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ON THE KY FAN K-NORM OF THE LI-MATRIX OF GRAPHS*

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Abstract. Let $A(G)$ and $D(G)$ be the adjacency matrix and the degree diagonal matrix of a graph G, respectively. Then $L(G) = D(G) - A(G)$ is called Laplacian matrix of the graph G. Let G be a graph with n vertices and m edges. Then the LI-matrix of G are defined as $LI(G) = L(G) - \frac{2m}{n} I_n$, where I_n is the identity matrix. In this paper, we are interested in extremal properties of the Ky Fan k-norm of the LI-matrix of graphs, which is closely related to the well known problems and results in spectral graph theory, such as the Laplacian spectral radius, the Laplacian spread, the sum of the k largest Laplacian eigenvalues, the Laplacian energy, and other parameters. Some bounds on the Ky Fan k-norm of the LI-matrix of graphs are given, and the extremal graphs are partly characterized. In addition, upper and lower bounds on the Ky Fan k-norm of LI-matrix of trees, unicyclic graphs, and bicyclic graphs are determined, and the corresponding extremal graphs are characterized.

Key words. LI-matrix, Ky Fan k-norm, Laplacian eigenvalues, Singular values, Extremal graph.

AMS subject classifications. 05C50.

1. Introduction. Let G be a simple finite undirected graph with vertex set $V(G)$ and edge set $E(G)$. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G, and its eigenvalues can be arranged as:

$$n \geq \mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0,$$

where $\mu_1(G)$ and $\mu_{n-1}(G)$ are called Laplacian spectral radius and algebraic connectivity of G, respectively. The investigation on the eigenvalues of Laplacian matrix of graphs is a topic of interest in spectral graph theory. There are amount of results on the eigenvalues of $L(G)$ in the literature, such as the Laplacian spectral radius [14, 38], the Laplacian spread $spr(L(G))$ [1, 8, 45], the sum of the k largest Laplacian eigenvalues $S_k(L(G))$ [10, 16, 20], the Laplacian energy $LE(G)$ [5, 6, 17], etc. The $spr(L(G))$, $S_k(L(G))$, and $LE(G)$ are defined as follows:

$$spr(L(G)) = \mu_1(G) - \mu_{n-1}(G), \quad S_k(L(G)) = \sum_{i=1}^{k} \mu_i(G), \quad LE(G) = \sum_{i=1}^{n} \left| \mu_i(G) - \frac{2m}{n} \right|.$$

Recently, the trace norm of the adjacency matrix $A(G)$ of a graph G, defined as the sum of the singular values of $A(G)$, has been extensively studied under the name of graph energy [27]. For generalizing and enriching the study of graph energy, Nikiforov [31, 34] investigated the Ky Fan k-norm of adjacency matrix of a graph G, that is

$$||A(G)||_{F_k} = \sigma_1(A(G)) + \sigma_2(A(G)) + \cdots + \sigma_k(A(G)),$$

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where $\sigma_1(A(G)) \geq \sigma_2(A(G)) \geq \cdots \geq \sigma_n(A(G))$ are the singular values of the adjacency matrix $A(G)$, i.e. the nonnegative square roots of the eigenvalues of $A(G)A^T(G)$. Since the singular values of a real symmetric matrix are the moduli of its eigenvalues, the Ky Fan k-norm of adjacency matrix of a graph G is also the sum of the k largest absolute values of the eigenvalues of $A(G)$. He showed that some well-known problems and results in spectral graph theory are best stated in terms of the Ky Fan k-norm, for example, this norm is related to energy, spread, spectral radius, and other parameters. Thus, he suggested to study arbitrary Ky Fan k-norm of graphs and proposed many interesting questions, especially the maximal Ky Fan k-norm of graphs of given order. Later, Nikiforov has done a series of systematic in-depth analyses and researches for the Ky Fan k-norm, which are not restricted to the adjacency matrix of graphs. One may refer to [31, 32, 33, 34, 36] for more details on the Ky Fan k-norm.

Motivated by the above works, we study the Ky Fan k-norm of the LI-matrix of graphs. Let G be a graph with n vertices and m edges. Then the LI-matrix of G is defined as

$$LI(G) = L(G) - \frac{2m}{n} I_n.$$

By the definition of the Ky Fan k-norm, we have $||LI(G)||_{F_k} = \sum_{i=1}^{k} \sigma_i(LI(G))$, where the singular values of $LI(G)$ are always indexed in decreasing order. Clearly, $||LI(G)||_{F_n} = LE(G)$. Thus, a close examination of $||LI(G)||_{F_k}$ further advances the study of Laplacian energy of graphs. In particular, $\sigma_1(LI(G))$ is called the spectral norm of the LI-matrix. Moreover, if G is a regular graph, then $||LI(G)||_{F_k} = ||A(G)||_{F_k}$. From a geometric perspective, the Ky Fan k-norm of the LI-matrix of graphs represents the ordered sum of the distance between Laplacian eigenvalues and the average of all Laplacian eigenvalues, which is relevant to the hard problem that distribution of Laplacian eigenvalues of graphs in spectral graph theory, see [22].

In this paper, the extremal properties of the Ky Fan k-norm of the LI-matrix of graphs are studied. Around the following Nikiforov's question, upper and lower bounds on the Ky Fan k-norm of LI-matrix of trees, unicyclic graphs, and bicyclic graphs are given, and the corresponding extremal graphs are characterized, which integrates previous results on the Laplacian spectral radius, the Laplacian spread and the sum of the k largest Laplacian eigenvalues of trees, unicyclic graphs, and bicyclic graphs.

QUESTION 1.1. (**[31]**) *Study the extrema of the Ky Fan k-norm of a graph G and their relations to the structure of G.*

The rest of the paper is organized as follows. In Section 2, we introduce some notions and lemmas which we need to use in the proofs of our results. In Section 3, some properties on $\sigma_1(LI(G))$, $\sigma_n(LI(G))$, and $||LI(G)||_{F_2}$ of a graph G are obtained. In Section 4, some bounds on $||LI(G)||_{F_k}$ of a graph G are presented, and the extremal graphs are partly characterized. In Section 5, lower and upper bounds on the Ky Fan k-norm of LI-matrix of trees, unicyclic graphs, and bicyclic graphs are obtained, and the corresponding extremal graphs are characterized.

2. Preliminaries. Denote by K_n, P_n, C_n, and $K_{1,\,n-1}$ the complete graph, path, cycle, and star with n vertices, respectively. For $v_i \in V(G)$, $d_G(v_i) = d_i(G)$ denotes the degree of vertex v_i in G. The minimum and the maximum degrees of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We assume that $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$ and say that $d = (d_1(G), d_2(G), \ldots, d_n(G))$ is the degree sequence of the graph G. The conjugate of a degree sequence d is the sequence $d^* = (d_1^*(G), d_2^*(G), \ldots, d_n^*(G))$ where $d_i^*(G) = |\{j : d_j(G) \geq i\}|$ is the number of vertices of G of degree at least i. For a graph G, the first Zagreb index $Z_1 = Z_1(G)$ is defined as the sum of the squares of the vertices degrees.

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A threshold graph may be obtained through an iterative process which starts with an isolated vertex, and at each step, either a new isolated vertex is added or a vertex adjacent to all previous vertices (dominating vertex) is added. The double star $S_{a,\,b}$ is the tree obtained from K_2 by attaching a pendant edges to a vertex and b pendant edges to the other. A connected graph is called a c-cyclic graph if it contains n vertices and $n+c-1$ edges. Specially, if $c = 0$, 1 or 2, then G is called a tree, a unicyclic graph, or a bicyclic graph, respectively. The $G_{m,\,n}$, shown in Fig. 1, is a graph with n vertices and m edges which has $m-n+1$ triangles with a common edge and $2n-m+3$ pendent edges incident with one end vertex of the common edge. The join graph $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

Lemma 2.1. ([15, 23]) *Let G be a graph with n vertices and at least one edge. Then $\Delta+1 \leq \mu_1(G) \leq n$. The left equality for connected graph holds if and only if $\Delta = n-1$, and the right equality holds if and only if the complement of G is disconnected.*

Lemma 2.2. ([40]) *Let A be an $m \times n$ matrix with singular values $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min\{m,\,n\}}$. Let B be a $p \times q$ submatrix of A, with singular values $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{\min\{p,\,q\}}$. Then, $\alpha_i \geq \beta_i \geq \alpha_{i+(m-p)+(n-q)}$ for $i = 1, 2, \ldots, \min\{p,\,q\}$ and $i \leq \min\{p+q-m, p+q-n\}$.*

Lemma 2.3. ([2]) *Let G be a graph and let H be a (not necessarily induced) subgraph of G with p vertices. Then, $\mu_i(G) \geq \mu_i(H)$ for $1 \leq i \leq p$.*

Lemma 2.4. ([3, 26]) *Let G be a graph on n vertices, with vertex degrees $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$. If G is not $K_2 \cup (n-2)K_1$, then $\mu_2(G) \geq d_2(G)$.*

Lemma 2.5. ([29]) *Let G be a threshold graph on n vertices with conjugate degree sequence $d^* = (d_1^*(G), d_2^*(G), \ldots, d_n^*(G))$. Then the Laplacian eigenvalue $\mu_i(G) = d_i^*(G) = d_i(G)+1$, $1 \leq i \leq n-1$.*

Lemma 2.6. ([42]) *Let $\mathcal{T}_n$ the set of trees on n vertices. Then,*

$$\mu_1(T_n) < \mu_1(S_{3,\,n-5}) < \mu_1(T_n^4) < \mu_1(T_n^3) < \mu_1(S_{2,\,n-4}) < \mu_1(S_{1,\,n-3}) < \mu_1(K_{1,\,n-1}),$$

for $T_n \in \mathcal{T}_n \setminus \{K_{1,\,n-1}, S_{1,\,n-3}, S_{2,\,n-4}, T_n^3, T_n^4, S_{3,\,n-5}\}$ and T_n^i $(i = 3, 4)$ shown in Fig. 1, where $\mu_1(S_{1,\,n-3})$, $\mu_1(S_{2,\,n-4})$, respectively, are the largest root of the following equations:

$$x^3 - (n+2)x^2 + (3n-2)x - n = 0,$$
$$x^3 - (n+2)x^2 + (4n-7)x - n = 0.$$

Lemma 2.7. ([18]) *For any tree T_n with $n \geq 4$ vertices, $S_2(L(T_n)) \leq S_2(L(S_{\lceil \frac{n-2}{2} \rceil,\, \lfloor \frac{n-2}{2} \rfloor}))$. The equality holds if and only if $T_n \cong S_{\lceil \frac{n-2}{2} \rceil,\, \lfloor \frac{n-2}{2} \rfloor}$.*

Lemma 2.8. ([24]) *Let G be a connected graph with $n \geq 12$. Then, $\mu_1(G)+\mu_2(G) \geq 4+2\left(\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}\right)$ with equality holding if and only if $G \cong P_n$.*

Lemma 2.9. ([12]) *Let T_n be a tree with $n \geq 5$ vertices. Then,*

$$spr(L(P_n)) \leq spr(L(T_n)) \leq spr(L(K_{1,\,n-1})).$$

The equality in the left-hand side holds if and only if $T_n \cong P_n$, and the equality in the right-hand side holds if and only if $T_n \cong K_{1,n-1}$.

Lemma 2.10. ([13, 43]) *Let $\mathcal{U}_n$ the set of unicyclic graphs on n vertices. Then,*

$$\mu_1(U_n) < \mu_1(G_{n,\,n}) \quad \textit{and} \quad S_2(U_n) < S_2(G_{n,\,n}),$$

for $U_n \in \mathcal{U}_n \setminus \{G_{n,\,n}\}$, $G_{n,\,n}$ shown in Fig. 1.

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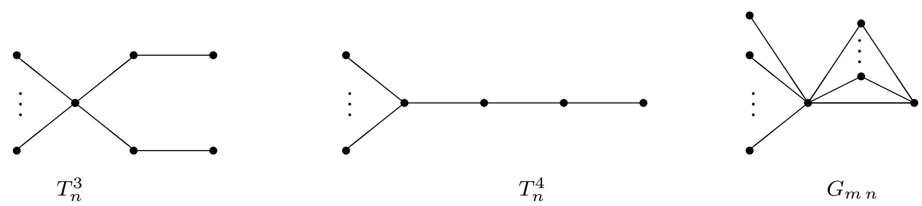

FIGURE 1. *Graphs T_n^3, T_n^4 and $G_{m\,n}$.*

LEMMA 2.11. ([4, 41]) *Let U_n be a unicyclic graph with $n \geq 4$. Then,*

$$spr(L(C_n)) \leq spr(L(U_n)) \leq spr(L(G_{n,\,n})).$$

The equality in the left-hand side holds if and only if $U_n \cong C_n$, and the equality in the right-hand side holds if and only if $U_n \cong G_{n,\,n}$.

LEMMA 2.12. ([11, 21, 44]) *Let $\mathcal{B}_n$ the set of bicyclic graphs on n vertices. Then,*

(i) *$\mu_1(B_n) < \mu_1(B_n^*)$ for $B_n \in \mathcal{B}_n \setminus \{B_n^*\}$, where B_n^* is obtained from a star of order n by adding two edges.*

(ii) *$spr(L(B_n)) < spr(L(B_n^*))$ for $B_n \in \mathcal{B}_n \setminus \{B_n^*\}$.*

(iii) *$S_2(L(B_n)) < S_2(L(G_{n+1,\,n}))$ for $B_n \in \mathcal{B}_n \setminus \{G_{n+1,\,n}\}$, $G_{n+1,\,n}$ shown in Fig. 1.*

3. Some properties on $\sigma_1(LI)$, $\sigma_n(LI)$ and $||LI||_{F_2}$ of a graph.

THEOREM 3.1. *Let G be a graph with n vertices and $m \geq 1$ edges.*

(i) *If $m \geq \frac{n^2}{4}$, then $\sigma_1(LI(G)) = \frac{2m}{n}$.*

(ii) *If $\Delta \geq \frac{4m}{n} - 1$, then $\sigma_1(LI(G)) = \mu_1(G) - \frac{2m}{n}$.*

(iii) *If G is a connected r-regular graph, then $\sigma_1(LI(G)) = \frac{2m}{n}$.*

(iv) *If G is a bipartite graph, then $\sigma_1(LI(G)) = \mu_1(G) - \frac{2m}{n}$.*

Proof. (i) If $m \geq \frac{n^2}{4}$, by Lemma 2.1, we have $\mu_1(G) \leq n \leq \frac{4m}{n}$. Thus,

$$\sigma_1(LI(G)) = \max\left\{\mu_1(G) - \frac{2m}{n}, \frac{2m}{n}\right\} = \frac{2m}{n}.$$

(ii) If $\Delta \geq \frac{4m}{n} - 1$, by Lemma 2.1, we have $\mu_1(G) \geq \Delta + 1 \geq \frac{4m}{n}$. Thus,

$$\sigma_1(LI(G)) = \max\left\{\mu_1(G) - \frac{2m}{n}, \frac{2m}{n}\right\} = \mu_1(G) - \frac{2m}{n}.$$

(iii) If G is a connected r-regular graph, then $\mu_1(G) = r - \lambda_n(G)$, where $\lambda_n(G)$ is the least eigenvalue of the adjacency matrix of G. If $\mu_1(G) = r - \lambda_n(G) > \frac{4m}{n} = 2r$, that is ,$|\lambda_n(G)| > r = \lambda_1(G)$, a contradiction. Thus, $\mu_1(G) \leq \frac{4m}{n}$. Further, $\sigma_1(LI(G)) = \frac{2m}{n}$.

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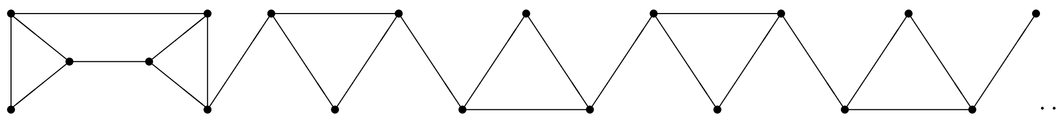

FIGURE 3.1. *The graph H_n ($n = 3t$, $t \geq 2$ is the number of triangles).*

(iv) It is well known that the spectra of Laplacian matrix and signless Laplacian matrix coincide if and only if the graph G is bipartite. From Lemma 2.1 in [35], we have $\mu_1(G) \geq \frac{4m}{n}$. Thus, $\sigma_1(LI(G)) = \mu_1(G) - \frac{2m}{n}$.

This completes the proof. ∎

COROLLARY 3.2. *Let G be a graph with n vertices and m edges. If $\mu_1(G) < \frac{4m}{n}$, then G contains odd cycles.*

REMARK 3.3. *There exist nonbipartite graphs for which the equality $\sigma_1(LI(G)) = \mu_1(G) - \frac{2m}{n}$ holds for $m < \frac{n^2}{4}$. A bicyclic graph with $n \geq 17$ vertices is an example. However, there also exist nonregular graphs for which the equality $\sigma_1(LI(G)) = \frac{2m}{n}$ holds for $m < \frac{n^2}{4}$. The graph H_n with n vertices and $\frac{4}{3}n$ edges, depicted in Fig. 3.1, is an example. By direct calculations, we have $\mu_1(H_n) = 5 < \frac{16}{3} = \frac{4m}{n}$. Thus, $\sigma_1(LI(G)) = \frac{2m}{n} = \frac{8}{3}$. It is interesting to characterize the graphs satisfying $\sigma_1(LI(G)) = \frac{2m}{n}$ for $m < \frac{n^2}{4}$.*

QUESTION 3.4. *Characterize all graphs G satisfying $\sigma_1(LI(G)) = \frac{2m}{n}$ for $m < \frac{n^2}{4}$.*

THEOREM 3.5. *Let G be a graph on $n > 4$ vertices. Then,*

$$\mu_1(G) + \mu_2(G) > \frac{4m}{n} + 1.$$

Proof. Let $(d_1(G), d_2(G), \ldots, d_n(G))$ be degree sequence of G. If G is not a r-regular graph, by Lemmas 2.1 and 2.4, we have

$$\mu_1(G) + \mu_2(G) \geq d_1(G) + d_2(G) + 1 > \frac{2(d_1(G) + d_2(G) + \cdots + d_n(G))}{n} + 1 = \frac{4m}{n} + 1.$$

If G is a r-regular graph, then $\mu_1(G) + \mu_2(G) = 2r - \lambda_{n-1}(G) - \lambda_n(G)$, where $\lambda_{n-1}(G)$ and $\lambda_n(G)$ are the second least eigenvalue and the least eigenvalue of $A(G)$, respectively. Clearly, $\lambda_{n-1}(G) \leq 0$ and $\lambda_n(G) \leq -\frac{1+\sqrt{5}}{2}$ (see, e.g. [19]). Thus,

$$\mu_1(G) + \mu_2(G) = 2r - \lambda_{n-1}(G) - \lambda_n(G) \geq 2r + \frac{1+\sqrt{5}}{2} > \frac{4m}{n} + 1.$$

From the above arguments, we have the proof. ∎

COROLLARY 3.6. *Let G be a graph on $n > 4$ vertices. If $\sigma_{n-1}(LI(G)) > \sigma_n(LI(G))$, then*

$$\sigma_n(LI(G)) \neq \mu_1(G) - \frac{2m}{n}.$$

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Proof. Suppose that $\sigma_n(LI(G)) = \mu_1(G) - \frac{2m}{n}$. Then,

$$\sigma_n(LI(G)) = \mu_1(G) - \frac{2m}{n} < \sigma_{n-1}(LI(G)) \le \left|\mu_2(G) - \frac{2m}{n}\right|.$$

If $\mu_2(G) \ge \frac{2m}{n}$, then $\mu_1(G) < \mu_2(G)$, a contradiction. If $\mu_2(G) < \frac{2m}{n}$, then $\mu_1(G) + \mu_2(G) < \frac{4m}{n}$, a contradiction. Therefore, $\sigma_n(LI(G)) \ne \mu_1(G) - \frac{2m}{n}$. This completes the proof. ∎

THEOREM 3.7. *Let G be a graph with n vertices and $m \ge 1$ edges. For any edge $uv \in E(G)$, we have*

$$||LI(G)||_{F_2} \ge \left|\frac{d_u + d_v + \sqrt{(d_u + d_v)^2 + 4}}{2} - \frac{2m}{n}\right| + \left|\frac{d_u + d_v - \sqrt{(d_u + d_v)^2 + 4}}{2} - \frac{2m}{n}\right|.$$

Proof. Let $uv \in E(G)$. By Lemma 2.2, we have $\sigma_1(LI(G)) \ge \sigma'_1$ and $\sigma_2(LI(G)) \ge \sigma'_2$, where σ'_1, σ'_2 are the singular values of the matrix

$$\begin{pmatrix} d_u - \frac{2m}{n} & -1 \\ -1 & d_v - \frac{2m}{n} \end{pmatrix}.$$

Thus,

$$\begin{aligned} ||LI(G)||_{F_2} &= \sigma_1(LI(G)) + \sigma_2(LI(G)) \\ &\ge \sigma'_1 + \sigma'_2 \\ &= \left|\frac{d_u + d_v + \sqrt{(d_u + d_v)^2 + 4}}{2} - \frac{2m}{n}\right| + \left|\frac{d_u + d_v - \sqrt{(d_u + d_v)^2 + 4}}{2} - \frac{2m}{n}\right|. \end{aligned}$$

This completes the proof. ∎

THEOREM 3.8. *Let G be a triangle-free graph with n vertices and $m \ge 1$ edges. For any edge $uv \in E(G)$, we have*

$$||LI(G)||_{F_2} \ge \sqrt{\Upsilon + 2\sqrt{\Psi}},$$

where $\Upsilon = \left(d_u - \frac{2m}{n}\right)^2 + \left(d_v - \frac{2m}{n}\right)^2 + d_u + d_v$, $\Psi = \left(\left(d_u - \frac{2m}{n}\right)^2 + d_u\right)\left(\left(d_v - \frac{2m}{n}\right)^2 + d_v\right) - \left(d_u + d_v - \frac{4m}{n}\right)^2$.

Proof. Let $uv \in E(G)$. By Lemma 2.2, we have $\sigma_1(LI(G)) \ge \sigma'_1$ and $\sigma_2(LI(G)) \ge \sigma'_2$, where σ'_1, σ'_2 are the singular values of the matrix

$$B = \begin{pmatrix} d_u - \frac{2m}{n} & -1 & * & \cdots & * \\ -1 & d_v - \frac{2m}{n} & * & \cdots & * \end{pmatrix}_{2\times n}.$$

Since G is a triangle-free graph, we have

$$BB^T = \begin{pmatrix} (d_u - \frac{2m}{n})^2 + d_u & \frac{4m}{n} - d_u - d_v \\ \frac{4m}{n} - d_u - d_v & (d_v - \frac{2m}{n})^2 + d_v \end{pmatrix}.$$

Thus, the eigenvalues x_1 and x_2 of BB^T are the roots of the following equations:

$$\begin{aligned} &x^2 - \left(\left(d_u - \frac{2m}{n}\right)^2 + \left(d_v - \frac{2m}{n}\right)^2 + d_u + d_v\right)x \\ &\quad + \left(\left(d_u - \frac{2m}{n}\right)^2 + d_u\right)\left(\left(d_v - \frac{2m}{n}\right)^2 + d_v\right) - \left(d_u + d_v - \frac{4m}{n}\right)^2 = 0. \end{aligned}$$

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By Lemma 2.2, we have

$$\begin{aligned}||LI(G)||_{F_2} &= \sigma_1(LI(G)) + \sigma_2(LI(G)) \\ &\geq \sigma_1' + \sigma_2' \\ &= \sqrt{x_1} + \sqrt{x_2} \\ &= \sqrt{x_1 + x_2 + 2\sqrt{x_1 x_2}} \\ &= \sqrt{\Upsilon + 2\sqrt{\Psi}},\end{aligned}$$

where $\Upsilon = \left(d_u - \frac{2m}{n}\right)^2 + \left(d_v - \frac{2m}{n}\right)^2 + d_u + d_v$, $\Psi = \left(\left(d_u - \frac{2m}{n}\right)^2 + d_u\right)\left(\left(d_v - \frac{2m}{n}\right)^2 + d_v\right) - \left(d_u + d_v - \frac{4m}{n}\right)^2$. This completes the proof. ∎

4. Bounds on $||LI||_{F_k}$ of a graph.

THEOREM 4.1. *Let G be a connected graph on $n \geq 3$ vertices and m edges, and let $\{v_1, v_2, \dots, v_k\} \subset V(G)$ and $d_i(G) \geq \frac{4m}{n}$ for $1 \leq i \leq k$. If $v_i v_j \notin E(G)$ for $1 \leq i, j \leq k$, or $v_i v_j \in E(G)$ and $v_i v_h \notin E(G)$ and $v_j v_g \notin E(G)$ for $1 \leq i, j, h, g \leq k$, then*

$$||LI(G)||_{F_k} = S_k(L(G)) - \frac{2km}{n}.$$

Proof. If $v_i v_j \notin E(G)$ for $1 \leq i, j \leq k$, then the graph $K_{1,s_1} \cup K_{1,s_2} \cup \cdots \cup K_{1,s_k}$ is a subgraph of G, where each star K_{1,s_i} centered on v_i, $s_i \geq \frac{4m}{n}$ and $1 \leq i \leq k$. By Lemma 2.3, we have

$$\mu_i(G) \geq \mu_i(K_{1,s_1} \cup K_{1,s_2} \cup \cdots \cup K_{1,s_k}) = \mu_1(K_{1,s_i}) \geq \frac{4m}{n} + 1,$$

for $i = 1, 2, \dots, k$. Thus, we have

$$||LI(G)||_{F_k} = \sum_{i=1}^{k} \sigma_i(LI(G)) = \sum_{i=1}^{k} \left|\mu_k(G) - \frac{2m}{n}\right| = S_k(L(G)) - \frac{2km}{n}.$$

If $v_i v_j \in E(G)$ and $v_i v_h \notin E(G)$ and $v_j v_g \notin E(G)$ for $1 \leq i, j, h, g \leq k$, then the graph $S_{a_1,b_1} \cup S_{a_2,b_2} \cup \cdots S_{a_p,b_p} \cup K_{1,s_1} \cup \cdots K_{1,s_q}$ is a subgraph of G, where each double star S_{a_p,b_p} centered on v_i and v_j, $a_p, b_p \geq \frac{4m}{n} - 1$, $s_q \geq \frac{4m}{n}$, $2p + q = k$, $1 \leq p \leq \frac{k}{2}$ and $0 \leq q \leq k-2$. By Lemmas 2.3 and 2.4, we have

$$\begin{aligned}\mu_i(G) &\geq \mu_i(S_{a_1,b_1} \cup S_{a_2,b_2} \cup \cdots S_{a_p,b_p} \cup K_{1,s_1} \cup \cdots K_{1,s_q}) \\ &\geq \min\{\mu_1(S_{a_p,b_p}), \mu_2(S_{a_p,b_p}), \mu_1(K_{1,s_q})\} \\ &\geq \frac{4m}{n},\end{aligned}$$

for $i = 1, 2, \dots, k$. Thus, we have

$$||LI(G)||_{F_k} = \sum_{i=1}^{k} \sigma_i(LI(G)) = \sum_{i=1}^{k} \left|\mu_k(G) - \frac{2m}{n}\right| = S_k(L(G)) - \frac{2km}{n}.$$

Combining the above arguments, we have the proof. ∎

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THEOREM 4.2. *Let G be a graph on n vertices and m edges, with vertex degrees $d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)$. If the subgraph H of G is a threshold graph with $d_k(H) \geq \frac{4m}{n} - 1$, then*

$$||LI(G)||_{F_k} = S_k(L(G)) - \frac{2km}{n}.$$

Proof. If $d_k(H) \geq \frac{4m}{n} - 1$, by Lemmas 2.3 and 2.5, then $\mu_k(G) \geq \mu_k(H) \geq \frac{4m}{n}$. Thus, we have

$$||LI(G)||_{F_k} = \sum_{i=1}^{k} \sigma_i(LI(G)) = \sum_{i=1}^{k} \left|\mu_k(G) - \frac{2m}{n}\right| = S_k(L(G)) - \frac{2km}{n}.$$

This completes the proof. $\square$

THEOREM 4.3. *Let G be a graph with n vertices and m edges. Then,*

$$||LI(G)||_{F_k} \leq S_k(L(\overline{G})) + \frac{2m}{n} + (k-1)(n - \frac{2m}{n}).$$

The equality holds if $G = K_n$.

Proof. Let J_n be the all ones matrix of size n. Then, $-L(\overline{G}) = LI(G) + W_n$ and $W_n = (\frac{2m}{n} - n)I_n + J_n$. Hence, the triangle inequality implies that

$$||LI(G)||_{F_k} \leq ||L(\overline{G})||_{F_k} + ||W_n||_{F_k},$$

that is,

$$||LI(G)||_{F_k} \leq S_k(L(\overline{G})) + \frac{2m}{n} + (k-1)(n - \frac{2m}{n}).$$

This completes the proof. $\square$

THEOREM 4.4. *Let $G \neq K_n$ be a graph on n vertices and $m \geq 1$ edges. If $m \geq \frac{n^2}{4}$ and $k \geq 2$, then*

$$||LI(G)||_{F_k} \leq \frac{2m}{n} + (k-1)spr(L(G)),$$

with equality if and only if G is a r-regular graph, and $G = G_1 \vee (K_1 \cup G_2)$, $|V(G_2)| = n - r - 1$, $\mu_{n-1}(G_1) \geq 2r - n$ and $\sigma_2(LI(G)) = \cdots = \sigma_k(LI(G)) = \mu_1(G) - r$.

Proof. Since $m \geq \frac{n^2}{4}$, by Theorem 3.1, we have $\sigma_1(LI(G)) = \frac{2m}{n}$. Let $x + y = k - 1$, $0 \leq x, y \leq k - 1$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
||LI(G)||_{F_k} &= \sum_{i=1}^{k} \sigma_i(LI(G)) \\
&\leq \frac{2m}{n} + x(\mu_1(G) - \frac{2m}{n}) + y(\frac{2m}{n} - \mu_{n-1}(G)) \\
&\leq \frac{2m}{n} + \sqrt{x^2 + y^2}\sqrt{(\mu_1(G) - \frac{2m}{n})^2 + (\frac{2m}{n} - \mu_{n-1}(G))^2} \\
&\leq \frac{2m}{n} + (k-1)spr(L(G)),
\end{aligned}$$

with equality if and only if $\sigma_2(LI(G)) = \cdots = \sigma_k(LI(G)) = \mu_1(G) - \frac{2m}{n}$, $x = k - 1$ and $\mu_{n-1}(G) = \frac{2m}{n}$. Since $\mu_{n-1}(G) = \frac{2m}{n}$ and $\mu_{n-1}(G) \leq \delta(G)$ for $G \neq K_n$ (see, e.g. [9]), we have that G is a r-regular graph

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with $\mu_{n-1}(G) = r$. From Theorem 1 in [25], we have $\mu_{n-1}(G) = r$ if and only if $G = G_1 \vee (K_1 \cup G_2)$, $|V(G_2)| = n - r - 1$ and $\mu_{n-1}(G_1) \geq 2r - n$. This completes the proof. ∎

REMARK 4.5. *If $G = C_4 \vee (K_1 \cup K_1)$, then $||LI(G)||_{F_k} = \frac{2m}{n} + (k-1)spr(L(G))$ for $k = 1, 2, 3$.*

THEOREM 4.6. *Let G be a graph with n vertices and m edges. If $k \geq 2$ and $m \geq \frac{n^2}{4}$, then*

$$||LI(G)||_{F_k} \leq \frac{2m}{n} + \sqrt{(k-1)\left(2m + Z_1 - \frac{4m^2}{n} - \frac{4m^2}{n^2}\right)}. \tag{4.2}$$

The equality holds in (4.2) if and only if G is a graph satisfying

$$\begin{cases} \sigma_2(LI(G)) = \sigma_3(LI(G)) = \cdots = \sigma_k(LI(G)), \\ \sigma_{k+1}(LI(G)) = \sigma_{k+2}(LI(G)) = \cdots = \sigma_n(LI(G)) = 0. \end{cases}$$

Proof. Since $m \geq \frac{n^2}{4}$, by Theorem 3.1, we have $\sigma_1(LI(G)) = \frac{2m}{n}$. Since $\sum_{i=1}^{n} \sigma_i^2(LI(G)) = 2m + Z_1 - \frac{4m^2}{n}$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} ||LI(G)||_{F_k} &= \sum_{i=1}^{k} \sigma_i(LI(G)) \\ &\leq \frac{2m}{n} + \sqrt{(k-1)\sum_{i=2}^{k} \sigma_i^2(LI(G))} \\ &\leq \frac{2m}{n} + \sqrt{(k-1)\sum_{i=2}^{n} \sigma_i^2(LI(G))} \\ &= \frac{2m}{n} + \sqrt{(k-1)\left(2m + Z_1 - \frac{4m^2}{n} - \frac{4m^2}{n^2}\right)}. \end{aligned}$$

Hence, the equality holds in (4.2) if and only if G is a graph satisfying

$$\begin{cases} \sigma_2(LI(G)) = \sigma_3(LI(G)) = \cdots = \sigma_k(LI(G)), \\ \sigma_{k+1}(LI(G)) = \sigma_{k+2}(LI(G)) = \cdots = \sigma_n(LI(G)) = 0. \end{cases}$$

The proof is completed. ∎

REMARK 4.7. *For a fixed k, we can use complete regular r-partite graphs or threshold graphs to construct graphs such that the equality holds in (4.2). However, it is an open problem to find all the graphs such that the equality holds in (4.2).*

THEOREM 4.8. *Let G be a graph with n vertices and $m > 1$ edges, and let d_2 be the second largest degree of G. If $d_2 \geq \frac{4m}{n}$, then*

$$||LI(G)||_{F_k} \leq kn - \frac{2km}{n}. \tag{4.3}$$

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Proof. Since $d_2 \geq \frac{4m}{n}$, by Lemma 2.4, we have $|\mu_2 - \frac{2m}{n}| \geq |\mu_i - \frac{2m}{n}|$ for $i = 3, 4, \ldots, n$. By Lemma 2.1, we have

$$\begin{aligned} ||LI(G)||_{F_k} &= \sum_{i=1}^{k} \sigma_i(LI(G)) \\ &\leq \mu_1(G) - \frac{2m}{n} + (k-1)\left(\mu_2(G) - \frac{2m}{n}\right) \\ &= \mu_1(G) + (k-1)\mu_2(G) - \frac{2km}{n} \\ &\leq kn - \frac{2km}{n}. \end{aligned}$$

This completes the proof. ∎

Remark 4.9. *If G is a complete split graph $K_k \vee (n-k)K_1$ with $k \leq \lfloor \frac{2n-1-\sqrt{2n^2-2n+1}}{2} \rfloor$, then the equality in* (4.3) *holds.*

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two nonincreasing sequences of real numbers. If $\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_i$ for $j = 1, 2, \ldots, n$, then we say that x is weakly majorized by y and denote $x \prec_w y$. If in addition to $x \prec_w y$, $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ holds, then we say that x is majorized by y and denote $x \prec y$. From [30], if $f(t)$ is a convex function, then $x \prec y$ implies $(f(x_1), f(x_2), \ldots, f(x_n)) \prec_w (f(y_1), f(y_2), \ldots, f(y_n))$.

Theorem 4.10. *Let G be a graph with n vertices and m edges. If $m \geq \frac{n^2}{4}$ and there is α such that $\mu_1(G) \geq \alpha \geq \frac{2m}{n-1}$, then*

$$||LI(G)||_{F_k} \geq \frac{2m}{n} + \left|\alpha - \frac{2m}{n}\right| + (k-2)\left|\frac{2m-\alpha}{n-2} - \frac{2m}{n}\right|,$$

for $k \geq 2$.

Proof. Let $x = (\alpha, \frac{2m-\alpha}{n-2}, \ldots, \frac{2m-\alpha}{n-2}, 0)$, $y = (\mu_1(G), \mu_2(G), \ldots, \mu_{n-1}(G), 0) \in \mathbb{R}^n$. Then, $x \prec y$. By Theorem 3.1, we have $\sigma_1(LI(G)) = \frac{2m}{n}$ for $m \geq \frac{n^2}{4}$. Since $f(t) = |t - \frac{2m}{n}|$ is a convex function, we have

$$\begin{aligned} ||LI(G)||_{F_k} &= \sigma_1(LI(G)) + \sigma_2(LI(G)) + \cdots + \sigma_k(LI(G)) \\ &\geq \frac{2m}{n} + \left|\mu_1(G) - \frac{2m}{n}\right| + \cdots + \left|\mu_{k-1}(G) - \frac{2m}{n}\right| \\ &\geq \frac{2m}{n} + \left|\alpha - \frac{2m}{n}\right| + (k-2)\left|\frac{2m-\alpha}{n-2} - \frac{2m}{n}\right|, \end{aligned}$$

for $k \geq 2$. This completes the proof. ∎

Corollary 4.11. *Let G be a graph with n vertices and m edges. If $m \geq \frac{n^2}{4}$, then*

$$||LI(G)||_{F_k} \geq \Delta + 1 + (k-2)\left|\frac{2m-\Delta-1}{n-2} - \frac{2m}{n}\right|,$$

for $k \geq 2$.

Proof. By Lemma 2.1 and Theorem 4.10, we have the proof. ∎

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5. On the $||LI||_{F_k}$ of c-cyclic graphs.

THEOREM 5.1. *Let T_n be a tree with n vertices. Then,*

$$\sigma_1(LI(P_n)) < \sigma_1(LI(T_n)) < \sigma_1(LI(S_{3,\,n-5})) < \sigma_1(LI(T_n^4)) < \sigma_1(LI(T_n^3)) \\ < \sigma_1(LI(S_{2,\,n-4})) < \sigma_1(LI(S_{1,\,n-3})) < \sigma_1(LI(K_{1,\,n-1})),$$

for $T_n \in \mathcal{T}_n \setminus \{K_{1,\,n-1}, S_{1,\,n-3}, S_{2,\,n-4}, T_n^3, T_n^4, S_{3,\,n-5}, P_n\}$.

Proof. By Theorems 3.1, we have $||LI(T_n)||_{F_1} = ||QI(T_n)||_{F_1} = \mu_1(T_n) - 2 + \frac{2}{n}$. It is well known that $\mu_1(P_n) \le \mu_1(T_n)$ for all trees with n vertices (see, e.g. [37]). By Lemma 2.6, we have

$$\sigma_1(LI(P_n)) < \sigma_1(LI(T_n)) < \sigma_1(LI(S_{3,\,n-5})) < \sigma_1(LI(T_n^4)) < \sigma_1(LI(T_n^3)) \\ < \sigma_1(LI(S_{2,\,n-4})) < \sigma_1(LI(S_{1,\,n-3})) < \sigma_1(LI(K_{1,\,n-1})),$$

for $T_n \in \mathcal{T}_n \setminus \{K_{1,\,n-1}, S_{1,\,n-3}, S_{2,\,n-4}, T_n^3, T_n^4, S_{3,\,n-5}, P_n\}$. This completes the proof. $\square$

THEOREM 5.2. *Let T_n be a tree with $n \ge 12$ vertices. Then,*

$$||LI(P_n)||_{F_2} < ||LI(T_n)||_{F_2} < ||LI(S_{\lceil \frac{n-2}{2} \rceil,\, \lfloor \frac{n-2}{2} \rfloor})||_{F_2} < ||LI(S_{1,\,n-3})||_{F_2} < ||LI(K_{1,\,n-1})||_{F_2}$$

for $T_n \in \mathcal{T}_n \setminus \{K_{1,\,n-1}, S_{1,\,n-3}, S_{\lceil \frac{n-2}{2} \rceil,\, \lfloor \frac{n-2}{2} \rfloor}, P_n\}$.

Proof. For the upper bounds, we consider two cases depending on $\sigma_2(LI(T_n))$.

Case 1. $\sigma_2(LI(T_n)) = \mu_2(T_n) - 2 + \frac{2}{n}$. Then $||LI(T_n)||_{F_2} = \mu_1(T_n) + \mu_2(T_n) - 4 + \frac{4}{n}$. By Lemma 2.7, we have

$$||LI(T_n)||_{F_2} \le S_2(L(S_{\lceil \frac{n-2}{2} \rceil,\, \lfloor \frac{n-2}{2} \rfloor})) - 4 + \frac{4}{n}.$$

Case 2. $\sigma_2(LI(T_n)) = 2 - \frac{2}{n}$. Then, $||LI(T_n)||_{F_2} = \mu_1(T_n)$. Let $d_2(T_n)$ be the second largest degree of T_n. If $d_2(T_n) \ge 4$, by Lemma 2.4, then $\left|\mu_2(T_n) - \frac{2m}{n}\right| \ge \left|\mu_i(T_n) - \frac{2m}{n}\right|$ for $i = 3, 4, \ldots, n$, that is, $\sigma_2(LI(T_n)) = \mu_2(T_n) - 2 + \frac{2}{n}$, a contradiction. Thus, $d_2(T_n) \le 3$. By Lemma 2.6, we have

$$\mu_1(T_n) < \mu_1(S_{3,\,n-5}) < \mu_1(T_n^4) < \mu_1(T_n^3) < \mu_1(S_{2,\,n-4}) < \mu_1(S_{1,\,n-3}) < \mu_1(K_{1,\,n-1}),$$

for $T_n \in \mathcal{T}_n \setminus \{K_{1,\,n-1}, S_{1,\,n-3}, S_{2,\,n-4}, T_n^3, T_n^4, S_{3,\,n-5}\}$ and T_n^i $(i = 3, 4)$ shown in Fig. 1, where $\mu_1(S_{1,\,n-3})$, $\mu_1(S_{2,\,n-4})$, respectively, are the largest root of the following polynomials:

$$f_1(x) = x^3 - (n+2)x^2 + (3n-2)x - n, \\ f_2(x) = x^3 - (n+2)x^2 + (4n-7)x - n.$$

By derivative, we know that $f_1'(x) > 0$ for $x \in (n-1, +\infty)$. Therefore, $f_1(x)$ is strictly increasing on $(n-1, +\infty)$. Since $f_1(n-1) = -1 < 0$ and $f_1(n) = n(n-3) > 0$ for $n \ge 10$, we have $n-1 < \mu_1(S_{1,\,n-3}) < n$.

By derivative, we know that $f_2'(x) > 0$ for $x \in (n-2, +\infty)$. Therefore, $f_2(x)$ is strictly increasing on $(n-2, +\infty)$. Since $f_2(n-2) = -2 < 0$ and $f_2(n - 2 + \frac{1}{n}) = \frac{1}{n^3}(n^4 - 10n^3 + 15n^2 - 8n + 1) > 0$ for $n \ge 10$, we have $n - 2 < \mu_1(S_{2,\,n-4}) < n - 2 + \frac{1}{n}$.

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By direct calculation, the Laplacian characteristic polynomial of $S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor}$ is

$$\phi(x) = x(x-1)^{n-4}[x^3 - (n+2)x^2 + (2n + \lceil \frac{n-2}{2} \rceil \lfloor \frac{n-2}{2} \rfloor + 1)x - n].$$

It is well known that $\mu_{n-1}(T_n) \le \delta(T_n) = 1$ (see, e.g. [9]). By Lemma 2.3, we have $\mu_2(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor}) \ge 2$. Thus, $\mu_1(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor})$, $\mu_2(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor})$ and $\mu_{n-1}(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor})$ are roots of the following polynomial

$$g(x) = x^3 - (n+2)x^2 + (2n + \lceil \frac{n-2}{2} \rceil \lfloor \frac{n-2}{2} \rfloor + 1)x - n.$$

Since $\mu_{n-1}(G) = n - \mu_1(\overline{G})$ (see, e.g. [28]), by the Vieta Theorem, we have

$$\mu_1(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor}) + \mu_2(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor}) = n + 2 - \mu_{n-1}(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor}) = \mu_1(\overline{S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor}}) + 2.$$

Thus,

$$||LI(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor})||_{F_2} = \mu_1(\overline{S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor}}) - 2 + \frac{4}{n} < n - 2 + \frac{4}{n} < \mu_1(S_{1,n-3}).$$

If n is an odd number, by direct calculation, we have that the Laplacian characteristic polynomial of $\overline{S_{\frac{n-1}{2}, \frac{n-3}{2}}}$ is

$$\varphi(x) = \frac{1}{4}x(x-n+1)^{n-4}[4x^3 - 8(n-1)x^2 + (5n^2 - 12n + 7)x - n^3 + 4n^2 - 3n].$$

It follows that $\mu_1(\overline{S_{\frac{n-1}{2}, \frac{n-3}{2}}})$ is the largest root of the following polynomial

$$h_1(x) = 4x^3 - 8(n-1)x^2 + (5n^2 - 12n + 7)x - n^3 + 4n^2 - 3n.$$

Thus, $||LI(S_{\frac{n-1}{2}, \frac{n-3}{2}})||_{F_2}$ is the largest root of the polynomial $h_1(x + 2 - \frac{4}{n})$. Noting that $f_2(x)$ and $h_1(x + 2 - \frac{4}{n})$ are strictly increasing on $(n-2, +\infty)$. Since

$$\begin{aligned} h_1(x + 2 - \frac{4}{n}) - f_2(x) &= 3x^3 - (7n + \frac{48}{n} - 34)x^2 + (5n^2 - 48n - \frac{256}{n} \\ &\quad + \frac{192}{n^2} + 158)x - n^3 + 14n^2 - 78n - \frac{476}{n} \\ &\quad + \frac{512}{n^2} - \frac{256}{n^3} + 254 \\ &< 0, \end{aligned}$$

for $x \in (n-2, n-2+1/n)$, we have $||LI(S_{\frac{n-1}{2}, \frac{n-3}{2}})||_{F_2} > \mu_1(S_{2,n-4})$.

If n is an even number, by a similar reasoning as the above, we can conclude that $||LI(S_{\frac{n-2}{2}, \frac{n-2}{2}})||_{F_2} > \mu_1(S_{2,n-4})$. Therefore, $||LI(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor})||_{F_2} > \mu_1(S_{2,n-4})$.

It is easy to see that $||LI(K_{1,n-1})||_{F_2} = \mu_1(K_{1,n-1})$, $||LI(S_{1,n-3})||_{F_2} = \mu_1(S_{1,n-3})$ and $||LI(S_{2,n-4})||_{F_2} = \mu_1(S_{2,n-4})$. Combining the above arguments, we have

$$||LI(T_n)||_{F_2} < ||LI(S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor})||_{F_2} < ||LI(S_{1,n-3})||_{F_2} < ||LI(K_{1,n-1})||_{F_2},$$

for $T_n \in \mathcal{T}_n \setminus \{K_{1,n-1}, S_{1,n-3}, S_{\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor}\}$.

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Now we will show that $||LI(P_n)||_{F_2} < ||LI(T_n)||_{F_2}$ for $T_n \in \mathcal{T}_n \setminus \{P_n\}$. By Lemma 2.8, we have $S_2(L(P_n)) \leq S_2(L(T_n))$. Since $\mu_1(P_n) \leq \mu_1(T_n)$, we have

$$\begin{aligned}||LI(T_n)||_{F_2} &= \sigma_1(LI(T_n)) + \sigma_2(LI(T_n))\\ &= \max\{\mu_1(T_n) + \mu_2(T_n) - 4 + \frac{4}{n}, \mu_1(T_n)\}\\ &= \max\{S_2(L(T_n)) - 4 + \frac{4}{n}, \mu_1(T_n)\}\\ &\geq ||LI(P_n)||_{F_2},\end{aligned}$$

with equality if and only if $T_n \cong P_n$. This completes the proof. ∎

THEOREM 5.3. *Let T_n be a tree with $n \geq 5$ vertices. Then,*

$$||LI(T_n)||_{F_3} \leq n + 1 - \frac{2}{n},$$

with equality if and only if $T_n \cong K_{1,\,n-1}$.

Proof. From Theorem 1.1 in [10], we have $S_2(L(T_n)) < n+2-\frac{2}{n}$ and $S_3(L(T_n)) < n+4-\frac{4}{n}$. By Lemma 2.9, we have $spr(L(T_n)) < spr(L(K_{1,\,n-1})) = n-1$ for $T_n \in \mathcal{T}_n \setminus \{K_{1,\,n-1}\}$. Thus,

$$\begin{aligned}||LI(T_n)||_{F_3} &= \sigma_1(LI(T_n)) + \sigma_2(LI(T_n)) + \sigma_3(LI(T_n))\\ &= \max\{S_2(L(T_n)) - 2 + \frac{2}{n}, S_3(L(T_n)) - 6 + \frac{6}{n}, spr(L(T_n)) + 2 - \frac{2}{n}\}\\ &\leq \max\{n, n-2+\frac{2}{n}, spr(L(T_n)) + 2 - \frac{2}{n}\}\\ &\leq ||LI(K_{1,\,n-1})||_{F_3}\\ &= n + 1 - \frac{2}{n},\end{aligned}$$

with equality if and only if $T_n \cong K_{1,\,n-1}$. This completes the proof. ∎

CONJECTURE 5.4. *Let T_n be a tree with $n \geq 12$ vertices. Then,*

$$||LI(P_n)||_{F_k} \leq ||LI(T_n)||_{F_k} \leq ||LI(K_{1,\,n-1})||_{F_k}.$$

The equality in the left-hand side holds if and only if $T_n \cong P_n$, and the equality in the right-hand side holds if and only if $T_n \cong K_{1,\,n-1}$.

THEOREM 5.5. *Let U_n be a unicyclic graph with $n \geq 5$ vertices. Then,*

$$\sigma_1(LI(C_n)) \leq \sigma_1(LI(U_n)) \leq n - 2.$$

The equality in the left-hand side holds if and only if $U_n \cong C_n$, and the equality in the right-hand side holds if and only if $U_n \cong G_{n,\,n}$.

Proof. If $\Delta(U_n) \geq 3$, by Lemma 2.1, we have $\mu_1(U_n) \geq \Delta(U_n)+1 \geq 4$. Thus, $\sigma_1(LI(U_n)) = \mu_1(U_n)-2$. By Lemma 2.10, we have $\sigma_1(LI(U_n)) \leq \sigma_1(LI(G_{n,\,n}))$ with equality if and only if $U_n \cong G_{n,\,n}$. From [39], it follows that U_n^2, shown in Fig. 5.1, is the smallest Laplacian spectral radii among all unicyclic graphs with $\Delta(U_n) \geq 3$. Hence, $\sigma_1(LI(U_n)) \geq \sigma_1(LI(U_n^2))$. By Lemma 2.3, we have $\sigma_1(LI(U_n^2)) = \mu_1(U_n^2) - 2 \geq \mu_1(U_5^2) - 2 > 2.17008$. Thus, $\sigma_1(LI(U_n)) > 2.17008$.

If $\Delta(U_n) = 2$, then $U_n = C_n$. Thus,$\sigma_1(LI(U_n)) = \sigma_1(LI(C_n)) = 2$.

Combining the above arguments, we have the proof. ∎

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THEOREM 5.6. *Let U_n be a unicyclic graph with $n \geq 12$ vertices. Then,*

$$||LI(C_n)||_{F_2} \leq ||LI(U_n)||_{F_2} \leq n.$$

The equality in the left-hand side holds if and only if $U_n \cong C_n$, and the equality in the right-hand side holds if and only if $U_n \cong G_{n,n}$.

Proof. Since $||LI(U_n)||_{F_2} = \max\{\mu_1(U_n), S_2(L(U_n)) - 4\}$, by Lemma 2.10, we have $||LI(U_n)||_{F_2} \leq ||LI(G_{n,n})||_{F_2}$ with equality if and only if $U_n \cong G_{n,n}$. It is well known that $S_2(L(C_n)) = 6 + 2\cos\frac{2\pi}{n}$ for even cycle and $S_2(L(C_n)) = 4 + 4\cos\frac{\pi}{n}$ for odd cycle. Thus, $S_2(L(C_n)) < 8$. From the proof of Lemma 4.4 in [24], it follows that $\mu_1(T_n) + \mu_2(T_n) = q_1(T_n) + q_2(T_n) \geq 8$ for $n \geq 12$. By Lemma 2.3, we have $S_2(L(U_n)) = \mu_1(U_n) + \mu_2(U_n) \geq 8$ for $n \geq 12$. Hence, $S_2(L(U_n)) > S_2(L(C_n))$ for $\mathcal{U}_n \setminus \{C_n\}$. By the proof of Theorem 5.5, we have $\mu_1(U_n) > \mu_1(C_n)$ for $\mathcal{U}_n \setminus \{C_n\}$. Therefore, $||LI(U_n)||_{F_2} = \max\{\mu_1(U_n), S_2(L(U_n)) - 4\} \geq ||LI(C_n)||_{F_2}$ with equality if and only if $U_n \cong C_n$. This completes the proof. ∎

THEOREM 5.7. *Let U_n be a unicyclic graph with $n \geq 12$ vertices. Then,*

$$||LI(U_n)||_{F_3} \leq n + 1,$$

with equality if and only if $U_n \cong G_{n,n}$.

Proof. From Corollary 4.1 in [7], we have $S_3(L(U_n)) \leq n + 6$. By Lemmas 2.10 and 2.11, we have $S_2(L(U_n)) < S_2(L(G_{n,n}))$ and $spr(L(U_n)) < spr(L(G_{n,n})) = n - 1$ for $U_n \in \mathcal{U}_n \setminus \{G_{n,n}\}$. Thus,

$$\begin{aligned}
||LI(U_n)||_{F_2} &= \sigma_1(LI(U_n)) + \sigma_2(LI(U_n)) + \sigma_3(LI(U_n)) \\
&= \max\{S_2(L(U_n)) - 2, S_3(L(U_n)) - 6, spr(L(U_n)) + 2\} \\
&\leq ||LI(G_{n,n})||_{F_3} \\
&= n + 1
\end{aligned}$$

with equality if and only if $U_n \cong G_{n,n}$. This completes the proof. ∎

THEOREM 5.8. *Let B_n be a bicyclic graph with $n \geq 17$ vertices. Then,*

$$\sigma_1(LI(B_n^1)) \leq \sigma_1(LI(B_n)) \leq \sigma_1(LI(B_n^*)).$$

The equality in the left-hand side holds if and only if $B_n \cong B_n^1$, and the equality in the right-hand side holds if and only if $B_n \cong B_n^*$.

Proof. Since $B_n^1 = U_n^2 + e$, by Lemma 2.3, we have

$$\mu_1(B_n^1) - 2 - \frac{2}{n} \geq \mu_1(U_n^2) - 2 - \frac{2}{n} \geq \mu_1(U_{10}^2) - 2 - \frac{2}{n} > 2.23566 - \frac{2}{n} > 2 + \frac{2}{n}$$

for $n \geq 17$. From [39], we know that B_n^1, shown in Fig. 5.1, is the smallest Laplacian spectral radii among all bicyclic graphs. Thus, $\sigma_1(LI(B_n)) = \mu_1(B_n) - 2 - \frac{2}{n} \geq \mu_1(B_n^1) - 2 - \frac{2}{n}$ with equality if and only if $G \cong B_n^1$. By Lemma 2.12, we have $\sigma_1(LI(B_n)) = \mu_1(B_n) - 2 - \frac{2}{n} \leq \mu_1(B_n^*) - 2 - \frac{2}{n} = \sigma_1(LI(B_n^*))$ with equality if and only if $B_n \cong B_n^*$. This completes the proof. ∎

THEOREM 5.9. *Let B_n be a bicyclic graph with n vertices. Then,*

$$||LI(B_n)||_{F_2} \leq ||LI(G_{n+1,n})||_{F_2},$$

with equality if and only if $B_n \cong G_{n+1,n}$.

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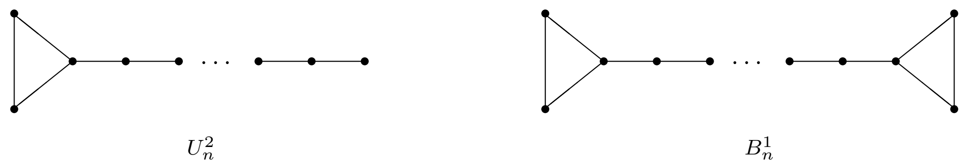

FIGURE 5.1. *Graphs U_n^2 and B_n^1.*

Proof. Since $||LI(B_n)||_{F_2} = \max\{\mu_1(B_n), S_2(L(B_n)) - 4 - \frac{4}{n}\}$, by Lemma 2.12, we have the proof. □

THEOREM 5.10. *Let B_n be a bicyclic graph with n vertices. Then*

$$||LI(B_n)||_{F_3} \leq n + 2 - \frac{2}{n},$$

with equality if and only if $B_n \cong G_{n+1,\,n}$.

Proof. From Corollary 4.2 in [7], we have $S_3(L(B_n)) \leq n + 7$. By Lemma 2.12, we have

$$\begin{aligned} ||LI(B_n)||_{F_2} &= \sigma_1(LI(U_n)) + \sigma_2(LI(U_n)) + \sigma_3(LI(U_n)) \\ &= \max\{S_2(L(B_n)) - 2 - \frac{2}{n}, S_3(L(B_n)) - 6 - \frac{6}{n}, spr(L(B_n)) + 2 + \frac{2}{n}\} \\ &\leq ||LI(G_{n+1,\,n})||_{F_3} \\ &= n + 2 - \frac{2}{n} \end{aligned}$$

with equality if and only if $B_n \cong G_{n+1,\,n}$. This completes the proof. □

Based on the conjecture of Guan et al. [18], we present the following conjecture on the uniqueness of the extremal graph.

CONJECTURE 5.11. *Among all connected graphs with n and m edges $n \leq m \leq 2n-3$, the $G_{m,\,n}$ is the unique graph with maximal value of $||LI(G)||_{F_2}$ and $||LI(G)||_{F_3}$.*

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