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MTH ROOTS OF H-SELFADJOINT MATRICES OVER THE QUATERNIONS*

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Abstract. The complex matrix representation for a quaternion matrix is used in this paper to find necessary and sufficient conditions for the existence of an H-selfadjoint mth root of a given H-selfadjoint quaternion matrix. In the process, when such an H-selfadjoint mth root exists, its construction is also given.

Key words. Quaternion matrices, Roots of matrices, Indefinite inner product, H-selfadjoint matrices, Canonical forms.

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1. Introduction.

Denote the skew-field of real quaternions by $\mathbb{H}$. The basic theory of quaternion linear algebra can be found in various books and papers, see for example the book by Rodman [10], and [13, 14]. Let m be any positive integer and let H be a square quaternion matrix which is invertible and Hermitian. We focus on the class of selfadjoint matrices relative to the indefinite inner product generated by H. In the general sense of the definition, a square quaternion matrix A is said to be H-selfadjoint if $HA = A^*H$.

If B is a square H-selfadjoint matrix with quaternion entries, we seek to find necessary and sufficient conditions for the existence of an H-selfadjoint matrix A such that $A^m = B$. This matrix A is referred to as an H-selfadjoint mth root of the matrix B.

It is well known that there exists a complex matrix representation for a matrix with quaternion entries, that is, there exists an isomorphism ω_n between the real algebra of all $n \times n$ quaternion matrices and a subalgebra Ω_{2n} of the algebra of all $2n \times 2n$ complex matrices. This isomorphism ω_n maps an $n \times n$ quaternion matrix $A = A_1 + \mathrm{j}A_2$, where $A_1, A_2 \in \mathbb{C}^{n\times n}$, to a $2n \times 2n$ matrix:

$$\begin{bmatrix} A_1 & \bar{A}_2 \\ -A_2 & \bar{A}_1 \end{bmatrix}.$$

This fact simplifies our problem. Therefore, we find necessary and sufficient conditions for the existence of an $\tilde{H}$-selfadjoint mth root in Ω_{2n} of an $\tilde{H}$-selfadjoint matrix in Ω_{2n}.

Quaternion matrices in indefinite inner product spaces, and specifically different canonical forms, were studied in [1, 4, 9, 11]. Not much research has been done on roots of quaternion matrices, although [7] does give a formula for obtaining mth roots of quaternion matrices of a particular form. On the other hand, in the complex case, H-selfadjoint mth roots of H-selfadjoint matrices have been studied extensively. Necessary

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and sufficient conditions for the existence of such roots can be found in [5]. For the case of H-selfadjoint square roots and applications to polar decompositions of H-selfadjoint matrices, see [2], and for the case of square roots of H-nonnegative matrices, see [8]. We refer to the introduction of a previous paper [5], for an overview of mth roots of matrices in general.

Note that in the process of finding an H-selfadjoint mth root $A \in \Omega_{2n}$ of $B \in \Omega_{2n}$, that is, $A^m = B$, the functional analytic approach via the Cauchy integral:

$$A := \frac{1}{2\pi \mathsf{i}} \int_{\Gamma} \sqrt[m]{\lambda}(\lambda I - B)^{-1} \, d\lambda,$$

which can be found for example in [6], would be sufficient in the case where there are no eigenvalues of B in $(-\infty, 0]$. However, in this paper, we prefer to take a more direct approach also in the case where the eigenvalues do not lie on the negative real line.

2. Preliminaries. We recap the basic theory for quaternions and matrices with quaternion entries as found in the book by Leiba Rodman, [10]. Every element in $\mathbb{H}$ is of the form:

$$x = x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k},$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the elements $\mathsf{i}, \mathsf{j}, \mathsf{k}$ satisfy the following formulas:

$$\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1, \quad \mathsf{ij} = -\mathsf{ji} = \mathsf{k}, \quad \mathsf{jk} = -\mathsf{kj} = \mathsf{i}, \quad \mathsf{ki} = -\mathsf{ik} = \mathsf{j}.$$

It is important to keep in mind the fact that multiplication in $\mathbb{H}$ is not commutative, that is, in general $xy \neq yx$ for $x, y \in \mathbb{H}$. Let $\bar{x} = x_0 - x_1\mathsf{i} - x_2\mathsf{j} - x_3\mathsf{k}$ denote the conjugate quaternion of x. For a quaternion matrix A, let $\bar{A}$ denote the matrix in which each entry is the conjugate of the corresponding entry in A.

Let A be an $n \times n$ quaternion matrix, that is, $A \in \mathbb{H}^{n\times n}$, and write A as $A = A_1 + \mathsf{j}A_2$ where $A_1, A_2 \in \mathbb{C}^{n\times n}$. The map $\omega_n : \mathbb{H}^{n\times n} \to \mathbb{C}^{2n\times 2n}$ is defined by:

$$\omega_n(A) = \begin{bmatrix} A_1 & \bar{A}_2 \\ -A_2 & \bar{A}_1 \end{bmatrix}.$$

Then ω_n is an isomorphism of the real algebra $\mathbb{H}^{n\times n}$ onto the real unital subalgebra:

$$\Omega_{2n} := \left\{ \begin{bmatrix} A_1 & \bar{A}_2 \\ -A_2 & \bar{A}_1 \end{bmatrix} \mid A_1, A_2 \in \mathbb{C}^{n\times n} \right\},$$

of $\mathbb{C}^{2n\times 2n}$. The following properties can be found in [10, Section 3.4] and [13]. Let $X, Y \in \mathbb{H}^{n\times n}$ and $s, t \in \mathbb{R}$ be arbitrary, then

(i) $\omega_n(I_n) = I_{2n}$;
(ii) $\omega_n(XY) = \omega_n(X)\omega_n(Y)$;
(iii) $\omega_n(sX + tY) = s\omega_n(X) + t\omega_n(Y)$;
(iv) $\omega_n(X^*) = (\omega_n(X))^*$;
(v) $\omega_n(X^{-1}) = (\omega_n(X))^{-1}$ if X is invertible.

Note that the matrix $X^* \in \mathbb{H}^{n\times m}$ is obtained from $X \in \mathbb{H}^{m\times n}$ by replacing each entry with its conjugate quaternion and then taking the transpose. This isomorphism between $\mathbb{H}^{n\times n}$ and Ω_{2n} ensures that results

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for matrices in $\mathbb{H}^{n\times n}$ that are purely algebraic are also true for matrices in the subalgebra Ω_{2n} since we can apply ω_n, and vice versa as long as we stay within the subalgebra Ω_{2n}. All definitions that follow could also be made with respect to matrices in Ω_{2n}.

An $n \times n$ quaternion matrix A has left eigenvalues and right eigenvalues but since we only work with right eigenvalues and right eigenvectors, we will refer to them as *eigenvalues* and *eigenvectors*.

Definition 2.1. *A nonzero vector $v \in \mathbb{H}^n$ is called an* eigenvector *of a matrix $A \in \mathbb{H}^{n\times n}$ corresponding to the* eigenvalue *$\lambda \in \mathbb{H}$ if the equality $Av = v\lambda$ holds.*

The *spectrum* of A, denoted by $\sigma(A)$, is the set of all eigenvalues of A. Note that $\sigma(A)$ is closed under similarity of quaternions, that is, if v is an eigenvector of A corresponding to the eigenvalue λ, then $v\alpha$ is an eigenvector of A corresponding to the eigenvalue $\alpha^{-1}\lambda\alpha$, for all nonzero $\alpha \in \mathbb{H}$. From [13], we see that A has exactly n eigenvalues which are complex numbers with nonnegative imaginary parts and the Jordan normal form of A has exactly these numbers on the diagonal. Let $\mathbb{C}_+ = \{a + ib \mid a \in \mathbb{R}, b > 0\}$.

Let a single Jordan block of size $n \times n$ at the eigenvalue λ be denoted by $J_n(\lambda)$. The $n \times n$ matrix with ones on the main anti-diagonal and zeros elsewhere, called a standard involutary permutation (sip) matrix, is denoted by Q_n.

Recall the following definition from [12].

Definition 2.2. *Let A be a square quaternion matrix with Jordan blocks $\bigoplus_{i=1}^{r} J_{n_i}(\lambda)$ at the eigenvalue λ in its Jordan normal form and assume that $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_r > 0$. The* Segre characteristic *of A corresponding to the eigenvalue λ is defined as the sequence:*

$$n_1, n_2, n_3, \ldots, n_r, 0, 0, \ldots.$$

We will use this definition mostly in the case where λ is equal to zero unless indicated otherwise and therefore sometimes will simply use *Segre characteristic* to refer to the Segre characteristic of A corresponding to the eigenvalue 0. Note that the Jordan normal form of matrices in the subalgebra Ω_{2n} can be found from the Jordan normal form of matrices in $\mathbb{H}^{n\times n}$ since $\omega_n(J_n(\lambda)) = J_n(\lambda) \oplus J_n(\bar{\lambda})$, for $\lambda \in \mathbb{C}$. Then it is easy to see the following result.

Corollary 2.3. *If a nilpotent matrix A is in Ω_{2n}, then each number in the Segre characteristic of A occurs twice.*

This actually holds for the Segre characteristic corresponding to any real eigenvalue in the case where $A \in \Omega_{2n}$ has real numbers in its spectrum. However, only the nilpotent case will be used later.

A matrix $X \in \mathbb{H}^{n\times n}$ is said to be *Hermitian* if $X^* = X$. Let $H \in \mathbb{H}^{n\times n}$ be an invertible Hermitian matrix. We consider the indefinite inner product $[\cdot, \cdot]$ generated by H:

$$[x, y] = \langle Hx, y\rangle = y^* Hx, \quad x, y \in \mathbb{H}^n,$$

where $\langle\cdot, \cdot\rangle$ denotes the standard inner product. A matrix $A \in \mathbb{H}^{n\times n}$ is called *H-selfadjoint* if $HA = A^*H$. Two pairs of matrices (A_1, H_1) and (A_2, H_2) are said to be *unitarily similar* if there exists an invertible quaternion matrix S such that $S^{-1}A_1S = A_2$ and $S^*H_1S = H_2$ hold.

As in the complex case, there exists a canonical form for the pair (A, H) where $A \in \mathbb{H}^{n\times n}$ is an H-selfadjoint matrix. This is given in, for example [1, Theorem 4.1], [9] and [10, Theorem 10.1.1].

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Theorem 2.4. *Let $H \in \mathbb{H}^{n\times n}$ be an invertible Hermitian matrix and $A \in \mathbb{H}^{n\times n}$ an H-selfadjoint matrix. Then there exists an invertible matrix $S \in \mathbb{H}^{n\times n}$ such that*

$$\begin{aligned} S^{-1}AS &= J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_\alpha}(\lambda_\alpha) \\ &\oplus \begin{bmatrix} J_{k_{\alpha+1}}(\lambda_{\alpha+1}) & 0 \\ 0 & J_{k_{\alpha+1}}(\overline{\lambda}_{\alpha+1}) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} J_{k_\beta}(\lambda_\beta) & 0 \\ 0 & J_{k_\beta}(\overline{\lambda}_\beta) \end{bmatrix}, \end{aligned} \tag{2.1}$$

where $\lambda_i \in \sigma(A)\cap\mathbb{R}$ for all $i = 1,\dots,\alpha$, $\lambda_i \in \sigma(A)\cap\mathbb{C}_+$ for all $i = \alpha+1,\dots,\beta$, and

$$S^*HS = \eta_1 Q_{k_1} \oplus \cdots \oplus \eta_\alpha Q_{k_\alpha} \oplus Q_{2k_{\alpha+1}} \oplus \cdots \oplus Q_{2k_\beta}, \tag{2.2}$$

where $\eta_i = \pm 1$. The form $(S^{-1}AS,\, S^*HS)$ in (2.1) and (2.2) is uniquely determined by the pair (A,H), up to a permutation of diagonal blocks.

We refer to the pair $(S^{-1}AS, S^*HS)$ in (2.1) and (2.2) as the *canonical form* of the pair (A,H) of quaternion matrices.

The following result from [5] holds for quaternion matrices as well, but by applying ω_n it also holds for matrices in the subalgebra Ω_{2n}. We use it in the proofs throughout this paper.

Lemma 2.5. *Let X and Y be $n\times n$ quaternion matrices such that the pair (X, H_X) is unitarily similar to the pair (Y, H_Y) where $H_X \in \mathbb{H}^{n\times n}$ and $H_Y \in \mathbb{H}^{n\times n}$ are invertible Hermitian matrices, that is, there exists an invertible matrix $P \in \mathbb{H}^{n\times n}$ such that*

$$P^{-1}XP = Y \quad \text{and} \quad P^*H_XP = H_Y.$$

Let the matrix $J \in \mathbb{H}^{n\times n}$ be an H_X-selfadjoint mth root of X, that is, $J^m = X$. Then the matrix $A := P^{-1}JP$ is an H_Y-selfadjoint mth root of Y.

From the properties of the map ω_n, we see that a matrix A is invertible if and only if $\omega_n(A)$ is invertible. Also, from Proposition 3.4.1 in [10], A is Hermitian if only if $\omega_n(A)$ is Hermitian.

To find a canonical form for a pair $(\tilde{A},\tilde{H})$ where $\tilde{A} \in \Omega_{2n}$ is $\tilde{H}$-selfadjoint, and $\tilde{H} \in \Omega_{2n}$ is invertible and Hermitian, we apply ω_n to the equations (2.1) and (2.2). Denote the right-hand side of (2.1) by J and the right-hand side of (2.2) by Q and then we have

$$(\omega_n(S))^{-1}\omega_n(A)\omega_n(S) = \omega_n(J), \quad \omega_n(S)^*\omega_n(H)\omega_n(S) = \omega_n(Q).$$

The uniqueness follows from the fact that the canonical form of (A,H) is unique (Theorem 2.4) and that ω_n is an isomorphism. Therefore, the canonical form of a pair of matrices in Ω_{2n} is as follows.

Theorem 2.6. *Let $\tilde{H} \in \Omega_{2n}$ be an invertible Hermitian matrix and $\tilde{A} \in \Omega_{2n}$ an $\tilde{H}$-selfadjoint matrix. Then there exists an invertible matrix $S \in \Omega_{2n}$ such that*

$$\begin{aligned} S^{-1}\tilde{A}S &= J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_\alpha}(\lambda_\alpha) \oplus \begin{bmatrix} J_{k_{\alpha+i}}(\lambda_{\alpha+i}) & 0 \\ 0 & J_{k_{\alpha+i}}(\overline{\lambda}_{\alpha+i}) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} J_{k_\beta}(\lambda_\beta) & 0 \\ 0 & J_{k_\beta}(\overline{\lambda}_\beta) \end{bmatrix} \\ &\oplus J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_\alpha}(\lambda_\alpha) \oplus \begin{bmatrix} J_{k_{\alpha+i}}(\overline{\lambda}_{\alpha+i}) & 0 \\ 0 & J_{k_{\alpha+i}}(\lambda_{\alpha+i}) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} J_{k_\beta}(\overline{\lambda}_\beta) & 0 \\ 0 & J_{k_\beta}(\lambda_\beta) \end{bmatrix}, \end{aligned} \tag{2.3}$$

where $\lambda_i \in \sigma(\tilde{A})\cap\mathbb{R}$ for all $i = 1,\dots,\alpha$, $\lambda_i \in \sigma(\tilde{A})\cap\mathbb{C}_+$ for all $i = \alpha+1,\dots,\beta$; and

$$\begin{aligned} S^*\tilde{H}S &= \eta_1 Q_{k_1} \oplus \cdots \oplus \eta_\alpha Q_{k_\alpha} \oplus Q_{2k_{\alpha+1}} \oplus \cdots \oplus Q_{2k_\beta} \\ &\oplus \eta_1 Q_{k_1} \oplus \cdots \oplus \eta_\alpha Q_{k_\alpha} \oplus Q_{2k_{\alpha+1}} \oplus \cdots \oplus Q_{2k_\beta}, \end{aligned} \tag{2.4}$$

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where $\eta_i = \pm 1$. The form $(S^{-1}\tilde{A}S, S^*\tilde{H}S)$ in (2.3) and (2.4) is uniquely determined by the pair $(\tilde{A}, \tilde{H})$ up to a permutation of diagonal blocks.

We note at this stage that the canonical form of the nonreal part of an H-selfadjoint matrix A must be of dimensions a multiple of 4 due to the direct sums of $J_{k_i}(\lambda_i)$ and $J_{k_i}(\overline{\lambda}_i)$ occurring twice. It is crucial to ensure that all matrices throughout the proofs are in Ω_{2n} and for this reason we give the following result to explain why we can study different Jordan blocks separately.

Lemma 2.7. *Let $H_1 = Q_1 \oplus \bar{Q}_1, B_1 = J_1 \oplus \bar{J}_1 \in \Omega_{2n}$ and $H_2 = Q_2 \oplus \bar{Q}_2, B_2 = J_2 \oplus \bar{J}_2 \in \Omega_{2p}$ where B_1 is H_1-selfadjoint and B_2 is H_2-selfadjoint. Let $A_1 \in \Omega_{2n}$ be an H_1-selfadjoint mth root of B_1 and $A_2 \in \Omega_{2p}$ an H_2-selfadjoint mth root of B_2, and let their entries be as follows:*

$$A_1 = \begin{bmatrix} A_{11}^{(1)} & \bar{A}_{12}^{(1)} \\ -A_{12}^{(1)} & \bar{A}_{11}^{(1)} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} A_{11}^{(2)} & \bar{A}_{12}^{(2)} \\ -A_{12}^{(2)} & \bar{A}_{11}^{(2)} \end{bmatrix}.$$

Let $\hat{B} = J_1 \oplus J_2 \oplus \bar{J}_1 \oplus \bar{J}_2 \in \Omega_{2(n+p)}$ and $\hat{H} = Q_1 \oplus Q_2 \oplus \bar{Q}_1 \oplus \bar{Q}_2 \in \Omega_{2(n+p)}$. Then $\hat{B}$ is $\hat{H}$-selfadjoint and the matrix:

$$\hat{A} = \begin{bmatrix} A_{11}^{(1)} & 0 & \bar{A}_{12}^{(1)} & 0 \\ 0 & A_{11}^{(2)} & 0 & \bar{A}_{12}^{(2)} \\ -A_{12}^{(1)} & 0 & \bar{A}_{11}^{(1)} & 0 \\ 0 & -A_{12}^{(2)} & 0 & \bar{A}_{11}^{(2)} \end{bmatrix} \in \Omega_{2(n+p)},$$

is an $\hat{H}$-selfadjoint mth root of the matrix $\hat{B}$.

Proof. Let P be the following permutation matrix:

$$P = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}.$$

This P produces a map from $\Omega_{2n} \oplus \Omega_{2p}$ to $\Omega_{2(n+p)}$ which satisfies $P(A_1 \oplus A_2)P^{-1} = \hat{A}$. We also then have $P(B_1 \oplus B_2)P^{-1} = \hat{B}$ and $P(H_1 \oplus H_2)P^{-1} = \hat{H}$. Therefore,

$$\hat{A}^m = \left(P(A_1 \oplus A_2)P^{-1}\right)^m = P(A_1 \oplus A_2)^m P^{-1} = P(B_1 \oplus B_2)P^{-1} = \hat{B}.$$

Now, by noting that $P^* = P^{-1}$ and using the facts that B_1 and A_1 are H_1-selfadjoint and B_2 and A_2 are H_2-selfadjoint, it follows that $\hat{H}\hat{B} = \hat{B}^*\hat{H}$ and $\hat{H}\hat{A} = \hat{A}^*\hat{H}$. Therefore, $\hat{A}$ is an $\hat{H}$-selfadjoint mth root of the $\hat{H}$-selfadjoint matrix $\hat{B}$. ∎

The matrix J_i in Lemma 2.7 can be the Jordan normal form $J_k(\lambda)$ in the case where λ is real or the Jordan normal form $J_k(\lambda) \oplus J_k(\bar{\lambda})$ in the case where λ is nonreal.

In general, let the permutation matrix P be a $2t \times 2t$ block matrix where the block in the ith row and jth column is defined by:

$$P_{ij} = \begin{cases} I_{2k_i} & \text{if } j = 2i-1,\ i \le t; \\ I_{2k_{i-t}} & \text{if } j = 2(i-t),\ i > t; \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

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Then we have, for example, that

$$P\bigoplus_{j=1}^{t}\left(Q_{k_j}\oplus -Q_{k_j}\oplus Q_{k_j}\oplus -Q_{k_j}\right)P^{-1}=\bigoplus_{j=1}^{t}\left(Q_{k_j}\oplus -Q_{k_j}\right)\oplus\bigoplus_{j=1}^{t}\left(Q_{k_j}\oplus -Q_{k_j}\right).$$

3. Existence of mth roots. We first present a very handy tool for working with quaternion matrices.

LEMMA 3.1. *Let H be an $n\times n$ quaternion matrix which is invertible and Hermitian, and let B be an $n\times n$ H-selfadjoint quaternion matrix. There exists an H-selfadjoint quaternion matrix A such that $A^m=B$ if and only if there exists an $\tilde{H}=\omega_n(H)$-selfadjoint matrix $\tilde{A}=\omega_n(A)$ such that $\tilde{A}^m=\tilde{B}$, where $\tilde{B}=\omega_n(B)$ is an $\tilde{H}$-selfadjoint matrix in the subalgebra Ω_{2n}.*

Proof. From the properties of the map ω_n and the fact that it is an isomorphism from $\mathbb{H}^{n\times n}$ to Ω_{2n}, we see that $A^m=B$ if and only if $(\omega_n(A))^m=\omega_n(B)$, and A is H-selfadjoint if and only if $\omega_n(A)$ is $\omega_n(H)$-selfadjoint. ∎

Because of this lemma, necessary and sufficient conditions for the existence of an H-selfadjoint mth root of an H-selfadjoint matrix B are the same as the necessary and sufficient conditions for the existence of an $\tilde{H}$-selfadjoint mth root of an $\tilde{H}$-selfadjoint matrix $\tilde{B}$ where $\tilde{B},\tilde{H}\in\Omega_{2n}$. If we could work in $\mathbb{C}^{2n\times 2n}$, we would now be done by simply referring to [5]. However, since ω_n is an isomorphism between $\mathbb{H}^{n\times n}$ and the subalgebra Ω_{2n} of $\mathbb{C}^{2n\times 2n}$, we have to be more careful.

We now first present in Theorem 3.2 our main theorem for the existence of H-selfadjoint mth roots of H-selfadjoint matrices in Ω_{2n}. The proof of this theorem may be split into the following separate parts due to Lemma 2.7, viz. the case where $\tilde{B}$ has only positive eigenvalues, the case where $\tilde{B}$ has only nonreal eigenvalues, the case where $\tilde{B}$ has only negative eigenvalues (separated into two cases for m even and for m odd), and lastly, the case where $\tilde{B}$ has only zero as an eigenvalue. These we state and prove in Theorems 3.3, 3.4, 3.6, 3.7, and 3.9.

THEOREM 3.2. *Let $\tilde{B}$ and $\tilde{H}$ be matrices in the subalgebra Ω_{2n} such that $\tilde{H}$ is invertible and Hermitian, and $\tilde{B}$ is $\tilde{H}$-selfadjoint. Then there exists an $\tilde{H}$-selfadjoint matrix in Ω_{2n}, say $\tilde{A}$, such that $\tilde{A}^m=\tilde{B}$ if and only if the canonical form of $(\tilde{B},\tilde{H})$ has the following properties:*

1. *The part of the canonical form corresponding to negative eigenvalues when m is even, say $(\tilde{B}_-,\tilde{H}_-)$, is given by:*

$$\tilde{B}_-=\bigoplus_{j=1}^{t}\left(J_{k_j}(\lambda_j)\oplus J_{k_j}(\lambda_j)\right)\oplus\bigoplus_{j=1}^{t}\left(J_{k_j}(\lambda_j)\oplus J_{k_j}(\lambda_j)\right),$$

$$\tilde{H}_-=\bigoplus_{j=1}^{t}\left(Q_{k_j}\oplus -Q_{k_j}\right)\oplus\bigoplus_{j=1}^{t}\left(Q_{k_j}\oplus -Q_{k_j}\right),$$

   where $\lambda_j<0$.

2. *The part of the canonical form corresponding to zero eigenvalues, say $(\tilde{B}_0,\tilde{H}_0)$, is given by:*

$$\tilde{B}_0=\bigoplus_{j=1}^{t}\left(\bigoplus_{i=1}^{r_j}J_{a_j+1}(0)\oplus\bigoplus_{i=r_j+1}^{m}J_{a_j}(0)\right)\oplus\bigoplus_{j=1}^{t}\left(\bigoplus_{i=1}^{r_j}J_{a_j+1}(0)\oplus\bigoplus_{i=r_j+1}^{m}J_{a_j}(0)\right),$$

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and

$$\tilde{H}_0 = \bigoplus_{j=1}^{t}\left(\bigoplus_{i=1}^{r_j}\varepsilon_i^{(j)}Q_{a_j+1} \oplus \bigoplus_{i=r_j+1}^{m}\varepsilon_i^{(j)}Q_{a_j}\right) \oplus \bigoplus_{j=1}^{t}\left(\bigoplus_{i=1}^{r_j}\varepsilon_i^{(j)}Q_{a_j+1} \oplus \bigoplus_{i=r_j+1}^{m}\varepsilon_i^{(j)}Q_{a_j}\right),$$

for some $a_j, r_j \in \mathbb{Z}$ with $0 < r_j \leq m$. The signs are as follows, given in terms of η_j, where η_j could be either 1 or -1: if r_j (respectively $m - r_j$) is even, half of $\varepsilon_i^{(j)}$ for $i = 1, \ldots, r_j$ (respectively for $i = r_j + 1, \ldots, m$) is equal to η_j and the other half is equal to $-\eta_j$. If r_j (respectively $m - r_j$) is odd, there is one more of $\varepsilon_i^{(j)}$ for $i = 1, \ldots, r_j$ (respectively for $i = r_j + 1, \ldots, m$) equal to η_j than those equal to $-\eta_j$.

Note that it follows from Lemma 2.5 (which also holds for matrices in Ω_{2n}) that it is sufficient to assume that the pair $(\tilde{B}, \tilde{H})$ is in canonical form.

For the positive eigenvalue case, we now prove the following:

THEOREM 3.3. *Let $\tilde{B}, \tilde{H} \in \Omega_{2n}$, where $\tilde{H}$ is invertible and Hermitian, and $\tilde{B}$ is $\tilde{H}$-selfadjoint with a spectrum consisting of only positive real numbers. Then there exists an $\tilde{H}$-selfadjoint matrix in Ω_{2n}, say $\tilde{A}$, such that $\tilde{A}^m = \tilde{B}$.*

Proof. Let $\tilde{B} \in \Omega_{2n}$ be $\tilde{H}$-selfadjoint with only positive real eigenvalues, where $\tilde{H} \in \Omega_{2n}$ is invertible and Hermitian. We can assume that $\tilde{B} = J_n(\lambda) \oplus J_n(\lambda)$, where λ is a positive real number, and that $\tilde{H} = \varepsilon Q_n \oplus \varepsilon Q_n$, $\varepsilon = \pm 1$. To construct an mth root of $\tilde{B}$ which is $\tilde{H}$-selfadjoint and also in Ω_{2n}, we let $J = J_n(\mu) \oplus J_n(\mu)$ where μ is the positive real mth root of λ. Then both J and $J^m = (J_n(\mu))^m \oplus (J_n(\mu))^m$ are $\tilde{H}$-selfadjoint, and the Jordan normal form of J^m is equal to $\tilde{B}$. We now wish to find an invertible matrix $P \in \Omega_{2n}$ such that equations:

$$P^{-1}J^mP = \tilde{B} \quad \text{and} \quad P^*\tilde{H}P = \tilde{H}, \tag{3.6}$$

hold. To ensure that the first equation holds, let $P = P_1 \oplus P_2$ with

$$P_1 = \left[((J_n(\mu))^m - \lambda I)^{n-1}y \quad \cdots \quad ((J_n(\mu))^m - \lambda I)\,y \quad y\right],$$
$$P_2 = \left[((J_n(\mu))^m - \lambda I)^{n-1}z \quad \cdots \quad ((J_n(\mu))^m - \lambda I)\,z \quad z\right],$$

where $y, z \in \operatorname{Ker}((J_n(\mu))^m - \lambda I)^n = \mathbb{C}^n$ but $y, z \notin \operatorname{Ker}((J_n(\mu))^m - \lambda I)^{n-1}$. From the choices of y and z, it is clear that P_1 and P_2 are invertible $n \times n$ matrices and hence the $2n \times 2n$ matrix P is invertible. Let $z = \bar{y}$, so that $P_2 = \bar{P}_1$; then P is in Ω_{2n}. Note that $P^*\tilde{H}P = \tilde{H}$ if and only if $P_1^*Q_nP_1 = Q_n$ and since $P_1^*Q_nP_1$ is a lower anti-triangular Hankel matrix, $P^*\tilde{H}P = \tilde{H}$ holds if and only if

$$\phi_{n,j}(y) = [p_j, p_n] = y^*Q_n\,((J_n(\mu))^m - \lambda I)^{n-j}\,y = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j = 2, \ldots, n, \end{cases}$$

where $\phi_{i,j}(y)$ denotes the entries in the matrix $P_1^*Q_nP_1$ and p_i denotes the columns in the matrix P_1. As illustrated in [5], one can easily find one solution to these equations by assuming that y is real. Therefore, there exists a matrix $P \in \Omega_{2n}$ satisfying the equations (3.6). Then by using Lemma 2.5, the matrix $\tilde{A} := P^{-1}JP$ is an $\tilde{H}$-selfadjoint mth root of $\tilde{B}$, and $\tilde{A}$ is also in Ω_{2n} since P is. ∎

Now, for the nonreal eigenvalue case.

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THEOREM 3.4. *Let $\tilde{B}, \tilde{H} \in \Omega_{4n}$, where $\tilde{H}$ is invertible and Hermitian, and $\tilde{B}$ is $\tilde{H}$-selfadjoint with a spectrum consisting of only nonreal numbers. Then there exists an $\tilde{H}$-selfadjoint matrix in Ω_{4n}, say $\tilde{A}$, such that $\tilde{A}^m = \tilde{B}$.*

Proof. Let $\tilde{B} \in \Omega_{4n}$ be $\tilde{H}$-selfadjoint with only nonreal eigenvalues, where $\tilde{H} \in \Omega_{4n}$ is invertible and Hermitian. We can assume that $\tilde{B} = J_n(\lambda) \oplus J_n(\bar{\lambda}) \oplus J_n(\bar{\lambda}) \oplus J_n(\lambda)$, where λ is a nonreal number, and that $\tilde{H} = Q_{2n} \oplus Q_{2n}$. To construct an mth root of $\tilde{B}$, let $J = J_n(\mu) \oplus J_n(\bar{\mu}) \oplus J_n(\bar{\mu}) \oplus J_n(\mu)$ where μ is any mth root of λ. Then the Jordan normal form of J^m is equal to $\tilde{B}$, and both J and J^m are $\tilde{H}$-selfadjoint. Let $P = P_1 \oplus \bar{P}_1 \oplus \bar{P}_1 \oplus P_1 \in \Omega_{4n}$ where

$$P_1 = \left[((J_n(\mu))^m - \lambda I)^{n-1} y \quad \cdots \quad ((J_n(\mu)^m - \lambda I) y \quad y \right],$$

with $y \in \operatorname{Ker}((J_n(\mu))^m - \lambda I)^n = \mathbb{C}^n$ but $y \notin \operatorname{Ker}((J_n(\mu))^m - \lambda I)^{n-1}$. Then $P^*\tilde{H}P = \tilde{H}$ if and only if $P_1^T Q_n P_1 = Q_n$, and according to the proof of Theorem 2.4 in [5], there exists a solution to the latter equation. Hence, there exists an invertible matrix $P \in \Omega_{4n}$ such that the equations:

$$P^{-1} J^m P = \tilde{B} \quad \text{and} \quad P^* \tilde{H} P = \tilde{H},$$

hold. Once again, by Lemma 2.5, the matrix $\tilde{A} := P^{-1}JP$ is an $\tilde{H}$-selfadjoint mth root of $\tilde{B}$ and is in Ω_{4n}. ∎

Before stating the results for the negative case, we give the following result which was obtained by applying ω_n to all matrices in Lemma 2.12 of [5].

LEMMA 3.5. *Let $T \in \Omega_{2n}$ be a diagonal block matrix consisting of an upper triangular Toeplitz matrix with diagonal entries $t_1, \ldots, t_n$ and its complex conjugate. Let the diagonal entries $\lambda = t_1$ be real, t_2 nonzero and $B = T \oplus \bar{T}$. Then B is $(Q_{2n} \oplus Q_{2n})$-selfadjoint, and the pair $(B, Q_{2n} \oplus Q_{2n})$ is unitarily similar to*

$$(J_n(\lambda) \oplus J_n(\lambda) \oplus J_n(\lambda) \oplus J_n(\lambda), Q_n \oplus -Q_n \oplus Q_n \oplus -Q_n).$$

Turning to the case of the negative eigenvalue for a matrix $\tilde{B}$, we first point out that if m is even, any mth root $\tilde{A}$ will necessarily have only complex eigenvalues, so in order for $\tilde{A}$ to be $\tilde{H}$-selfadjoint, by Theorem 2.6, it (and so also $\tilde{B}$) would have to be in Ω_{4n}. We now first prove a result for m even.

THEOREM 3.6. *Let $\tilde{H} \in \Omega_{4n}$ be an invertible Hermitian matrix and let $\tilde{B} \in \Omega_{4n}$ be an $\tilde{H}$-selfadjoint matrix with a spectrum consisting of only negative real numbers. Then, for an even positive integer m, there exists an $\tilde{H}$-selfadjoint matrix in Ω_{4n}, say $\tilde{A}$, such that $\tilde{A}^m = \tilde{B}$ if and only if the canonical form of $(\tilde{B}, \tilde{H})$ is given by:*

$$S^{-1}\tilde{B}S = \bigoplus_{j=1}^{t} \left(J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j) \right) \oplus \bigoplus_{j=1}^{t} \left(J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j) \right), \tag{3.7}$$

and

$$S^*\tilde{H}S = \bigoplus_{j=1}^{t} \left(Q_{k_j} \oplus -Q_{k_j} \right) \oplus \bigoplus_{j=1}^{t} \left(Q_{k_j} \oplus -Q_{k_j} \right), \tag{3.8}$$

where $\lambda_j < 0$ and S is some invertible matrix in Ω_{4n}.

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Proof. Let $\tilde{B}$ be an $\tilde{H}$-selfadjoint matrix where both $\tilde{B}$ and $\tilde{H}$ are in Ω_{4n} and $\tilde{B}$ has only negative eigenvalues. Assume that there exists an $\tilde{H}$-selfadjoint mth root $\tilde{A} \in \Omega_{4n}$ of $\tilde{B}$, that is, $\tilde{A}^m = \tilde{B}$. Denote the eigenvalues of $\tilde{B}$ by λ_j, and let μ_j be any mth root of λ_j. Since m is even, μ_j is nonreal. Thus from Theorem 2.6, we know that the canonical form of $(\tilde{A}, \tilde{H})$ is (J, Q) where

$$J = \bigoplus_{j=1}^{t} \left(J_{k_j}(\mu_j) \oplus J_{k_j}(\bar{\mu}_j)\right) \oplus \bigoplus_{j=1}^{t} \left(J_{k_j}(\bar{\mu}_j) \oplus J_{k_j}(\mu_j)\right),$$

and

$$Q = \bigoplus_{j=1}^{t} Q_{2k_j} \oplus \bigoplus_{j=1}^{t} Q_{2k_j},$$

for some t. Hence, there exists an invertible matrix $P \in \Omega_{4n}$ such that the equations $P^{-1}\tilde{A}P = J$ and $P^*\tilde{H}P = Q$ hold. Consider

$$P^{-1}\tilde{B}P = (P^{-1}\tilde{A}P)^m = J^m = \bigoplus_{j=1}^{t} \left((J_{k_j}(\mu_j))^m \oplus (J_{k_j}(\bar{\mu}_j))^m\right) \oplus \bigoplus_{j=1}^{t} \left((J_{k_j}(\bar{\mu}_j))^m \oplus (J_{k_j}(\mu_j))^m\right),$$

and note that this matrix is $P^*\tilde{H}P$-selfadjoint. Next, by taking

$$T_j = (J_{k_j}(\mu_j))^m \oplus (J_{k_j}(\bar{\mu}_j))^m,$$

for $j = 1, \ldots, t$, and applying Lemma 3.5 we have that $(T_j \oplus \bar{T}_j,\ Q_{2k_j} \oplus Q_{2k_j})$ is unitarily similar to

$$\left(J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j) \oplus J_{k_j}(\lambda_j),\ Q_{k_j} \oplus -Q_{k_j} \oplus Q_{k_j} \oplus -Q_{k_j}\right),$$

for all $j = 1, \ldots, t$. Therefore, by using the permutation matrix defined in (2.5), we see that the canonical form of $(\tilde{B}, \tilde{H})$ is given by (3.7) and (3.8).

Conversely, let $\tilde{B}$ be $\tilde{H}$-selfadjoint and such that the canonical form of $(\tilde{B}, \tilde{H})$ is given by (3.7) and (3.8), and let m be even. Assume that

$$\tilde{B} = J_n(\lambda) \oplus J_n(\lambda) \oplus J_n(\lambda) \oplus J_n(\lambda) \quad \text{and} \quad \tilde{H} = Q_n \oplus -Q_n \oplus Q_n \oplus -Q_n,$$

where $\lambda < 0$. Then let $J = J_n(\mu) \oplus J_n(\bar{\mu}) \oplus J_n(\bar{\mu}) \oplus J_n(\mu)$ where μ is any mth root of λ. The number μ is nonreal and therefore J is Q-selfadjoint where $Q = Q_{2n} \oplus Q_{2n}$. Note that the matrix J^m has λ on its main diagonal and satisfies the conditions of $T \oplus \bar{T}$ as in Lemma 3.5. Thus, it follows that the pair $(J^m, Q_{2n} \oplus Q_{2n})$ is unitarily similar to the pair $(\tilde{B}, \tilde{H})$, that is, there exists an invertible matrix $P \in \Omega_{4n}$ such that the equations $P^{-1}J^mP = \tilde{B}$ and $P^*QP = \tilde{H}$ hold. Finally, from Lemma 2.5, the matrix $\tilde{A} := P^{-1}JP \in \Omega_{4n}$ is an $\tilde{H}$-selfadjoint mth root of $\tilde{B}$. ∎

Next, we give the negative eigenvalue case where m is odd. This case is similar to the positive eigenvalue case.

THEOREM 3.7. *Let $\tilde{H} \in \Omega_{2n}$ be an invertible Hermitian matrix and $\tilde{B} \in \Omega_{2n}$ an $\tilde{H}$-selfadjoint matrix with a spectrum consisting of only negative real numbers. Then, for m odd, there exists an $\tilde{H}$-selfadjoint matrix in Ω_{2n}, say $\tilde{A}$, such that $\tilde{A}^m = \tilde{B}$.*

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Proof. Let $\tilde{B} \in \Omega_{2n}$ be $\tilde{H}$-selfadjoint with only negative real eigenvalues, where $\tilde{H} \in \Omega_{2n}$ is invertible and Hermitian, and let m be odd. Assume that $\tilde{B} = J_n(\lambda) \oplus J_n(\lambda)$ and $\tilde{H} = \varepsilon Q_n \oplus \varepsilon Q_n$, where $\lambda < 0$ and $\varepsilon = \pm 1$. Since m is odd, we can take μ to be the real mth root of λ and let $J = J_n(\mu) \oplus J_n(\mu)$. Then the Jordan normal form of J^m is equal to $\tilde{B}$, and both J and J^m are $\tilde{H}$-selfadjoint. Similarly to Theorem 3.3, we can construct an invertible matrix $P \in \Omega_{2n}$ such that the equations:

$$P^{-1} J^m P = \tilde{B} \quad \text{and} \quad P^* \tilde{H} P = \tilde{H},$$

hold. Therefore, from Lemma 2.5, the matrix $\tilde{A} := P^{-1} J P \in \Omega_{2n}$ is an $\tilde{H}$-selfadjoint mth root of $\tilde{B}$. ◻

In the case where the matrix $\tilde{B}$ has only the number zero as an eigenvalue, we instantly notice a necessary condition for the existence of an mth root $\tilde{A}$ of $\tilde{B}$. Note that $\tilde{A}$ is not necessarily $\tilde{H}$-selfadjoint. Since each number in the Segre characteristic of $\tilde{A}$ corresponding to the zero eigenvalue occurs twice (see Corollary 2.3) and from the way mth roots are formed for nilpotent matrices, see for example [3, 5], the m-tuples in the Segre characteristic (or some reordering thereof) corresponding to the zero eigenvalue of $\tilde{B}$ have to exist in pairs, that is, there are two of each m-tuple. For example, with $m = 4$, if the nonzero part of the Segre characteristic of a nilpotent matrix $\tilde{B} \in \Omega_{20}$ is $(3, 3, 2, 2), (3, 3, 2, 2)$, then the nonzero part of the Segre characteristic of any mth root $\tilde{A} \in \Omega_{20}$ is $(10, 10)$.

Here we need another result from [5] which was obtained by applying ω_n to all matrices:

Lemma 3.8. *Let A be equal to $J_n(0) \oplus J_n(0)$. Then A^m has Jordan normal form:*

$$\bigoplus_{i=1}^{r} J_{a+1}(0) \oplus \bigoplus_{i=1}^{m-r} J_a(0) \oplus \bigoplus_{i=1}^{r} J_{a+1}(0) \oplus \bigoplus_{i=1}^{m-r} J_a(0),$$

where $n = am + r$, for $a, r \in \mathbb{Z}$, $0 < r \leq m$.

We now give the result for this last case.

Theorem 3.9. *Let $\tilde{H} \in \Omega_{2n}$ be an invertible Hermitian matrix and $\tilde{B} \in \Omega_{2n}$ an $\tilde{H}$-selfadjoint matrix with a spectrum consisting of only the number zero. Then there exists an $\tilde{H}$-selfadjoint matrix in Ω_{2n}, say $\tilde{A}$, such that $\tilde{A}^m = \tilde{B}$ if and only if the following properties hold:*

1. *There exists a reordering of the Segre characteristic of $\tilde{B}$ such that each m-tuple occurs twice and the difference between any two numbers in each m-tuple is at most one.*
2. *By using a reordering satisfying the first property, the canonical form of $(\tilde{B}, \tilde{H})$ is given by $(J_B \oplus J_B, H_B \oplus H_B)$ where*

$$J_B = \bigoplus_{j=1}^{t} \left(\bigoplus_{i=1}^{r_j} J_{a_j+1}(0) \oplus \bigoplus_{i=r_j+1}^{m} J_{a_j}(0) \right),$$

and

$$H_B = \bigoplus_{j=1}^{t} \left(\bigoplus_{i=1}^{r_j} \varepsilon_i^{(j)} Q_{a_j+1} \oplus \bigoplus_{i=r_j+1}^{m} \varepsilon_i^{(j)} Q_{a_j} \right), \tag{3.9}$$

for some $a_j, r_j \in \mathbb{Z}$ with $0 < r_j \leq m$, and the signs are as follows, given in terms of η_j, where η_j could be either 1 or -1: If r_j (respectively $m - r_j$) is even, half of the $\varepsilon_i^{(j)}$ for $i = 1, \ldots, r_j$ (respectively

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for $i = r_j+1,\dots,m$) are equal to η_j and the other half are equal to $-\eta_j$. If r_j (respectively $m-r_j$) is odd, there is one more of the $\varepsilon_i^{(j)}$ for $i=1,\dots,r_j$ (respectively for $i=r_j+1,\dots,m$) equal to η_j than to $-\eta_j$.

Proof. Let $\tilde{B} \in \Omega_{2n}$ be $\tilde{H}$-selfadjoint with only zero in its spectrum, where $\tilde{H} \in \Omega_{2n}$ is invertible and Hermitian. Assume there exists an $\tilde{H}$-selfadjoint mth root $\tilde{A} \in \Omega_{2n}$, that is, $\tilde{A}^m = \tilde{B}$. From Theorem 2.6, there exists an invertible matrix $S \in \Omega_{2n}$ such that the canonical form of $(\tilde{A}, \tilde{H})$ is given by:

$$S^{-1}\tilde{A}S = \bigoplus_{j=1}^{t} J_{k_j}(0) \oplus \bigoplus_{j=1}^{t} J_{k_j}(0) \quad \text{and} \quad S^*\tilde{H}S = \bigoplus_{j=1}^{t} \eta_j Q_{k_j} \oplus \bigoplus_{j=1}^{t} \eta_j Q_{k_j},$$

for some t, k_j and signs $\eta_j = \pm 1$. Consider

$$S^{-1}\tilde{B}S = (S^{-1}\tilde{A}S)^m = \bigoplus_{j=1}^{t} (J_{k_j}(0))^m \oplus \bigoplus_{j=1}^{t} (J_{k_j}(0))^m.$$

Therefore by using Lemma 3.8 and the permutation matrix defined in (2.5), the matrix $\tilde{B}$ has Jordan normal form:

$$\bigoplus_{j=1}^{t} \left(\bigoplus_{i=1}^{r_j} J_{a_j+1}(0) \oplus \bigoplus_{i=1}^{m-r_j} J_{a_j}(0) \right) \oplus \bigoplus_{j=1}^{t} \left(\bigoplus_{i=1}^{r_j} J_{a_j+1}(0) \oplus \bigoplus_{i=1}^{m-r_j} J_{a_j}(0) \right), \tag{3.10}$$

where $k_j = a_j m + r_j$, $a_j, r_j \in \mathbb{Z}$ and $0 < r_j \le m$. Hence, from (3.10), we see that there exists a reordering of the Segre characteristic of $\tilde{B}$ in which each m-tuple occurs twice and where the difference of any two numbers in each m-tuple is at most one.

Since the Jordan normal form of $\tilde{B}$ is given in (3.10), by Theorem 2.6 the corresponding matrix in the canonical form is given by $H_B \oplus H_B$ where

$$H_B = \bigoplus_{j=1}^{t} \left(\bigoplus_{i=1}^{r_j} \varepsilon_i^{(j)} Q_{a_j+1} \oplus \bigoplus_{i=r_j+1}^{m} \varepsilon_i^{(j)} Q_{a_j} \right),$$

for $\varepsilon_i^{(j)} = \pm 1$, $i = 1,\dots,m$, $j = 1,\dots,t$. From the assumption that an $\tilde{H}$-selfadjoint mth root exists, we know that the two properties given in [5, Theorem 2.5] hold. The second property is used to find the signs of H_B. Thus, it follows: if r_j (respectively $m-r_j$) is even, half of the $\varepsilon_i^{(j)}$ for $i=1,\dots,r_j$ (respectively for $i=r_j+1,\dots,m$) are equal to η_j and the other half are equal to $-\eta_j$. If r_j (respectively $m-r_j$) is odd, there is one more of the $\varepsilon_i^{(j)}$ for $i=1,\dots,r_j$ (respectively for $i=r_j+1,\dots,m$) equal to η_j than to $-\eta_j$.

Conversely, let $\tilde{B}$ be an $\tilde{H}$-selfadjoint matrix and suppose that the two properties in the theorem hold. Assume from the first property that $\tilde{B} = B_1 \oplus B_1 \in \Omega_{2n}$ where

$$B_1 = \bigoplus_{j=1}^{t} \left(\bigoplus_{i=1}^{r_j} J_{a_j+1}(0) \oplus \bigoplus_{i=1}^{m-r_j} J_{a_j}(0) \right),$$

and assume that $\tilde{H} = H_1 \oplus H_1$ where $H_1 = H_B$ is as in (3.9). Let

$$J = \bigoplus_{j=1}^{t} J_{t_j}(0) \oplus \bigoplus_{j=1}^{t} J_{t_j}(0),$$

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where t_j is the sum of the sizes of the blocks in $\tilde{B}$ which correspond to one m-tuple, that is, $t_j = r_j(a_j + 1) + (m - r_j)(a_j) = a_j m + r_j$. Thus, by using Lemma 3.8 and the permutation matrix defined by (2.5), the Jordan normal form of J^m is equal to $\tilde{B}$. Also, let $Q = \bigoplus_{j=1}^t \varepsilon_j Q_{t_j} \oplus \bigoplus_{j=1}^t \varepsilon_j Q_{t_j}$ where $\varepsilon_j = \eta_j$ is obtained from the signs of H_B, then J is Q-selfadjoint. According to [5, Theorem 2.5], there exists an invertible matrix P_1 such that

$$P_1^{-1}\Big(\bigoplus_{j=1}^{t} J_{t_j}(0)\Big)^m P_1 = B_1 \quad \text{and} \quad P_1^*\Big(\bigoplus_{j=1}^{t} \varepsilon_j Q_{t_j}\Big) P_1 = H_1.$$

Let $P = P_1 \oplus \bar{P}_1$, then P is an invertible matrix in Ω_{2n} and the equations:

$$P^{-1} J^m P = \tilde{B} \quad \text{and} \quad P^* Q P = \tilde{H},$$

hold. From Lemma 2.5, the matrix $\tilde{A} := P^{-1} J P \in \Omega_{2n}$ is an $\tilde{H}$-selfadjoint mth root of $\tilde{B}$. ∎

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