a_n s^n, \tag{1}$$

is *Hurwitz stable* (or shortly *stable*) if all its zeros have negative real parts. It is well known (and easily verified) that a necessary condition for the stability of a real polynomial is that its coefficients are all of the same sign; without losing generality we will assume in the sequel that they are positive. The entire family of Hurwitz stable polynomials of degree n with positive coefficients will be denoted by $\mathcal{H}_n^+$.

Let $\triangle_i(f)$ denote the i–th leading principal minor of the Hurwitz matrix $H_f \in \mathbb{R}^{n\times n}$ associated with polynomial (1):

$$H_f = \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & \ldots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & \ldots & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & \ldots & 0 \\ \vdots & \vdots & a_{n-5} & a_{n-4} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_0 \end{bmatrix}, \tag{2}$$

in particular $\triangle_1(f) = a_{n-1}$ and $\triangle_n(f) = \det H_f = a_0\triangle_{n-1}(f)$. It follows from the Routh–Hurwitz stability criterion (see, e.g., Gantmacher [4]) that polynomial (1) with positive coefficients is stable if and only if $\triangle_i(f) > 0$, for $i = 1, 2, \ldots, n-1$. Among many other properties of the Hurwitz matrix, one can also find the following given by Kemperman [7] (see Theorem 2 therein).

THEOREM 1. *If $f \in \mathcal{H}_n^+$, then every square submatrix of the Hurwitz matrix H_f has positive determinant if and only if all its diagonal elements are positive.*

Suppose now that $n \geq 4$ and that $\{H_1, \ldots, H_{m_n}\} \subset \mathbb{R}^{3\times 3}$ is a set of all 3×3 submatrices of the form:

$$H_{f,k} = \begin{bmatrix} h_{11}^{(k)} & h_{12}^{(k)} & 0 \\ h_{21}^{(k)} & h_{22}^{(k)} & h_{23}^{(k)} \\ 0 & h_{32}^{(k)} & h_{33}^{(k)} \end{bmatrix} \quad \text{with } h_{ij}^{(k)} > 0, \tag{3}$$

of the Hurwitz matrix H_f. Entries $h_{ij}^{(k)}$ occurring in (3) depend on the polynomial f but, to simplify the notation, throughout this paper we do not make this dependence explicit.

EXAMPLE 1. For $n = 4$ we have $m_4 = 2$ and

$$H_{f,1} = \begin{bmatrix} a_3 & a_4 & 0 \\ a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 \end{bmatrix} \quad \text{and} \quad H_{f,2} = \begin{bmatrix} a_3 & a_4 & 0 \\ a_1 & a_2 & a_4 \\ 0 & a_0 & a_2 \end{bmatrix}.$$

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For $n = 5$ we have $m_5 = 4$ and

$$H_{f,1} = \begin{bmatrix} a_3 & a_4 & 0 \\ a_1 & a_2 & a_4 \\ 0 & a_0 & a_2 \end{bmatrix}, \quad H_{f,2} = \begin{bmatrix} a_3 & a_5 & 0 \\ a_1 & a_3 & a_4 \\ 0 & a_1 & a_2 \end{bmatrix},$$

$$H_{f,3} = \begin{bmatrix} a_2 & a_4 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_0 & a_2 \end{bmatrix}, \quad H_{f,4} = \begin{bmatrix} a_2 & a_5 & 0 \\ a_0 & a_3 & a_4 \\ 0 & a_1 & a_2 \end{bmatrix}.$$

Theorem 1 allows us to conclude that if polynomial (1) of degree $n \geq 4$ is stable then

$$h_{11}^{(k)} h_{22}^{(k)} - h_{12}^{(k)} h_{21}^{(k)} > 0, \qquad h_{22}^{(k)} h_{33}^{(k)} - h_{23}^{(k)} h_{32}^{(k)} > 0, \tag{4}$$

and

$$h_{11}^{(k)} h_{22}^{(k)} h_{33}^{(k)} - h_{11}^{(k)} h_{23}^{(k)} h_{32}^{(k)} - h_{12}^{(k)} h_{21}^{(k)} h_{33}^{(k)} > 0, \tag{5}$$

for $k = 1, \ldots, m_n$. These inequalities will play a key role in our further considerations.

2.3. Polynomials admitting a Hadamard factorization. Together with polynomial (1), we will consider a polynomial $g \in \pi_n^+$:

$$g(s) = b_0 + b_1 s + \ldots + b_{n-1} s^{n-1} + b_n s^n, \tag{6}$$

their Hadamard product $f \circ g \in \pi_n^+$ defined as an element-wise multiplication, that is,

$$(f \circ g)(s) = a_0 b_0 + a_1 b_1 s + \ldots + a_{n-1} b_{n-1} s^{n-1} + a_n b_n s^n,$$

and their Hadamard quotient $f \diamond g \in \pi_n^+$ defined as an element-wise division, that is,

$$(f \diamond g)(s) = \frac{a_0}{b_0} + \frac{a_1}{b_1} s + \ldots + \frac{a_{n-1}}{b_{n-1}} s^{n-1} + \frac{a_n}{b_n} s^n.$$

Garloff and Wagner [6] proved that the Hadamard product of two real Hurwitz stable polynomials is again Hurwitz stable, and thus it seems to be quite natural to say that the polynomial $f \in \mathcal{H}_n^+$ *has a Hadamard factorization* (or *is Hadamard factorizable*) if there exist two polynomials $f_1, f_2 \in \mathcal{H}_n^+$ such that $f = f_1 \circ f_2$. Equivalently, the polynomial $f \in \mathcal{H}_n^+$ has a Hadamard factorization if there exists a polynomial $g \in \mathcal{H}_n^+$ such that $f \diamond g \in \mathcal{H}_n^+$. A stable polynomial which is not Hadamard factorizable is said to be *Hadamard irreducible.*

3. Main results. Let $f \in \pi_n^+$ and let $w_{f,1}^{(k)}$, $w_{f,2}^{(k)}$ be positive numbers given by:

$$w_{f,1}^{(k)} = \frac{h_{21}^{(k)} h_{12}^{(k)}}{h_{11}^{(k)} h_{22}^{(k)}}, \qquad w_{f,2}^{(k)} = \frac{h_{32}^{(k)} h_{23}^{(k)}}{h_{22}^{(k)} h_{33}^{(k)}}, \tag{7}$$

for $k = 1, \ldots, m_n$. We start with the following simple observation.

Lemma 2. *Let $n \geq 4$.*

(a) *If $f \in \mathcal{H}_n^+$, then for $k = 1, \ldots, m_n$*

$$w_{f,1}^{(k)} < 1, \quad w_{f,2}^{(k)} < 1, \quad w_{f,1}^{(k)} + w_{f,2}^{(k)} < 1. \tag{8}$$

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(b) *If $f \in \mathcal{H}_n^+$ and $g \in \pi_n^+$ are such that $f \diamond g \in \mathcal{H}_n^+$, then for $k = 1, \dots, m_n$*

(9) $$w_{f,1}^{(k)} < w_{g,1}^{(k)}, \quad w_{f,2}^{(k)} < w_{g,2}^{(k)},$$

and

(10) $$w_{f,1}^{(k)}/w_{g,1}^{(k)} + w_{f,2}^{(k)}/w_{g,2}^{(k)} < 1.$$

Proof. The above inequalities follow from the stability of f and $f \diamond g$: conditions (8) are equivalent to (4)–(5) and conditions (9)–(10) follow from (8) and from the identities:

$$w_{f\diamond g,1}^{(k)} = w_{f,1}^{(k)}/w_{g,1}^{(k)} \quad \text{and} \quad w_{f\diamond g,2}^{(k)} = w_{f,2}^{(k)}/w_{g,2}^{(k)}.$$ □

3.1. A necessary condition for Hadamard factorizability of real polynomials. We are now ready to prove the main result of this section.

Theorem 3. *Let $n \geq 4$. If a polynomial $f \in \pi_n^+$ has a Hadamard factorization, then*

(11) $$\Delta_k(f) = \left(w_{f,1}^{(k)} - w_{f,2}^{(k)}\right)^2 - 2\left(w_{f,1}^{(k)} + w_{f,2}^{(k)}\right) + 1 > 0,$$

for $k = 1, \dots, m_n$.

Proof. It follows from the assumption that there exists a polynomial $g \in \mathcal{H}_n^+$ for which the polynomial $f \diamond g$ is stable. Then, by Lemma 2, we conclude that for any fixed $k \in \{1, \dots, m_n\}$ it holds

$$w_{g,1}^{(k)} + w_{g,2}^{(k)} < 1,$$

and

$$w_{f,1}^{(k)}/w_{g,1}^{(k)} + w_{f,2}^{(k)}/w_{g,2}^{(k)} < 1.$$

In other words, the Hadamard factorizability of f implies that there exist $x_1 > 1$ and $x_2 > 1$ satisfying the following system of inequalities:

(12) $$\begin{cases} \frac{1}{x_1} + \frac{1}{x_2} < 1 \\ b_1 x_1 + b_2 x_2 < 1 \end{cases},$$

where, to simplify the notations, we put $b_i = w_{f,i}^{(k)}$, for $i = 1, 2$. By Lemma 2, we know that $b_1, b_2 \in (0, 1)$. It is easy to note (see Fig. 1) that system of inequalities (12) has a solution if and only if the system:

(13) $$\begin{cases} \frac{1}{x_1} + \frac{1}{x_2} = 1 \\ b_1 x_1 + b_2 x_2 = 1 \\ x_1 > 1, \ x_2 > 1 \end{cases},$$

has two different solutions (x_1', x_2') and (x_1'', x_2''). This condition holds, in turn, if and only if the equation:

(14) $$b_2 x_2^2 + (b_1 - b_2 - 1)\, x_2 + 1 = 0,$$

has two different solutions $x_2' > 1$ and $x_2'' > 1$. This, by (8), is equivalent to (11) completing the proof. □

From Theorem 3 one can draw the following sufficient condition for Hadamard irreducibility of a polynomial.

Conclusion 1. *Let $n \geq 4$. If $f \in \mathcal{H}_n^+$ and for some $k \in \{1, \dots, m_n\}$:*

$$\Delta_k(f) \leq 0,$$

where $\Delta_k(f)$ is as in (11), then f is Hadamard irreducible.

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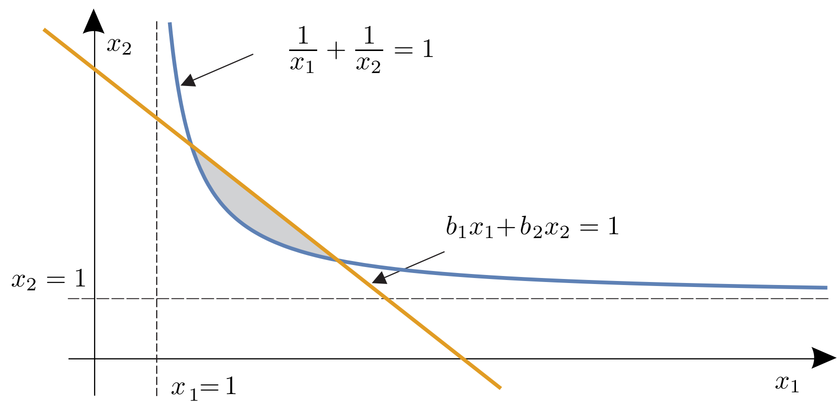

FIGURE 1. *Solutions to system of inequalities* (12) *filled with gray color.*

3.2. How to get a Hadamard irreducible polynomial? Garloff and Srinivasan considered in [5] the Hadamard factorization problem and gave, among others, an example of the Hadamard irreducible polynomial of degree $n = 4$. Now, using Theorem 3 and Conclusion 1, we will show a fairly universal way of construction of further examples of Hadamard irreducible polynomials of arbitrary degree $n \geq 4$. First, we need to introduce some additional notations.

Let, for $f \in \mathcal{H}_n^+$ and $a > 0$, $F_a \in \pi_{n+1}^+$ be a polynomial of the form:

$$F_a(s) = as^{n+1} + f(s),$$

and let $A, A_1, \ldots, A_{m_{n+1}} \subset \mathbb{R}$ be sets defined as follows:

$$A = \{a > 0 : F_a \in \mathcal{H}_{n+1}^+\},$$

and, for $k = 1, \ldots, m_{n+1}$,

$$A_k = \{a > 0 : \Delta_k(F_a) \leq 0\}.$$

The following theorem holds.

THEOREM 4. *Let $n \geq 4$ and let $f \in \mathcal{H}_n^+$. Under the above notations, if for some $k \in \{1, \ldots, m_{n+1}\}$ we have*

$$A \cap A_k \neq \emptyset, \tag{15}$$

then for every $a \in A \cap A_k$ the polynomial F_a is stable and Hadamard irreducible.

Proof. The thesis is a simple consequence of Conclusion 1.

It is known (see, e.g., Lemma 3.1 in Białas and Góra [3]) that if $f \in \mathcal{H}_n^+$ then there exists $a^* > 0$ such that $F_a \in \mathcal{H}_{n+1}^+$ for every $a \in (0, a^*)$. This means that the set A is always non-empty. On the other hand, one can show that if the polynomial f is stable, then sets $A_1, \ldots, A_{m_{n+1}}$ are all separated from zero and, hence, it is possible that in some cases assumption (15) of Theorem 4 will not be satisfied.

3.2.1. A numerical example. Consider a Hurwitz stable polynomial $f \in \pi_3^+$ of the form:

$$f(s) = s^3 + 3s^2 + s + 1,$$

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and let, for $a > 0$,

$$F_a(s) = as^4 + s^3 + 3s^2 + s + 1.$$

It follows from the Routh–Hurwitz stability criterion that the polynomial F_a is stable if and only if

$$0 \le a < 2. \tag{16}$$

On the other hand, submatrices (3) of the Hurwitz matrix H_{F_a} have the form (see Example 1):

$$H_{F_a,1} = \begin{bmatrix} 1 & a & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad H_{F_a,2} = \begin{bmatrix} 1 & a & 0 \\ 1 & 3 & a \\ 0 & 1 & 3 \end{bmatrix}.$$

Thus, according to (7),

$$w^{(1)}_{F_a,1} = \frac{a}{3},\ w^{(1)}_{F_a,2} = \frac{1}{3} \quad \text{and} \quad w^{(2)}_{F_a,1} = \frac{a}{3},\ w^{(2)}_{F_a,2} = \frac{a}{9}.$$

Simple calculations show that the condition $\Delta_1(F_a) \le 0$ is satisfied if and only if

$$a_{\min} \le a \le a_{\max}, \tag{17}$$

where $a_{\min} = 4 - 2\sqrt{3} \approx 0.5359$ and $a_{\max} = 4 + 2\sqrt{3} \approx 7.4641$. It follows from Theorem 4 that for every $a \in [a_{\min}, 2)$ the polynomial F_a is Hadamard irreducible; for $a = 1$ we get a Hadamard irreducible polynomial $g(s) = s^4 + s^3 + 3s^2 + s + 1$ obtained previously by Garloff and Srinivasan [5].

To construct a Hadamard irreducible polynomial of degree $n = 5$, we put $a = 3/2$ (unfortunatly, for $a = 1$ assumption (15) of Theorem 4 is not satisfied) and repeat the above reasoning. As previously, we begin with considering the polynomial:

$$G_a(s) = as^5 + 3/2s^4 + s^3 + 3s^2 + s + 1,$$

and showing that it is stable if and only if

$$0 \le a < a^* = -3/2 + \sqrt{3} \approx 0.2320. \tag{18}$$

One of the submatrices (3) of the Hurwitz matrix H_{G_a} has the form:

$$H_{G_a,1} = \begin{bmatrix} 1 & a & 0 \\ 1 & 1 & 3/2 \\ 0 & 1 & 3 \end{bmatrix},$$

and the condition $\Delta_1(G_a) \le 0$ is satisfied if and only if

$$a_{\min} \le a \le a_{\max}, \tag{19}$$

where $a_{\min} = 3/2 - \sqrt{2} \approx 0.0858$ and $a_{\max} = 3/2 + \sqrt{2} \approx 2.9142$. Combining (18) and (19), we conclude that for every $a \in [a_{\min}, a^*)$ the polynomial G_a is Hadamard irreducible; taking for example $a = 1/5$ we get a quintic polynomial:

$$h(s) = 1/5s^5 + 3/2s^4 + s^3 + 3s^2 + s + 1.$$

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When considering the submatrix:

$$\begin{bmatrix} 1/5 & a & 0 \\ 1 & 3/2 & 1/5 \\ 0 & 1 & 1 \end{bmatrix},$$

of the Hurwitz matrix for the polynomial $H_a(s) = as^6 + h(s)$ and performing similar calculations as above, we show that the polynomial:

$$k\left(s\right) = 1/5s^6 + 1/5s^5 + 3/2s^4 + s^3 + 3s^2 + s + 1,$$

is stable but not Hadamard factorizable.

3.3. A necessary and sufficient condition for Hadamard factorizability of fourth-degree real polynomials. In this last part of our work, we would like to focus the attention on polynomials of degree $n = 4$. We will show that in that case a necessary condition for the Hadamard factorizability of a polynomial given in Theorem 3 is also sufficient.

To do this, let $f, g \in \pi_4^+$ be two polynomials of the form:

$$f\left(s\right) = a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0,$$
$$g\left(s\right) = b_4s^4 + b_3s^3 + b_2s^2 + b_1s + b_0,$$

and let

$$w_{f,1} = \frac{a_1a_4}{a_2a_3}, \quad w_{f,2} = \frac{a_0a_3}{a_2a_1}, \quad w_{g,1} = \frac{b_1b_4}{b_2b_3}, \quad w_{g,2} = \frac{b_0b_3}{b_2b_1}. \tag{20}$$

Remark 1. It follows from the Routh–Hurwitz stability criterion that

(a) $f \in \pi_4^+$ is stable if and only if $w_{f,1} + w_{f,2} < 1$;
(b) for $f, g \in \pi_4^+$, $f \diamond g$ is stable if and only if $w_{f,1}/w_{g,1} + w_{f,2}/w_{g,2} < 1$.

The following theorem holds.

Theorem 5. *A polynomial $f \in \mathcal{H}_4^+$ is Hadamard factorizable if and only if*

$$\left(w_{f,1} - w_{f,2}\right)^2 - 2\left(w_{f,1} + w_{f,2}\right) + 1 > 0, \tag{21}$$

where $w_{f,1}$ and $w_{f,2}$ are as in (20).

Before proceeding to the proof, we need an auxiliary lemma.

Lemma 6. *Suppose that $0 < b_1 < 1$, $0 < b_2 < 1$ and that*

$$\left(b_1 - b_2\right)^2 - 2\left(b_1 + b_2\right) + 1 > 0. \tag{22}$$

Then

1. *there exist $x_1 > 1$ and $x_2 > 1$ satisfying the system of inequalities:*

$$\begin{cases} b_1x_1 + b_2x_2 < 1 \\ \frac{1}{x_1} + \frac{1}{x_2} < 1 \end{cases}; \tag{23}$$

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2. *it holds*

(24) $$\sqrt{b_1} + \sqrt{b_2} < 1.$$

Proof. The first part of the thesis follows from the proof of Theorem 3 where the equivalence of conditions (12) and (14) was derived.

To prove the second part note that the first one implies that $b_1 + b_2 < 1$. Then, (24) is a simple consequence of the following equality:

$$(b_1 - b_2)^2 - 2(b_1 + b_2) + 1 = \left(1 - \left(\sqrt{b_1} + \sqrt{b_2}\right)^2\right)\left(1 - b_1 - b_2 + 2\sqrt{b_1 b_2}\right).$$ □

***Proof of Theorem* 5.** The necessity of (21) for the Hadamard factorizability of f follows from Theorem 3. To prove the sufficiency, we will show that condition (21) implies that there exists a polynomial $g \in \mathcal{H}_4^+$ such that $f \diamond g \in \mathcal{H}_4^+$.

It follows from (21) and from Lemma 6 that there exist $s_1 > 1$, $s_2 > 1$ such that

(25) $$\begin{cases} w_{f,1} \cdot s_1 + w_{f,2} \cdot s_2 < 1 \\ \frac{1}{s_1} + \frac{1}{s_2} < 1 \end{cases}.$$

Let, for any solutions $s_1 > 1$ and $s_2 > 1$ of system (25),

(26) $$g(s) = s^4 + 2s^3 + s^2 + \frac{2}{s_1}s + \frac{1}{s_1 s_2}.$$

According to (20), we have $w_{g,1} = 1/s_1$ and $w_{g,2} = 1/s_2$. It follows from Remark 1(a) and from (25) that polynomial (26) is stable. Besides, as follows from (25),

$$w_{f,1}/w_{g,1} + w_{f,2}/w_{g,2} < 1.$$

It means, by Remark 1(b), that $f \diamond g$ is stable too. This completes the proof. □

As a conclusion, we present the following theorem collecting the necessary and sufficient conditions for Hadamard factorizability of polynomials of degree $n = 4$ derived in this section.

THEOREM 7. *For $f \in \mathcal{H}_4^+$, the following conditions are equivalent:*

(a) *the polynomial $f(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$ is Hadamard factorizable;*
(b) *it holds*
$$(w_{f,1} - w_{f,2})^2 - 2(w_{f,1} + w_{f,2}) + 1 > 0;$$
(c) *it holds*
$$\sqrt{w_{f,1}} + \sqrt{w_{f,2}} < 1;$$
(d) *it holds*
$$\frac{a_0 a_3}{a_1} < \left(\sqrt{\frac{a_1 a_4}{a_3}} - \sqrt{a_2}\right)^2;$$
(e) *the polynomial $g(s) = \sqrt{a_4} s^4 + \sqrt{a_3} s^3 + \sqrt{a_2} s^2 + \sqrt{a_1} s + \sqrt{a_0}$ is stable.*

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Proof. The equivalence $(a) \Leftrightarrow (b)$ follows from Theorem 5, the implication $(b) \Rightarrow (c)$ follows from Lemma 6, the equivalence $(c) \Leftrightarrow (d)$ is straightforward and follows from the stability of f, the equivalence $(c) \Leftrightarrow (e)$ follows from Remark 1 and from the identities $w_{g,1} = \sqrt{w_{f,1}}$ and $w_{g,2} = \sqrt{w_{f,2}}$, and the last implication $(e) \Rightarrow (a)$ follows from the obvious equality $f = g \circ g$. This completes the proof. ∎

Let us point out that alternative proofs of two equivalences stated in Conclusion 7 can be found in some earlier works: the equivalence $(a) \Leftrightarrow (d)$ was previously proven by Loredo–Villalobos and Aguirre–Hernández [8] (see Lemma 3 therein) and $(a) \Leftrightarrow (e)$ by Aguirre–Hernández *et al.* [1] (see Theorem 7 therein).

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