THE STRUCTURE OF LINEAR PRESERVERS OF LEFT MATRIX
MAJORIZATION ON \( \mathbb{R}^p \)

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Abstract. For vectors \( X, Y \in \mathbb{R}^n \), \( Y \) is said to be left matrix majorized by \( X \) (\( Y \prec_\ell X \)) if for some row stochastic matrix \( R \), \( Y = RX \). A linear operator \( T: \mathbb{R}^p \to \mathbb{R}^n \) is said to be a linear preserver of \( \prec_\ell \) if \( Y \prec_\ell X \) on \( \mathbb{R}^p \) implies that \( TY \prec_\ell TX \) on \( \mathbb{R}^n \). The linear operators \( T: \mathbb{R}^p \to \mathbb{R}^n \) (\( n < p(p-1) \)) which preserve \( \prec_\ell \) have been characterized. In this paper, linear operators \( T: \mathbb{R}^p \to \mathbb{R}^n \) which preserve \( \prec_\ell \) are characterized without any condition on \( n \) and \( p \).

Key words. Row stochastic matrix, Doubly stochastic matrix, Matrix majorization, Weak matrix majorization, Left (right) multivariate majorization, Linear preserver.

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1. Introduction. Let \( M_{nm} \) be the algebra of all \( n \times m \) real matrices. A matrix \( R = [r_{ij}] \in M_{nm} \) is called a row stochastic (resp., row substochastic) matrix if \( r_{ij} \geq 0 \) and \( \sum_{k=1}^m r_{ik} = 1 \) (resp., \( \leq 1 \)) for all \( i, j \). For \( A, B \) in \( M_{nm} \), \( A \) is said to be left matrix majorized by \( B \) (\( A \prec_\ell B \)), if \( A = RB \) for some \( n \times n \) row stochastic matrix \( R \). These notions were introduced in [11]. If \( A \prec_\ell B \prec_\ell A \), we write \( A \sim_\ell B \). Let \( T: \mathbb{R}^p \to \mathbb{R}^n \) be a linear operator. \( T \) is said to be a linear preserver of \( \prec_\ell \) if \( Y \prec_\ell X \) on \( \mathbb{R}^p \) implies that \( TY \prec_\ell TX \) on \( \mathbb{R}^n \). For more information about types of majorization see [1], [5] and [10]; for their preservers see [2]-[4], [6] and [9].

We shall use the following conventions throughout the paper: Let \( T: \mathbb{R}^p \to \mathbb{R}^n \) be a nonzero linear operator and let \( [T] = [t_{ij}] \) denote the matrix representation of \( T \) with respect to the standard bases \( \{e_1, e_2, \ldots, e_p\} \) of \( \mathbb{R}^p \) and \( \{f_1, f_2, \ldots, f_n\} \) of \( \mathbb{R}^n \). If \( p = 1 \), then all linear operators on \( \mathbb{R}^1 \) are preservers of \( \prec_\ell \). Thus, we assume \( p \geq 2 \). Let \( A_i \) be \( m_i \times p \) matrices, \( i = 1, \ldots, k \). We use the notation \([A_1/A_2/\ldots/A_k]\) to denote the corresponding \((m_1 + m_2 + \ldots + m_k) \times p \) matrix. We let \( e = (1,1,\ldots,1)^t \in \mathbb{R}^p \), and denote

\[
a := \max \{\max T(e_1), \ldots, \max T(e_p)\}, \\
b := \min \{\min T(e_1), \ldots, \min T(e_p)\}.
\]
Linear Preservers of Left Matrix Majorization

Theorem 1.1. ([9, Theorem 2.2]) Let $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a nonzero linear preserver of $\prec_{\ell}$ and suppose $p \geq 2$. Then $p \leq n$, $b \leq 0 \leq a$ and for each $i \in \{1, \ldots, p\}$, $a = \max T(e_i)$ and $b = \min T(e_i)$. In particular, every column of $[T]$ contains at least one entry equal to $a$ and at least one entry equal to $b$.

Definition 1.2. Let $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator. We denote by $P_i$ (resp., $N_i$) the sum of the nonnegative (resp., non positive) entries in the $i^{th}$ row of $[T]$. If all the entries in the $i^{th}$ row are positive (resp., negative), we define $N_i = 0$ (resp., $P_i = 0$).

We know that $T$ is a linear preserver of $\prec_{\ell}$ if and only if $\alpha T$ is also a linear preserver of $\prec_{\ell}$ for some nonzero real number $\alpha$. Without loss of generality we make the following assumption.

Assumption 1.3. Let $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a nonzero linear preserver of $\prec_{\ell}$. Let $a$ and $b$ be as in (1.1). We assume that $0 \leq -b \leq 1 = a$.

Definition 1.4. Let $P$ be the permutation matrix such that $P(e_i) = e_{i+1}$, $1 \leq i \leq p-1$, $P(e_p) = e_1$. Let $I$ denote the $p \times p$ identity matrix, and let $r, s \in \mathbb{R}$ be such that $rs < 0$. Define the $p(p-1) \times p$ matrix $P_p(r, s) = [P_1/P_2/ \ldots /P_{p-1}]$, where $P_j = rI + sP^j$, for all $j = 1, 2, \ldots, p-1$. It is clear that up to a row permutation, the matrices $P_p(r, s)$ and $P_p(s, r)$ are equal. Also define $P_p(r, 0) := rI$, $P_p(0, s) := sI$ and $P_p(0, 0)$ as a zero row.

The structure of all linear operators $T: M_{nm} \rightarrow M_{nm}$ preserving matrix majorizations was considered in [6, 7, 8]. Also the linear operators $T$ from $\mathbb{R}^p$ to $\mathbb{R}^n$ that preserve the left matrix majorization $\prec_{\ell}$ were characterized in [9] for $n < p(p-1)$. In the present paper, we will characterize all linear preservers of $\prec_{\ell}$ mapping $\mathbb{R}^p$ to $\mathbb{R}^n$ without any additional conditions.

2. Left matrix majorization. In this section we obtain a key condition that is necessary for $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ to be a linear preserver of $\prec_{\ell}$. We first need the following.

Lemma 2.1. Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator such that $\min T(Y) \leq \min T(X)$ for all $X \prec_{\ell} Y$. Then $T$ is a preserver of $\prec_{\ell}$.

Proof. Let $X \prec_{\ell} Y$. It is enough to show that $\max T(X) \leq \max T(Y)$. Since $X \prec_{\ell} Y$, $-X \prec_{\ell} -Y$, and hence $\min T(-Y) \leq \min T(-X)$. This means that $\max T(X) \leq \max T(Y)$. Then $T$ is a preserver of $\prec_{\ell}$. \[QED\]

Remark 2.2. Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear preserver of $\prec_{\ell}$ and let $a$ and $b$ be as in Assumption 1.3. By Theorem 1.1 we know that in each column of $[T] = [t_{ij}]$ there is at least one entry equal to $a(= 1)$ and at least one entry equal to $b$. For $1 \leq k \leq p$, \[QED\]
we define

$$I_k = \{ i : 1 \leq i \leq n, t_{ik} = 1 \}, \quad J_k = \{ j : 1 \leq j \leq n, t_{jk} = b \}.$$ 

Next we state the key theorem of this paper.

**Theorem 2.3.** Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear preserver of $\prec _\ell$ and let $a$ and $b$ be as in Assumption 1.3. Then there exist $0 \leq \alpha \leq 1$ and $b \leq 0$ such that $\mathcal{P}_p(1, \beta)$ and $\mathcal{P}_p(\alpha, b)$ are submatrices of $[T]$, where $\mathcal{P}_p(r, s)$ is as in Definition 1.4.

**Proof.** Let $1 \leq k \leq p$ be a fixed number and let $I_k$ and $J_k$ be as in Remark 2.2. Since $T$ is a linear preserver of $\prec _\ell$, it follows that $I_k$ and $J_k$ are nonempty sets. Also $e_k + e_l \prec _\ell e_k$, $l \neq k$. Thus, the other entries in the $i^{th}$ row, $i \in I_k$ (resp., $j^{th}$ row, $j \in J_k$) are non positive (resp., nonnegative). Hence, $t_{il} \leq 0$, $t_{jl} \geq 0$, $l \neq k$, $i \in I_k$, and $j \in J_k$. Let $\beta_k = \sum_{i \neq k} t_{il} \leq 0$, $i \in I_k$ and $\alpha_k = \sum_{i \neq k} t_{jl} \geq 0$, $j \in J_k$. Set

$$\beta_k := \min \{ \beta_k, i \in I_k \}, \quad \alpha_k := \max \{ \alpha_k, j \in J_k \}. \tag{2.1}$$

Define $X_k = -(N + 1)e_k + e$. Choose $N_0$ large enough such that for all $N \geq N_0$ and $1 \leq i \leq n$,

$$\min T(X_k) = -N + \beta_k \leq -N t_{ik} + \sum_{i \neq k} t_{il} \leq -N b + \alpha_k = \max T(X_k). \tag{2.2}$$

We know that $X_k \prec _\ell X_r = -(N + 1)e_r + e$, $1 \leq r \leq p$ and $T$ is a linear preserver of $\prec _\ell$. Hence by (2.2), $\alpha := \alpha_k = \alpha_r$ and $\beta := \beta_k = \beta_r$, $1 \leq r \leq p$. Also, $X_k \prec _\ell -N e_i + e_j$, $i \neq j$. For each $N \geq N_0$, there exists $1 \leq h \leq n$ such that $-N t_{hi} + t_{hj} = \min T(-N e_i + e_j) = \min T(X_k) = -N + \beta$ and for each $1 \leq i \leq p$, $1 \leq j \leq p$ and $N \geq N_0$, there exists $1 \leq h \leq n$ such that $-N(1 - t_{hi}) = t_{hj} - \beta$. It follows that $t_{hi} = 1$, $t_{hj} = \beta$. Hence $\mathcal{P}_p(1, \beta)$ is a submatrix of $[T]$. Similarly, there exists $N_1$, such that for each $N \geq N_1$ there exists $1 \leq h \leq n$ so that $-N t_{hi} + t_{hj} = \max T(-N e_i + e_j) = \max T(X_k) = -N b + \alpha$ and $-N(b - t_{hi}) = t_{hj} - \alpha$. Thus, $t_{hi} = b$ and $t_{hj} = \alpha$. Since $1 \leq i \neq j \leq p$ was arbitrary, $\mathcal{P}_p(b, \alpha)$ is a submatrix of $[T]$. Therefore, $\mathcal{P}_p(1, \beta)$ and $\mathcal{P}_p(b, \alpha)$ are submatrices of $[T]$. □

**Remark 2.4.** Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ and $\tilde{T} : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be two linear operators such that $[T] = [T_1, T_2, \ldots, T_n]$ and let $[\tilde{T}] = [\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_m]$ be the matrix representation of these operators with respect to the standard basis. Let $\mathcal{R}(T) = \{ T_1, T_2, \ldots, T_n \}$ be the set of all rows of $[T]$. If $\mathcal{R}(T) \subset \mathcal{R}(\tilde{T})$, then $T$ preserves $\prec _\ell$ if and only if $\tilde{T}$ preserves $\prec _\ell$.

**Lemma 2.5.** Let $T$ be a linear operator on $\mathbb{R}^p$. If $[T] = \mathcal{P}_p(\alpha, \beta)$, $\alpha \beta \leq 0$, then $T$ is a preserver of $\prec _\ell$. 


Proof. Without loss of generality, let $\beta \leq 0 \leq \alpha$ and let $X = (x_1, \ldots, x_p)^t$, $Y = (y_1, \ldots, y_p)^t \in \mathbb{R}^p$ such that $X \prec_\ell Y$. Then $y_m = \min Y \leq x_i \leq \max Y = y_M$, for all $1 \leq i \leq p$. It is easy to check that $\alpha y_m + \beta y_M \leq \alpha x_i + \beta x_j$, for all $i \neq j \in \{1, \ldots, p\}$, which implies $\min TY \leq \min TX$. Hence by Lemma 2.1, $TX \prec_\ell TY$. \hfill $\Box$

3. Left matrix majorization on $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator and let $a, b$, be as in Assumption 1.3. We consider the square $S = [b, 1] \times [b, 1]$ in $\mathbb{R}^2$.

**Definition 3.1.** Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator and let $[T] = [T_1/\ldots/T_n]$, where $T_i = (t_{i1}, t_{i2})$, $1 \leq i \leq n$. Define

$$\Delta := \text{Conv} \{ (t_{i1}, t_{i2}) | (t_{i1}, t_{i2}) \neq (t_{j1}, t_{j2}) \} \subseteq \mathbb{R}^2.$$ 

Also, let $C(T)$ denote the set of all corners of $\Delta$.

**Lemma 3.2.** Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and $[T] = [T_1/\ldots/T_n]$, where $T_j = (t_{j1}, t_{j2})$, $1 \leq j \leq n$. If for some $1 \leq i \leq n$, $t_{i1}t_{i2} > 0$, then $T_i \notin C(T)$, where $C(T)$ is as in Definition 3.1.

**Proof.** Assume that, if possible, there exists $1 \leq i \leq n$ such that $T_i \in C(T)$ and $t_{i1}t_{i2} > 0$. By Remark 2.4 we can assume that $[T]$ has no identical rows. Without loss of generality, we assume that there exist $1 \leq i \leq n$ and real numbers $m \leq M$ such that $t_{i1} > 0, t_{i2} > 0$ and $m t_{i1} + M t_{i2} < m t_{j1} + M t_{j2}, j \neq i$. Choose $\varepsilon > 0$ small enough so that $m t_{i1} + (M + \varepsilon) t_{i2} < m t_{j1} + (M + \varepsilon) t_{j2}, j \neq i$. Since $(m, M)^t \lessdot_\ell (m, M + \varepsilon)^t, T(m, M)^t \lessdot_\ell T(m, M + \varepsilon)^t$. But $\min (T(m, M + \varepsilon)^t) = m t_{i1} + (M + \varepsilon) t_{i2} > m t_{i1} + M t_{i2} = \min (T(m, M)^t)$, a contradiction. \hfill $\Box$

Next we shall characterize all linear operators $T : \mathbb{R}^2 \to \mathbb{R}^n$ which preserve $\prec_\ell$.

**Theorem 3.3.** Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator. Then $T$ is a linear preserver of $\prec_\ell$ if and only if $P_2(x, y)$ is a submatrix of $[T]$ and $xy \leq 0$ for all $(x, y) \in C(T)$.

**Proof.** Let $T$ be a linear preserver of $\prec_\ell$ with $0 \leq -b \leq 1 = a$. Let $(x, y) \in C(T)$, then by Lemma 3.2, $xy \leq 0$. Without loss of generality, let $T_i = (t_{i1}, t_{i2}) \in C(T)$ and $t_{i1}t_{i2} \leq 0$. By Remark 2.4, we assume that $[T]$ has no identical rows. Then there exist real numbers $m, M \in \mathbb{R}$ such that $m t_{i1} + M t_{i2} < m t_{j1} + M t_{j2}, j \neq i$. Choose $\varepsilon_0 > 0$ small enough so that $m( - \varepsilon) t_{i1} + (M + \varepsilon) t_{i2} < m( - \varepsilon) t_{j1} + (M + \varepsilon) t_{j2}, j \neq i, 0 < \varepsilon \leq \varepsilon_0$. Since $(M + \varepsilon, m - \varepsilon)^t \lessdot_\ell (m - \varepsilon, M + \varepsilon)^t, T(M + \varepsilon, m - \varepsilon)^t \lessdot_\ell T(m - \varepsilon, M + \varepsilon)^t$.

Hence, for all $0 \leq \varepsilon \leq \varepsilon_0$, there exist $1 \leq k \leq n$ such that $T_k = (t_{k1}, t_{k2}) \in C(T)$ and $(m - \varepsilon) t_{i1} + (M + \varepsilon) t_{i2} = \min T(m - \varepsilon, M + \varepsilon)^t = \min (T(M + \varepsilon, m - \varepsilon)^t) = (M + \varepsilon) t_{k1} + (m - \varepsilon) t_{k2}$. Since $k \in \{1, 2, \ldots, n\}$ is a finite set, there exists $k$ such that $t_{k1} = t_{i2}$ and $t_{k2} = t_{i1}$. Therefore, $P_2(t_{i1}, t_{i2})$ is a submatrix of $[T]$.

Conversely, let $P_2(x, y)$ be a submatrix of $[T]$ and suppose for all $(x, y) \in C(T)$,
Since $xy \leq 0$. Define the linear operator $\hat{T}$ on $\mathbb{R}^2$ such that $[\hat{T}] = [P_2(x_1, y_1) / \cdots / P_2(x_r, y_r)]$, where $(x_i, y_i) \in C(T), 1 \leq i \leq r$. By elementary convex analysis, we know that $\max X = \max \hat{T}(X)$ and $\min X = \min \hat{T}(X)$ for all $X \in \mathbb{R}^2$. Hence it is enough to show that $\hat{T}$ is a linear preserver of $\prec_\ell$. By Lemma 2.5, each $P_2(x_i, y_i)$ is a linear preserver of $\prec_\ell$. Thus, $\hat{T}$ is a linear preserver of $\prec_\ell$. $\square$

4. Left matrix majorization on $\mathbb{R}^p$. In this section we shall characterize all linear operators $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ which preserve $\prec_\ell$. We shall prove several lemmas and prove the main theorem of this paper.

**Definition 4.1.** Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator and let $[T] = [T_1 / \cdots / T_n]$. Define

$$\Omega := \text{Conv}([T_i = (t_{i1}, \ldots, t_{ip}), 1 \leq i \leq n]) \subseteq \mathbb{R}^p.$$  

Also, let $C(T)$ be the set of all corners of $\Omega$.

**Lemma 4.2.** Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and $[T] = [T_1 / \cdots / T_n]$, where $T_i = (t_{i1}, t_{i2}, \ldots, t_{ip}), 1 \leq i \leq n$. Suppose there exists $1 \leq i \leq n$ such that $t_{ij} > 0, \forall 1 \leq j \leq p$, or $t_{ij} < 0, \forall 1 \leq j \leq p$. Then $T_i \notin C(T)$, where $C(T)$ is as in Definition 4.1.

**Proof.** Assume that, if possible, there exists $1 \leq i \leq n$ such that $T_i \in C(T)$ and $t_{ij} > 0$, for all $1 \leq j \leq p$, or $t_{ij} < 0$, for all $1 \leq j \leq p$. By Remark 2.4, without loss of generality, we can assume that $[T]$ has no identical rows and there exists $1 \leq i \leq n$ such that $t_{ij} > 0$, for all $1 \leq j \leq p$. Since $T_i \in C(T)$, there exists $X = (x_1, \ldots, x_p)^T$ such that $x_1t_{i1} + x_2t_{i2} + \cdots + x_pl_{ip} < x_1t_{j1} + x_2t_{j2} + \cdots + x_pl_{jp}, 1 \neq i$. Let $x_k = \max \{x_i, 1 \leq i \leq p\}$. Choose $\varepsilon > 0$ small enough so that $x_1t_{i1} + \cdots + (x_k + \varepsilon)t_{ik} + \cdots + x_pl_{ip} < x_1t_{j1} + \cdots + (x_k + \varepsilon)t_{jk} + \cdots + x_pl_{jp}, 1 \neq i$. Define $\tilde{X} = (x_1, \ldots, x_k + \varepsilon, \ldots, x_p)^T$. Since $t_{ik} > 0$, hence $\min T(X) = x_1t_{i1} + x_2t_{i2} + \cdots + x_pl_{ip} < x_1t_{i1} + \cdots + (x_k + \varepsilon)t_{ik} + \cdots + x_pl_{ip} = \min T(\tilde{X})$. But $X \prec_\ell \tilde{X}$, a contradiction. $\square$

Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator. Without loss of generality, we assume that $[T] = [T^p/T^n/\bar{T}]$, where all entries of $T^p$ (resp., $T^n$) are positive (resp., negative) and each row of $\bar{T}$ has nonnegative and non positive entries.

**Corollary 4.3.** Let $T$ and $\bar{T}$ be as above. Then $T$ preserves $\prec_\ell$ if and only if $C(T) = C(\bar{T})$ and $\bar{T}$ preserves $\prec_\ell$, where $C(T)$ is as in Definition 4.1.

**Proof.** Let $T$ preserve $\prec_\ell$. By Lemma 4.2, $C(T) = C(\bar{T})$. Thus, if $X \in \mathbb{R}^p$, then $\max T(X) = \max \bar{T}(X)$ and $\min T(X) = \min \bar{T}(X)$. Therefore $\bar{T}$ preserves $\prec_\ell$. Conversely, let $C(T) = C(\bar{T})$. Then $\max T(X) = \max \bar{T}(X)$ and $\min T(X) = \min \bar{T}(X)$. Since $\bar{T}$ preserves $\prec_\ell$, $T$ preserves $\prec_\ell$. $\square$
**Definition 4.4.** Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. Define

$$\Delta = \text{Conv} \{ (P_i, N_i), (N_i, P_i) : 1 \leq i \leq n \},$$

where $P_i, N_i$ be as in (1.2). Let $E(T) = \{ (P_i, N_i) : (P_i, N_i) \text{ is a corner of } \Delta \}$. Let $1 \leq i \leq n$, define $[i] = \{ j : 1 \leq j \leq n, P_i = P_j \text{ and } N_i = N_j \}$.

**Lemma 4.5.** Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and let $C(T), E(T)$ be as in Definitions 4.1, 4.4, respectively. If $(P_r, N_r) \in E(T)$ for some $1 \leq r \leq n$, then there exists $k \in [r]$ such that $T_k \in C(T)$.

**Proof.** Suppose there exist $1 \leq r \leq n$ such that $(P_r, N_r) \in E(T)$. Then there exists $m \leq M$ such that

$$P_r m + N_r M < P_j m + N_j M, \quad j \notin [r]. \quad (4.1)$$

Let $X \in \mathbb{R}^p$ such that $\min(X) = m$ and $\max(X) = M$. Then there exists $1 \leq k \leq n$ such that $\min TX = \sum_{l=1}^p t_{kl} x_l$. Hence

$$P_r m + N_r M \leq P_k m + N_k M \leq \sum_{l=1}^p t_{kl} x_l = \min T X. \quad (4.2)$$

Define $Y \in \mathbb{R}^p$ by $y_l = m$, if $t_{rl} > 0$ and $y_l = M$, if $t_{rl} \leq 0$. Obviously $Y \prec_\ell X$. Since $T$ preserves $\prec_\ell$, $TY \prec_\ell TX$ which implies that

$$P_k m + N_k M \leq \sum_{l=1}^p t_{kl} x_l = \min T X \leq \min T Y \leq P_r m + N_r M. \quad (4.3)$$

Now, by (4.2) and (4.3), we have $P_r m + N_r M = P_k m + N_k M$. Thus by (4.1), $k \in [r]$ and $\min TX = \sum_{l=1}^p t_{kl} x_l$. Hence $T_k \in C(T)$ for some $k \in [r]$.

Next we state the main result in this paper.

**Theorem 4.6.** Let $T$ and $E(T)$ be as in Definition 4.4. Then $T$ preserves $\prec_\ell$ if and only if $P_\ell (\alpha, \beta)$ is a submatrix of $[T]$ for all $(\alpha, \beta) \in E(T)$.

**Proof.** Let $T$ be a preserver of $\prec_\ell$ and let $(P_r, N_r) \in E(T)$. Then there exists $m \leq M$ such that $P_r m + N_r M < P_j m + N_j M, \quad j \notin [r]$. Choose $\varepsilon_0$ small enough so that for all $0 < \varepsilon < \varepsilon_0$,

$$P_r (m - \varepsilon) + N_r (M + \varepsilon) < P_j (m - \varepsilon) + N_j (M + \varepsilon), \quad j \notin [r], \quad (4.4)$$

If $j \in [r]$, then $P_j = P_r$ and $N_j = N_r$. Thus

$$P_r (m - \varepsilon) + N_r (M + \varepsilon) \leq P_j (m - \varepsilon) + N_j (M + \varepsilon), \quad 1 \leq j \leq n.$$


Let $0 < \varepsilon < \varepsilon_0$, be fixed and let $X^\varepsilon = (x_1^\varepsilon, \ldots, x_p^\varepsilon)^t \in \mathbb{R}^p$ with $\min X^\varepsilon = m - \varepsilon$ and $\max X^\varepsilon = M + \varepsilon$. As in the proof of Lemma 4.5, there exists $k \in [r]$ such that

$$P_r(m - \varepsilon) + N_r(M + \varepsilon) = \min T(X^\varepsilon) = \sum_{l=1}^p t_{kl}x_l^\varepsilon.$$ 

Fix $i \neq j \in \{1, \ldots, p\}$ and define $Y^\varepsilon = (y_1^\varepsilon, \ldots, y_p^\varepsilon)^t \in \mathbb{R}^p$ such that $y_i^\varepsilon = m - \varepsilon$, $y_j^\varepsilon = M + \varepsilon$ and $y_l^\varepsilon = \gamma_l$, $m - \varepsilon < \gamma_l < M + \varepsilon$, $l \neq i, j$. Since $X^\varepsilon \sim_T Y^\varepsilon$, $TX^\varepsilon \sim_T TY^\varepsilon$, there exists $q \in [r]$ such that $t_{qi}(m - \varepsilon) + t_{qj}(M + \varepsilon) + \sum_{l\neq i,j} \gamma_l t_{ql} = P_r(m - \varepsilon) + N_r(M + \varepsilon)$. Since $0 < \varepsilon < \varepsilon_0$ and $m - \varepsilon < \gamma_l \leq M + \varepsilon$, $l \neq r, s$ are arbitrary, it is easy to show that there exists $s \in [r]$ such that $t_{si} = P_r$ and $t_{sj} = N_r$ and $t_{st} = 0, l \neq i, j$. Therefore $[T]$ has $P_r(P_r, N_r)$ as a submatrix.

Conversely, Let $E(T) = \{(P_{i_1}, N_{i_1}), \ldots, (P_{i_s}, N_{i_s})\}$. Then up to a row permutation $[T] = [P_p(P_{i_1}, N_{i_1})/ \ldots / P_p(P_{i_s}, N_{i_s})]/Q]$. Let $T_i \in Q$ and suppose there exists $X \in \mathbb{R}^p$ such that

$$\min T(X) = \sum_{i=1}^p t_{il}x_i \leq \sum_{l=1}^p t_{ij}x_l, 1 \leq j \leq n.$$ 

Obviously, $P_im + N_iM \leq \sum_{l=1}^p t_{il}x_i \leq \sum_{l=1}^p t_{jl}x_l, 1 \leq j \leq n$, where $m = \min X$ and $M = \max X$. We know that $(P_i, N_i) \in \Delta$ and $\Delta$ is convex. Hence there is $1 \leq k \leq n$ such that $(P_k, N_k) \in E(T)$ and $P_km + N_kM \leq P_im + N_iM$. As in the proof of Lemma 4.5, $\min TX = P_km + N_kM$. Then $\min \tilde{T}X \leq \min TX$. But we know that $\min T(X) \leq \min T\tilde{T}X$ and thus $\min T\tilde{T}X = \min TX$. Similarly, $\max T\tilde{T}X = \max TX$. Therefore, $T$ is a preserver of $\prec_\varepsilon$ if and only if $T$ preserves $\prec_\varepsilon$. By Lemma 2.5 each $P_p(P_{i_1}, N_{i_1})$ is a preserver of $\prec_\varepsilon$, $1 \leq l \leq k$. Hence $\tilde{T}$ is a preserver of $\prec_\varepsilon$ and the theorem is proved. 

Next we state necessary conditions for $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ to be a linear preserver of $\prec_\varepsilon$. We use the notation of Theorem 2.3 in the following corollary.

**Corollary 4.7.** Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator and let $a$ and $b$ be as given in (1.1). If the following conditions hold, then $T$ is a linear preserver of $\prec_\varepsilon$.

1. $[T]$ has $[P_p(a, 0)/P_p(0, b)/P_p(a, b)]$ as a submatrix.
2. $0 \leq P_i \leq a$ and $b \leq N_i \leq 0, \ 1 \leq i \leq n$.

where $P_i$ and $N_i$, $1 \leq i \leq n$ are as in Definition 1.2.

**Proof.** It is clear that $E(T) = \{(a, 0), (0, b), (a, b)\}$. Since $[T]$ has $P_p(a, 0), P_p(0, b)$ and $P_p(a, b)$ as submatrices, it follows by Theorem 4.6 that $T$ is a linear preserver of $\prec_\varepsilon$. 


Linear Preservers of Left Matrix Majorization

Let \( T : \mathbb{R}^p \to \mathbb{R}^n \) be a linear preserver of \( \prec_\ell \), and let \([T] = [T^1|T^2|\ldots|T^p]\), where \( T^i \) is the \( i^{th} \) column of \([T]\). For \( i \neq j \in \{1, \ldots, p\} \) define \( T^{ij} : \mathbb{R}^2 \to \mathbb{R}^n \) such that \([T^{ij}] = [T^i|T^j]\).

**Lemma 4.8.** Let \( T : \mathbb{R}^p \to \mathbb{R}^n \) be a linear preserver of \( \prec_\ell \), and let \( T^{ij} \) be as above. Then \( T^{ij} \) is a linear preserver of \( \prec_\ell \) for all \( i \neq j \in \{1, \ldots, p\} \).

**Proof.** Let \( i \neq j \in \{1, \ldots, p\} \) and let \( x = (x_1, x_2)^t \), \( y = (y_1, y_2)^t \in \mathbb{R}^2 \) such that \( x \prec_\ell y \). Define \( X, Y \in \mathbb{R}^p \) such that \( X_i = x_1, X_j = x_2, Y_i = y_1, Y_j = y_2 \) and \( X_k = Y_k = 0 \), for all \( k \neq i, j \). It is obvious that \( X \prec_\ell Y \) in \( \mathbb{R}^p \) and hence \( TX \prec_\ell TY \) in \( \mathbb{R}^n \). But \( T^{ij}x = x_1T^i + x_2T^j = TX \prec_\ell TY = y_1T^i + y_2T^j = T^{ij}y \). Therefore, \( T^{ij} \) is a linear preserver of \( \prec_\ell \). \( \square \)

The following example shows that the converse of Lemma 4.8 is not necessarily true.

**Example 4.9.** Assume \([T] = [P_3(1,-0.5)/0.25 0.25 0.25]\). Consider \( X = (-1,-1,-1)^t \) and \( Y = (-1,-1,-0.75)^t \), we know that \( X \prec_\ell Y \) and \( \min TX < \min TY \). Thus \( T \) is not a linear preserver of \( \prec_\ell \). However, by Corollary 4.7, for all \( i \neq j \in \{1,2,3\} \), \( T^{ij} \) preserves \( \prec_\ell \).

5. Additional results. In this section we give short proofs of some Theorems from [6, 9].

**Theorem 5.1.** [6] Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear operator. Then \( T \) preserves \( \prec_\ell \) if and only if \( T \) has the form \( T(X) = (aI + bP)X \) for all \( X \in \mathbb{R}^2 \), where \( P \) is the \( 2 \times 2 \) permutation matrix not equal to \( I \), and \( ab \leq 0 \).

**Proof.** Let \( T \) be a preserver of \( \prec_\ell \). By Assumption 1.3, \( a = 1 \). By Theorem 2.3, there exist \( 0 \leq \alpha \leq 1 \) and \( b \leq \beta \leq 0 \) such that \( P(1, \beta) \) and \( P(b, \alpha) \) are submatrices of \([T]\). Since \([T]\) is a \( 2 \times 2 \) matrix, \( \beta = b \) and \( \alpha = 1 \). Therefore, \([T] = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} \) and hence \( T(X) = (I + bP)X \), for all \( X \in \mathbb{R}^2 \). Conversely, up to a row permutation, \([T] = P_2(1, b)\) and by Lemma 2.5, \( T \) preserves \( \prec_\ell \). \( \square \)

**Theorem 5.2.** [6] Let \( p \geq 3 \). Then \( T : \mathbb{R}^p \to \mathbb{R}^p \) is a linear preserver of left matrix majorization if and only if \( T \) is of the form \( X \mapsto aPX \) for some \( a \in \mathbb{R} \) and some permutation matrix \( P \).

**Proof.** By Assumption 1.3, we have \( a = 1 \). Let \( T \) be a preserver of \( \prec_\ell \). By Theorem 2.3, \( b = 0 \) and \([T] \) has \( P_p(1, 0) \) as a submatrix; hence, up to a row permutation, \([T] = P_p(1, 0) = I \). Conversely, by a row permutation, \([T] = P_p(1, 0)\); hence by Lemma 2.5, \( T \) preserves \( \prec_\ell \). \( \square \)
Theorem 5.3. ([9, Theorem 3.1]) For a linear preserver $T$ of $\mathbb{R}^p$ to $\mathbb{R}^n$ the following assertions hold.

(a) If $n < 2p$ and $p \geq 3$, then $T$ is nonnegative.

(b) If $T$ is nonnegative, then there exists an $n \times n$ permutation matrix $Q$ such that $[T] = Q[I/W]$, where $W$ is a (possibly vacuous) $(n - p) \times p$ matrix of one of the following forms (i), (ii) or (iii):

(i) $W$ is row stochastic;

(ii) $W$ is row substochastic and has a zero row;

(iii) $W = [(cI)/B]$, where $0 < c < 1$ and $B$ is an $(n - 2p) \times p$ row substochastic matrix with row sums at least $c$.

(c) Let $Q$ be an $n \times n$ permutation matrix, and let $W$ be an $(n - p) \times p$ matrix of the form (i), (ii), or (iii) in part (b). Then the operator $X \mapsto Q[X/(WX)]$ from $\mathbb{R}^p$ into $\mathbb{R}^n$ is a nonnegative linear preserver of $\prec_e$.

Proof.

(a) Assume that, if possible, $b < 0$. By Theorem 2.3 $n \geq p(p - 1)$. Since $p \geq 3$, $n \geq 2p$, a contradiction.

(b) Since $T$ is nonnegative, $N_i = 0, 1 \leq i \leq n$, and $0 \leq P_i \leq 1$. By Theorem 2.3, $[T]$ has $P_p(1, 0)$ as its submatrix and therefore up to a row permutation $[T] = [I/W]$. Let $c = \min\{P_i, 1 \leq i \leq n\}$. Then $E(T) = \{(1, 0), (c, 0)\}$. By Theorem 4.6, $P_p(c, 0)$ is a submatrix of $[T]$. If $c = 1$ then (i) holds; if $c = 0$ then (ii) holds and if $0 < c < 1$, then (iii) holds.

(c) Let $[T] = [I/W]$, where $W$ is an $(n - p) \times p$ matrix of the form (i), (ii), or (iii) in part (b). Then $E(T) = \{(1, 0), (c, 0)\}$. By Theorem 4.6, $T$ is a nonnegative linear preserver of $\prec_e$.

Theorem 5.4. ([9, Theorem 4.5]) Assume $T : \mathbb{R}^p \to \mathbb{R}^n$ is a linear preserver of $\prec_e$, $b < 0$ and $2p \leq n < p(p - 1)$. Let $P_i$ (resp., $N_i$) denote the sum of the positive (resp., negative) entries of the $i$th row of $[T]$. Then, up to a row permutation, $[T] = [I/bI/B]$ and $\min(N_i + bP_i) = b$, $(i = 1, 2, \ldots, n)$.

Proof. By Theorem 2.3, $P_p(1, \beta)$ and $P_p(\alpha, b)$ are submatrices of $[T]$. Since $n < p(p - 1)$, $\beta = \alpha = 0$ and $E(T) = \{(1, 0), (0, b)\}$, where $E(T)$ is as in Definition 4.4. Then up to a row permutation, $[T] = [I/bI/B]$ and $\min\{(bx + y) : (x, y) \in \Delta\} = \min\{(bx + y) : (x, y) \in E(T)\} = b$. Therefore, $\min(N_i + bP_i) = b$, $(i = 1, 2, \ldots, n)$. 

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