Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

ELA

SCHUR COMPLEMENTS OF GENERALLY DIAGONALLY DOMINANT MATRICES AND A CRITERION FOR IRREDUCIBILITY OF MATRICES*

CHENG-YI ZHANG†, SHUANGHUA LUO‡, CHENGXIAN XU§, AND HONGYING JIANG¶

Abstract. As is well known, the Schur complements of strictly or irreducibly diagonally dominant matrices are $H-$matrices; however, the same is not true of generally diagonally dominant matrices. This paper proposes some conditions on the generally diagonally dominant matrix A and the subset $\alpha \subset \{1, 2, \ldots, n\}$ so that the Schur complement matrix A/α is an $H-$matrix. These conditions are then applied to decide whether a matrix is irreducible or not.

Key words. Schur complement, Diagonally dominant matrices, $H-$matrices; Irreducible, Reducible.

AMS subject classifications. 15A15, 15F10.

1. Introduction. Recently, considerable interest appears in the work on the Schur complements of some families of matrices and several significant results are proposed. As is shown in [4], [5], [8], [2], [7], [10], [16] and [9], the Schur complements of positive semidefinite matrices are positive semidefinite (see, e.g., [16]); the same is true of $M-$matrices, inverse $M-$matrices (see, e.g., [5]), $H-$matrices (see, e.g., [8]), diagonally dominant matrices (see, e.g., [2] and [7]), Dashnic-Zusmanovich matrices (see, e.g., [10]) and generalized doubly diagonally dominant matrices (see, e.g., [9]).

Since $M-$matrices, Dashnic-Zusmanovich matrices, strictly generalized doubly diagonally dominant matrices and strictly or irreducibly diagonally dominant matrices are all $H-$matrices (see, e.g., [1, pp. 132-161], [10], [9] and [12, p. 92), so are their Schur complements. This very property has been repeatedly used for the convergence of the Gauss-Seidel iterations and stability of Gaussian elimination in

*Received by the editors March 27, 2008. Accepted for publication January 7, 2008. Handling Editor: Joao Filipe Queiro.

†Department of Mathematics, School of Science, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, P.R. China (zhangchengyi_2004@163.com).

‡Department of Mathematics, School of Science, Lanzhou University of Technology, Lanzhou, 730050, P.R. China (iwantflyluo@163.com).

§SKLMSE Lab and Department of Mathematics, School of Science, Xi'an Jiaotong University, Xi'an, 710049, P.R. China (mxxu@mail.xjtu.edu.cn). The work of this author was supported by the National Natural Science Foundation of China (10671152).

¶School of Civil Engineering, Lanzhou University of Technology, Lanzhou, 730050, P.R. China. The work of this author was supported by the Special Foundation of Lanzhou University of Technology (SB04200701).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

numerical analysis (see, e.g., [6, pp.58], [3, pp. 508], [13, pp. 122-123] and [14, pp. 281-330]). However, for generally diagonally dominant matrices, their Schur complements are not necessarily $H-$matrices. But, it is found that for a diagonally dominant matrix which is not a strictly or irreducibly diagonally dominant matrix, its Schur complement can be an $H-$matrix. Thus, there arises an open problem: how do we decide whether the Schur complements of generally diagonally dominant matrices are $H-$matrices or not?

In this paper some conditions on the generally diagonally dominant matrix A and the subset $\alpha \subset N$ will be proposed such that the Schur complement matrix A/α is an $H-$matrix. These conditions are then applied to decide whether a matrix is irreducible or not.

The paper is organized as follows. Some notation and preliminary results about special matrices are given in Section 2. Some lemmas are presented in Section 3. The main results of this paper are given in Section 4, where we give some conditions such that the Schur complement of a generally diagonally dominant matrix is an $H-$matrix. These results are applied to determine the irreducibility of a matrix. Conclusions are given in Section 5.

2. Preliminaries. In this section we present some notions and preliminary results about special matrices that are used in this paper.

$C^{m\times n}$ ($R^{m\times n}$) will be used to denote the set of all $m\times n$ complex (real) matrices. Let $A=(a_{ij})\in R^{m\times n}$ and $B=(b_{ij})\in R^{m\times n}$, we write $A\geq B$, if $a_{ij}\geq b_{ij}$ holds for all $i=1,2,\cdots,m,\ j=1,2,\cdots,n$. A matrix $A=(a_{ij})\in R^{n\times n}$ is called a $Z-$*matrix* if $a_{ij}\leq 0$ for $i\neq j$, $i,j=1,2,\cdots,n$. We will use Z_n to denote the set of all $n\times n$ $Z-$matrices. A matrix $A=(a_{ij})\in R^{n\times n}$ is called an $M-$*matrix* if $A\in Z_n$ and $A^{-1}\geq 0$. M_n will be used to denote the set of all $n\times n$ $M-$matrices.

For a given matrix $A=(a_{ij})\in C^{n\times n}$, the *comparison matrix* $\mu(A)=(\mu_{ij})$ is given by

$$\mu_{ij}=\begin{cases} |a_{ii}|, & \text{if } i=j,\\ -|a_{ij}|, & \text{if } i\neq j.\end{cases}$$

It is clear that $\mu(A)\in Z_n$ for a matrix $A\in C^{n\times n}$. A matrix $A\in C^{n\times n}$ is called $H-$*matrix* if $\mu(A)\in M_n$. H_n will denote the set of all $n\times n$ $H-$matrices.

For $n\geq 2$, an $n\times n$ complex matrix A is *reducible* if there exists an $n\times n$ permutation matrix P such that

$$PAP^T=\begin{bmatrix} A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}, \tag{2.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

where A_{11} is an $r \times r$ submatrix and A_{22} is an $(n-r) \times (n-r)$ submatrix, where $1 \leq r < n$. If no such permutation matrix exists, then A is called *irreducible*. If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Let $|\alpha|$ denote the cardinality of the set $\alpha \subseteq N$. For nonempty index sets $\alpha, \beta \subseteq N$, $A(\alpha, \beta)$ is the submatrix of $A \in C^{n\times n}$ with row indices in α and column indices in β. The submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Let $\alpha \subset N$, $\alpha' = N - \alpha$, and $A(\alpha)$ be nonsingular. Then, the matrix

$$A/\alpha = A/A(\alpha) = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha') \tag{2.2}$$

is called the *Schur complement* of A with respect to $A(\alpha)$, where indices in both α and α' are arranged with increasing order.

DEFINITION 2.1. Let $A \in C^{n\times n}$, $N = \{1, 2, \cdots, n\}$, and define sets

$$J(A) = \left\{ i \mid |a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i \in N \right\} \tag{2.3}$$

and

$$K(A) = \left\{ i \mid |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|. \quad i \in N \right\} \tag{2.4}$$

If $J(A) = N$, A is called diagonally dominant by row. If A is diagonally dominant by row with $K(A) = N$, A is called *strictly diagonally dominant by row.* If A is irreducible and diagonally dominant by row with $K(A) \neq \emptyset$, A is called [*irreducibly diagonally dominant by row. If A is diagonally dominant with* $J(A) \cap K(A) = \emptyset$, *A is called* diagonally equipotent by row.

D_n, SD_n, ID_n and DE_n will be used to denote the sets of $n \times n$ matrices which are diagonally dominant by row, strictly diagonally dominant by row, irreducibly diagonally dominant by row and diagonally equipotent by row, respectively.

REMARK 2.2. Let $A = (a_{ij}) \in D_n$ and $\alpha = N - \alpha' \subset N$. If $A(\alpha)$ is a diagonally equipotent principal submatrix of A, then the following hold:

- $A(\alpha, \alpha') = 0$;
- $A(i_1) = (a_{i_1 i_1})$ being diagonally equipotent implies $a_{i_1 i_1} = 0$.

Remark 2.2 implies the following lemmas.

LEMMA 2.3. *If a matrix $A \in D_n$ has a diagonally equipotent principal submatrix $A(\alpha)$ for $\alpha \subset N$, then A is reducible.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

LEMMA 2.4. *If $A \in SD_n \cup ID_n$, then $A \in H_n$ and is nonsingular.*

DEFINITION 2.5. A *directed graph* or *digraph* Γ is an ordered pair $\Gamma := (V, E)$ that is subject to the following conditions: (i) V is a set whose elements are called *vertices* or *nodes*; (ii) E is a set of ordered pairs of vertices, called *directed edges, arcs,* or *arrows*. Let $A \in C^{n\times n}$, where $n < +\infty$. Then, we define the digraph $\Gamma(A)$ of A as the directed graph with vertex set $V = \{1, 2, \cdots, n\}$ and edge set $E = \{(i, j) \mid i, j \in V,\ i \neq j\}$, where $(i, j) \in E$ is an edge of $\Gamma(A)$ connecting the vertex i to the vertex j if $a_{ij} \neq 0,\ i \neq j$. Clearly, $\Gamma(A)$ is a finite directed graph without loops and multiple edges. A digraph $\Gamma(A)$ is called *strongly connected* if for any ordered pair of vertices i and j, there exists a directed path $(i, i_1), (i_1, i_2), \cdots, (i_{r-1}, j)$ connecting i and j.

LEMMA 2.6. *$A \in C^{n\times n}$ is irreducible if and only if its digraph $\Gamma(A)$ is strongly connected.*

DEFINITION 2.7. Given $A = (a_{ij}) \in C^{n\times n}$, we define the matrix

$$\pi(A) = (m_{ij}) \in R^{n\times n},$$

where

$$m_{ij} = \begin{cases} -|a_{ij}|, & a_{ij} \neq 0 \ \text{ and } \ i > j, \\ |a_{ij}|, & a_{ij} \neq 0 \ \text{ and } \ i < j, \\ \sum_{j=1, j\neq i}^{n} |a_{ij}|, & i = j, \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}$$

According to Definition 2.5 and 2.7, it is clear that

$$\Gamma(\pi(A)) = \Gamma(A). \tag{2.6}$$

LEMMA 2.8. *$A \in C^{n\times n}$ is irreducible if and only if $\pi(A)$ is irreducible.*

Proof. It follows from Lemma 2.6 and (2.6) that the conclusion of this lemma is true. □

3. Some lemmas. In this section, some lemmas concerning several properties of diagonally dominant matrices and the Schur complements of matrices will be presented. They will be of use in the sequel.

LEMMA 3.1. (see [15]) *A matrix $A \in D_n$ is singular if and only if the matrix A has at least either one zero principal submatrix or one irreducible and diagonally equipotent principal submatrix $A_k = A(i_1, i_2, \cdots, i_k), 1 < k \leq n$, which satisfies condition that there exists a $k \times k$ unitary diagonal matrix U_k such that*

$$U_k^{-1} D_{A_k}^{-1} A_k U_k = \mu(D_{A_k}^{-1} A_k), \tag{3.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

where $D_{A_k} = diag(a_{i_1 i_1}, a_{i_2 i_2}, \cdots, a_{i_k i_k})$.

LEMMA 3.2. *Let* $A \in D_n$. *Then* A *is singular if and only if* A *has at least one singular principal submatrix.*

Proof. Necessity is obvious since A is a principal submatrix of itself. For sufficiency, suppose that $A(i_1, i_2, \cdots, i_k)$ is a singular principal submatrix of A where $1 \leq k \leq n$. Then it follows from the necessity of Lemma 3.1 that there exists a zero principal submatrix or a singular irreducible diagonally equipotent principal submatrix $B = A(j_1, j_2, \cdots, j_{k\prime})(1 < k\prime \leq k)$ in $A(i_1, i_2, \cdots, i_k)$, which satisfies condition (3.1). Obviously, B is also a singular equipotent principal submatrix of A which satisfies condition (3.1). So it follows from Lemma 3.1 that A is singular. □

LEMMA 3.3. (see [15]) *Given a matrix* $A \in R^{n\times n}$, *if* $A \in D_n$ *and is nonsingular, then* A *has* $|J_+(A)|$ *eigenvalues with positive real part and* $|J_-(A)|$ *eigenvalues with negative real part, where* $J_+(A) = \{i \mid a_{ii} > 0,\ i \in N\},\ J_-(A) = \{i \mid a_{ii} < 0,\ i \in N\}$.

LEMMA 3.4. (see [1]) *Let* $A \in Z_n$. *Then* $A \in M_n$ *if and only if the real part of each eigenvalue of* A *is positive.*

LEMMA 3.5. *Let* $A \in D_n$. *Then* $A \in H_n$ *if and only if* A *has neither zero principal submatrices nor irreducibly diagonally equipotent principal submatrices.*

Proof. First we prove sufficiency. Assume that A has neither zero principal submatrices nor irreducibly diagonally equipotent principal submatrices. Then neither does $\mu(A)$. Since $\mu(A)$ has no zero principal submatrices and $\mu(A) \in D_n$ for $A \in D_n$, it follows from Lemma 3.1 that $\mu(A)$ is singular if and only if $\mu(A)$ has at least one irreducible and diagonally equipotent principal submatrix $A_k = A(i_1, i_2, \cdots, i_k), 1 < k \leq n$ satisfying condition (3.1). But, $\mu(A)$ has not any irreducibly diagonally equipotent principal submatrices. Consequently, $\mu(A)$ is nonsingular. Again, following Lemma 3.3, the real part of each eigenvalue of $\mu(A)$ is positive. Therefore, Lemma 3.4 indicates that $\mu(A) \in M_n$, i.e., $A \in H_n$.

Next, necessity can be proved by contradiction. Suppose A has either one zero principal submatrix or one irreducibly diagonally equipotent principal submatrix. So does $\mu(A)$. Such principal matrix is assumed as $\mu(A_k)$, where A_k is either one zero principal submatrix or one irreducibly diagonally equipotent principal submatrix of A. Then it follows from of Lemma 3.1 that $\mu(A)$ is singular. According to the definition of $M-$matrix, $\mu(A)$ is not an $M-$matrix. This shows that A is not an $H-$matrix, which contradicts $A \in H_n$. Thus, necessity is true, which completes the proof. □

LEMMA 3.6. (see [11]) *Let* $A \in C^{n\times n}$ *be partitioned as*

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

where $A_{21} = (a_{21}, a_{31}, \cdots, a_{n1})^T$ *and* $A_{12} = (a_{12}, a_{13}, \cdots, a_{1n})$. *If* A_{22} *is nonsingular, then*

$$\frac{det\ A}{det\ A_{22}} = a_{11} - (a_{12}, a_{13}, \cdots, a_{1n})[A_{22}]^{-1} \begin{pmatrix} a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

LEMMA 3.7. (see [1,4]) *If* $A \in D_n \cap Z_n$ *with* $a_{ii} \geq 0$ *for all* $i \in N$, *then* $det\ A \geq 0$. *Furthermore, if* $A \in D_n \cap M_n$, *then* $det\ A > 0$.

LEMMA 3.8. *Given an irreducible matrix* $A = (a_{ij}) \in Z_n$ *with* $a_{ii} \geq 0$ *for all* $i \in N$ *and a set* $\alpha \subset N$, *if* $A \in DE_n$, *then* $A/\alpha \in DE_{n-|\alpha|} \cap Z_{n-|\alpha|}$.

Proof. Since $A \in DE_n$ is irreducible and $\alpha \subset N$, $A(\alpha) \in H_{|\alpha|}$. Otherwise, Lemma 3.5 shows that $A(\alpha)$ has a zero principal submatrix or diagonally equipotent principal submatrix, say $A(\gamma)$ for $\gamma \subseteq \alpha$. Then, $A(\gamma)$ is also a zero principal submatrix or diagonally equipotent principal submatrix of A. It then follows from Lemma 2.3 that A is reducible which contradicts the irreducibility of A. Thus, $A(\alpha) \in H_{|\alpha|}$. Since $A = (a_{ij}) \in Z_n$ with $a_{ii} \geq 0$ for all $i \in N$, $A(\alpha) = (a'_{ij}) \in Z_{|\alpha|}$ with $a'_{ii} \geq 0$ for all $i \in \alpha$. As a result, $A(\alpha) \in M_{|\alpha|}$ and consequently $[A(\alpha)]^{-1} \geq 0$. Therefore, A/α exists. Suppose $\alpha = \{i_1, i_2, \cdots, i_k\}$, $\alpha' = N - \alpha = \{j_1, j_2, \cdots, j_m\}$, $k + m = n$. Let $A/\alpha = (\widetilde{a}_{j_l, j_t})_{m \times m}$. According to definition (2.2) of the matrix A/α, we have the off-diagonal entries of the Schur complement matrix A/α,

$$\begin{aligned} \widetilde{a}_{j_l,j_t} &= a_{j_l,j_t} - \left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_t} \\ a_{i_2,j_t} \\ \vdots \\ a_{i_k,j_t} \end{pmatrix} \right] \\ &= -\left[|a_{j_l,j_t}| + (|a_{j_l,i_1}|, \cdots, |a_{j_l,i_k}|)[A(\alpha)]^{-1} \begin{pmatrix} |a_{i_1,j_t}| \\ |a_{i_2,j_t}| \\ \vdots \\ |a_{i_k,j_t}| \end{pmatrix} \right] \leq 0, \\ & \quad l \neq t, \quad l, t = 1, 2, \cdots, m \end{aligned} \tag{3.2}$$

which shows $A/\alpha \in Z_{n-|\alpha|}$, and the diagonal entries (obtained by Lemma 3.6)

$$\begin{aligned} \widetilde{a}_{j_l,j_l} &= a_{j_l,j_l} - \left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \\ &= \frac{det\ C_l}{det\ A(\alpha)}, \quad l = 1, 2, \cdots, m, \end{aligned} \tag{3.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

where

$$C_l = \begin{pmatrix} a_{j_l,j_l} & h \\ s & A(\alpha) \end{pmatrix}_{(k+1)\times(k+1)},$$
$$s = (a_{i_1,j_l}, \cdots, a_{i_k,j_l})^T,$$
$$h = (a_{j_l,i_1}, \cdots, a_{j_l,i_k}).$$

Since C_l is a $(k+1)\times(k+1)$ principal submatrix of A, $C_l \in D_{k+1} \cap Z_{k+1}$ with all its diagonal entries being nonnegative and it follows from Lemma 3.7 that $det\ C_l \geq 0$. In the same way, $det\ A(\alpha) > 0$. Thus,

$$\widetilde{a}_{j_l,j_l} = \frac{det\ C_l}{det\ A(\alpha)} \geq 0.$$

Therefore,

$$|\widetilde{a}_{j_l,j_l}| = |a_{j_l,j_l}| - \left[(|a_{j_l,i_1}|, \cdots, |a_{j_l,i_k}|)[A(\alpha)]^{-1} \begin{pmatrix} |a_{i_1,j_l}| \\ |a_{i_2,j_l}| \\ \vdots \\ |a_{i_k,j_l}| \end{pmatrix} \right] \geq 0, \tag{3.4}$$
$$l = 1, 2, \cdots, m.$$

Since $A \in Z_n$ with $a_{ii} \geq 0$ for all $i \in N$ and $[A(\alpha)]^{-1} \geq 0$, using (3.2), (3.4) and Lemma 3.6, we have

$$\begin{aligned} |\widetilde{a}_{j_l,j_l}| - \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}| &= \left| a_{j_l,j_l} - (a_{j_l,i_1}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right| \\ &\quad - \sum_{i=1,i\neq l}^{m} \left| \left[a_{j_l,j_i} - (a_{j_l,i_1}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_i} \\ a_{i_2,j_i} \\ \vdots \\ a_{i_k,j_i} \end{pmatrix} \right] \right| \\ &= |a_{j_l,j_l}| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| \\ &\quad - \sum_{i=1}^{m} \left[(|a_{j_l,i_1}|, |a_{j_l,i_2}|, \cdots, |a_{j_l,i_k}|)|[A(\alpha)]^{-1}| \begin{pmatrix} |a_{i_1,j_i}| \\ |a_{i_2,j_i}| \\ \vdots \\ |a_{i_k,j_i}| \end{pmatrix} \right] \\ &= \frac{detB_l}{det\ A(\alpha)}, \quad l = 1, 2, \cdots, m, \end{aligned} \tag{3.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

where

$$B_l = \begin{pmatrix} |a_{j_l,j_l}| - \sum\limits_{i=1,i\neq l}^{m} |a_{j_l,j_i}| & h^T \\ g & A(\alpha) \end{pmatrix}_{(k+1)\times(k+1)},$$
$$g = (-\sum_{i=1}^{m} |a_{i_1,j_i}|, \cdots, -\sum_{i=1}^{m} |a_{i_k,j_i}|)^T,$$
$$h = (-|a_{j_l,i_1}|, \cdots, -|a_{j_l,i_k}|)^T.$$

It is clear that $B_l \in Z_{k+1}$. Since $A \in DE_n$, we have

$$|a_{j_l,j_l}| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| = \sum_{t=1}^{k} |a_{j_l,i_t}|. \tag{3.6}$$

and

$$|a_{i_r,i_r}| = \sum_{t=1,t\neq r}^{k} |a_{i_r,i_t}| + \sum_{i=1}^{m} |a_{i_r,j_i}|,\ r = 1, 2, \cdots, k. \tag{3.7}$$

Equalities (3.6) and (3.7) indicate $B_l \in DE_{k+1}$. Since $B_l \in DE_{k+1} \cap Z_{k+1}$, it follows from Lemma 3.1 that B_l is singular and $det\ B_l = 0$. Since $A(\alpha) \in D_{|\alpha|} \cap M_{|\alpha|}$, Lemma 3.7 gives $det\ A(\alpha) > 0$. Thus, by (3.5), we have

$$|\widetilde{a}_{j_l,j_l}| - \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}| = \frac{detB_l}{det\ A(\alpha)} = 0$$

and thus

$$|\widetilde{a}_{j_l,j_l}| = \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}|,\ l = 1, 2, \cdots, m, \tag{3.8}$$

which shows $A/\alpha \in DE_m$. This completes the proof. □

LEMMA 3.9. (see [16]) *Given a matrix $A = (a_{ij}) \in C^{n\times n}$ and two diagonal matrices $E = diag(e_1, \cdots, e_n)$ and $F = diag(f_1, \cdots, f_n)$ with $e_i \neq 0$ and $f_i \neq 0$ for all $i \in N$. Assume $B = EAF$ and $\alpha = N - \alpha' \subset N$. If $A(\alpha)$ is nonsingular, then $B/\alpha = E_{\alpha'}(A/\alpha)F_{\alpha'}$, where $\alpha' = N - \alpha = \{j_1, \cdots, j_m\}$, $E_{\alpha'} = diag(e_{j_1}, \cdots, e_{j_m})$ and $F_{\alpha'} = diag(f_{j_1}, \cdots, f_{j_m})$.*

LEMMA 3.10. (see [7]) *Given a matrix $A \in D_n$ and a set $\alpha \subset N$, if $A(\alpha)$ is nonsingular, then $A/\alpha \in D_{n-|\alpha|}$.*

LEMMA 3.11. (see [8]) *Let $A \in H_n$ and $\alpha \subset N$. Then $A/\alpha \in H_{n-|\alpha|}$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

ELA

LEMMA 3.12. *Let an irreducible matrix $A \in D_n$. Then for each $\alpha \subset N$, $A/\alpha \in DE_{n-|\alpha|}$ if and only if A is singular.*

Proof. Sufficiency will be proved firstly. Assume that A is singular. Since $A \in D_n$ is irreducible, $a_{ii} \neq 0$ for $i = 1, 2, \cdots, n$ and Lemma 3.1 yields that there exists a $n \times n$ unitary diagonal matrix $U = diag(u_1, \cdots, u_n)$ such that

$$D^{-1}A = U\mu(D^{-1}A)U^{-1} \in DE_n, \tag{3.9}$$

where $D = diag(a_{11}, a_{22}, \cdots, a_{nn})$. Let $B = D^{-1}A \in DE_n$, then $B = U\mu(B)U^{-1}$. According to Lemma 3.9, we have

$$B/\alpha = U_{\alpha'}[\mu(B)/\alpha]U_{\alpha'}^{-1}, \tag{3.10}$$

where $\alpha' = N - \alpha = \{j_1, \cdots, j_m\}$ and $U_{\alpha'} = diag(u_{j_1}, \cdots, u_{j_m})$. Since $B = D^{-1}A \in DE_n$, $\mu(B) \in DE_n$. It then follows from Lemma 3.8 that $\mu(B)/\alpha \in DE_{n-|\alpha|}$. Since $U_{\alpha'}$ is an $m \times m$ unitary diagonal matrix, (3.10) implies $B/\alpha \in DE_{n-|\alpha|}$. As is assumed, $A = DB$. Thus, Lemma 3.9 gives that

$$A/\alpha = D_{\alpha'}[B/\alpha], \tag{3.11}$$

where $D_{\alpha'} = diag(d_{j_1}, \cdots, d_{j_m})$. Since $B/\alpha \in DE_{n-|\alpha|}$, (3.11) implies $A/\alpha \in DE_{n-|\alpha|}$, which completes sufficiency.

Next, we will prove necessity by contradiction. Assume that $A \in D_n$ is nonsingular. If $A \in H_n$, Lemma 3.10 and Lemma 3.11 show $A/\alpha \in D_{n-|\alpha|} \cap H_{n-|\alpha|}$. It follows from Lemma 3.5 that $A/\alpha \notin DE_{n-|\alpha|}$, which shows that the assumption is not true. Thus, A is singular.

If $A \notin H_n$ but $A \in D_n$ is irreducible, Lemma 3.5 indicates $A \in DE_n$ is irreducible. Again, since $A \in DE_n$ is nonsingular, $a_{ii} \neq 0$ for $i = 1, 2, \cdots, n$ and Lemma 3.1 implies that there is not any $n \times n$ unitary diagonal matrix U such that (3.9) holds. Without loss of generality, let $D = I$. Then $D^{-1}A = A \in DE_n$ and $U^{-1}D^{-1}AU = U^{-1}AU \notin Z_n$ for all $n \times n$ unitary diagonal matrix U. Suppose $\alpha = \{i_1, i_2, \cdots, i_k\}$, $\alpha' = N - \alpha = \{j_1, j_2, \cdots, j_m\}$, $k+m = n$. Let $A/\alpha = (\widetilde{a}_{j_l, j_t})_{m \times m}$. Since $A \in DE_n$ is irreducible and $\alpha \subset N$, it follows from the proof of Lemma 3.8 that $A(\alpha) \in H_{|\alpha|}$. As a result, $[\mu(A(\alpha))]^{-1} \geq 0$. Since $U^{-1}D^{-1}AU = U^{-1}AU \notin Z_n$ for all $n \times n$ unitary diagonal matrix U, there exists at least one l, $l = 1, 2, \cdots, m$ such that

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

(3.5) becomes

$$
\begin{aligned}
|\widetilde{a}_{j_l,j_l}| \quad - \quad & \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}| = \left| a_{j_l,j_l} - (a_{j_l,i_1},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right| \\
& - \sum_{i=1,i\neq l}^{m} \left| \left[a_{j_l,j_i} - (a_{j_l,i_1},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_i} \\ a_{i_2,j_i} \\ \vdots \\ a_{i_k,j_i} \end{pmatrix} \right] \right| \\
> \quad & |a_{j_l,j_l}| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| \\
& - \sum_{i=1}^{m} \left[(|a_{j_l,i_1}|,|a_{j_l,i_2}|,\cdots,|a_{j_l,i_k}|)[\mu(A(\alpha))]^{-1} \begin{pmatrix} |a_{i_1,j_i}| \\ |a_{i_2,j_i}| \\ \vdots \\ |a_{i_k,j_i}| \end{pmatrix} \right] \\
= \quad & \frac{detC_l}{det\ \mu[A(\alpha)]},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
C_l &= \begin{pmatrix} |a_{j_l,j_l}| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| & h^T \\ g & \mu[A(\alpha)] \end{pmatrix}_{(k+1)\times(k+1)}, \\
g &= (-\sum_{i=1}^{m} |a_{i_1,j_i}|,\cdots,-\sum_{i=1}^{m} |a_{i_k,j_i}|)^T, \\
h &= (-|a_{j_l,i_1}|,\cdots,-|a_{j_l,i_k}|)^T.
\end{aligned}
$$

It is clear that $C_l \in Z_{k+1}$. Since $A \in DE_n$, we have that (3.6) and (3.7) hold. Equalities (3.6) and (3.7) indicate $C_l \in DE_{k+1}$. Since $C_l \in DE_{k+1} \cap Z_{k+1}$, it follows from Lemma 3.1 that C_l is singular and $det\ B_l = 0$. Since $A(\alpha) \in H_{|\alpha|}$, $\mu[A(\alpha)] \in M_{|\alpha|}$ and Lemma 3.7 gives $det\ \mu[A(\alpha)] > 0$. Thus, by (3.12), we have

$$
|\widetilde{a}_{j_l,j_l}| - \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}| > \frac{detB_l}{det\ A(\alpha)} = 0
$$

and thus there exists at least one $l,\ l = 1,2,\cdots,m$ such that

$$
|\widetilde{a}_{j_l,j_l}| > \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}| \tag{3.13}
$$

which shows $A/\alpha \notin DE_{n-|\alpha|}$. Therefore, the assumption is not true and A is singular. This completes necessity. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

LEMMA 3.13. *Let $A \in D_n$. Then A is nonsingular if and only if $A(\alpha)$ and A/α are both nonsingular for each $\alpha \subset N$.*

Proof. Assume $\alpha' = N - \alpha$. Since A is nonsingular, it follows from Lemma 3.2 that $A(\alpha)$ is nonsingular for each $\alpha \subset N$. Thus, A/α exists for each $\alpha \subset N$ and there exists an $n \times n$ permutation matrix P such that

$$B = P^T AP = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{bmatrix}.$$

Let $P_1 = \begin{bmatrix} I_{|\alpha|} & 0 \\ -A(\alpha', \alpha)[A(\alpha)]^{-1} & I_{|\alpha'|} \end{bmatrix}$, $P_2 = \begin{bmatrix} I_{|\alpha|} & -[A(\alpha)]^{-1}A(\alpha', \alpha) \\ 0 & I_{|\alpha'|} \end{bmatrix}$, where $I_{|\alpha|}$ and $I_{|\alpha'|}$ are the $|\alpha| \times |\alpha|$ identity matrix and the $|\alpha'| \times |\alpha'|$ identity matrix, respectively. Then, the product

$$\begin{aligned} P_1 B P_2 &= P_1 P^T A P P_2 \\ &= \begin{bmatrix} A(\alpha) & 0 \\ 0 & A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha', \alpha) \end{bmatrix} \\ &= \begin{bmatrix} A(\alpha) & 0 \\ 0 & A/\alpha \end{bmatrix}. \end{aligned} \tag{3.14}$$

As P, P_1 , P_2 and A are all nonsingular, so is $P_1 B P_2$. Again, $A(\alpha)$ is nonsingular, then A/α is nonsingular from (3.14).

If $A(\alpha)$ and A/α are nonsingular, then $P_1 B P_2 = P_1 P^T A P P_2$ is nonsingular from (3.14). So is A. □

LEMMA 3.14. *Given an $n \times n$ matrix $A \in C^{n \times n}$ and two sets α, α' satisfying $\alpha \subset N$ and $\alpha' = N - \alpha$, assume that $A(\alpha)$is nonsingular and there exists an $n \times n$ permutation matrix P such that*

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ 0 & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{ss} \end{bmatrix}, \tag{3.15}$$

where A_{ii} is irreducible, $A_{ij} = A(\alpha_i, \alpha_j)$ is the submatrix of A for $i, j = 1, 2, \cdots, s$ $(1 \le s \le n)$, with row indices in α_i and column indices in α_j, $\bigcup_{j=1}^{s} \alpha_j = N$ and $\alpha_i \cap \alpha_j = \emptyset$ for $i \ne j$. If A/α is irreducible, then for some i, $i = 1, 2, \cdots, s$, $\alpha' \subseteq \alpha_i$ and $A/\alpha = [A(\alpha_i)]/\eta$, where $\eta = \alpha_i - \alpha'$.

Proof. If A is irreducible, then $A_{11} = A(\alpha_1) = A$. As a result, $\alpha_1 = N$, $\eta = \alpha_1 - \alpha' = N - \alpha' = \alpha$. Hence $A/\alpha = [A(\alpha_1)]/\eta$. The conclusion of this lemma is true.

If A is reducible, then $2 \le s \le n$. For this case, the conclusion of this lemma will be proved by contradiction. Assume that the conclusion of this lemma is not true,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

that is, $\alpha' \subseteq \alpha_i$ doesn't hold for any $i, i = 1, 2, \cdots, s$. Then, let $\alpha' = \bigcup_{j=1}^{s} \beta_j$, where $\emptyset \subseteq \beta_j \subseteq \alpha_j$ for $j = 1, 2, \cdots, s$ and the number of nonempty set β_j is at least equal to 2. Let $\gamma_j = \alpha_j - \beta_j$ for $j = 1, 2, \cdots, s$, then

$$\alpha = N - \alpha' = \bigcup_{j=1}^{s} \alpha_j - \bigcup_{j=1}^{s} \beta_j = \bigcup_{j=1}^{s} \gamma_j.$$

Then there exists an $n \times n$ permutation matrix P_1 such that

$$\begin{aligned} C = P_1 P A P^T P_1^T &= P_1 \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ 0 & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{ss} \end{bmatrix} P_1^T \\ &= \begin{bmatrix} A'(\alpha_1) & A'(\alpha_1, \alpha_2) & \cdots & A'(\alpha_1, \alpha_s) \\ 0 & A'(\alpha_2) & \cdots & A'(\alpha_2, \alpha_s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A'(\alpha_s) \end{bmatrix}, \end{aligned} \tag{3.16}$$

where $A'(\alpha_i, \alpha_j) = \begin{bmatrix} A(\gamma_i, \gamma_j) & A(\gamma_i, \beta_j) \\ A(\beta_i, \gamma_j) & A(\beta_i, \beta_j) \end{bmatrix}$, $i < j$ and $A'(\alpha_i) = \begin{bmatrix} A(\gamma_i) & A(\gamma_i, \beta_i) \\ A(\beta_i, \gamma_i) & A(\beta_i) \end{bmatrix}$ for $i, j = 1, 2, \cdots, s$. Therefore, there exists an $n \times n$ permutation matrix Q such that

$$QCQ^T = QP_1 P A P^T P_1^T Q^T = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{bmatrix}, \tag{3.17}$$

where

$$\begin{aligned} A(\alpha) &= \begin{bmatrix} A(\gamma_1) & A(\gamma_1, \gamma_2) & \cdots & A(\gamma_1, \gamma_s) \\ 0 & A(\gamma_2) & \cdots & A(\gamma_2, \gamma_s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(\gamma_s) \end{bmatrix}, \\ A(\alpha') &= \begin{bmatrix} A(\beta_1) & A(\beta_1, \beta_2) & \cdots & A(\beta_1, \beta_s) \\ 0 & A(\beta_2) & \cdots & A(\beta_2, \beta_s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(\beta_s) \end{bmatrix}, \\ A(\alpha', \alpha) &= \begin{bmatrix} A(\beta_1, \gamma_1) & A(\beta_1, \gamma_2) & \cdots & A(\beta_1, \gamma_s) \\ 0 & A(\beta_2, \gamma_2) & \cdots & A(\beta_2, \gamma_s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(\beta_s, \gamma_s) \end{bmatrix}, \\ A(\alpha, \alpha') &= \begin{bmatrix} A(\gamma_1, \beta_1) & A(\gamma_1, \beta_2) & \cdots & A(\gamma_1, \beta_s) \\ 0 & A(\gamma_2, \beta_2) & \cdots & A(\gamma_2, \beta_s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A(\gamma_s, \beta_s) \end{bmatrix}. \end{aligned} \tag{3.18}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

Direct computation yields

$$\begin{aligned} A/\alpha &= A(\alpha') - A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha') \\ &= \begin{bmatrix} [A(\alpha_1)]/\gamma_1 & * & \cdots & * \\ 0 & A[A(\alpha_2)]/\gamma_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [A(\alpha_s)]/\gamma_s \end{bmatrix}, \end{aligned} \tag{3.19}$$

where $*$ denotes some unknown matrices. The equality (3.19) shows that A/α is reducible. This contradicts the irreducibility of A/α. Therefore, the assumption is incorrect and the conclusion of this lemma is true. □

4. Main results. The main theorems of this paper are presented in this section. They concern when a Schur complement of a diagonally dominant matrix is an $H-$matrix. Then these theorems are applied to decide whether a matrix is an irreducible matrix or not.

THEOREM 4.1. *Let $A \in DE_n$ be nonsingular. Then $A/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$ if and only if A is irreducible.*

Proof. Sufficiency will be proved firstly. Assume that A is irreducible. Since $A \in DE_n \subseteq D_n$ is nonsingular, it follows from Lemma 3.10 and Lemma 3.12 that $A/\alpha \in D_{n-|\alpha|}$, but $A/\alpha \notin DE_{n-|\alpha|}$. It follows that sufficiency can be proved by the following two cases.

Case (i). If A/α is irreducible, then $A/\alpha \in ID_{n-|\alpha|}$. Lemma 2.4 yields $A/\alpha \in H_{n-|\alpha|}$ for each αN which completes the sufficiency of this theorem.

Case (ii). If A/α is reducible, there exists an $|\alpha'| \times |\alpha'|$ permutation matrix P such that

$$P[A/\alpha]P^T = P[\widetilde{A}(\alpha')]P^T = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & \cdots & \widetilde{A}_{1s} \\ 0 & \widetilde{A}_{22} & \cdots & \widetilde{A}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{A}_{ss} \end{bmatrix}, \tag{4.1}$$

where $\widetilde{A}_{ii}$ is irreducible for $i = 1, 2, \cdots, s$ $(s \geq 2)$, and correspondingly,

$$P[A(\alpha')]P^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{bmatrix}, \tag{4.2}$$

$$P[A(\alpha',\alpha)] = \begin{bmatrix} A_{10} & A_{20} & \cdots & A_{s0} \end{bmatrix}^T, \tag{4.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

and

$$[A(\alpha,\alpha')]P^T = \left[\begin{array}{cccc} A_{01} & A_{02} & \cdots & A_{0s} \end{array}\right], \tag{4.4}$$

where $\widetilde{A}_{ii}$ is irreducible for $i = 1, 2, \cdots, s$ $(s \geq 2)$, $\widetilde{A}_{ij} = \widetilde{A}(\alpha_i, \alpha_j)$ and $A_{ij} = A(\alpha_i, \alpha_j)$ are the submatrices of the matrix A and $A/\alpha = \widetilde{A}(\alpha')$, respectively, with row indices in α_i and column indices in α_j, $\bigcup_{j=1}^{s} \alpha_j = \alpha'$ and $\alpha_i \cap \alpha_j = \emptyset$ for $i \neq j$, $i, j = 0, 1, 2, \cdots, s$ $(s \geq 2)$, $\alpha_0 = \alpha$. Using (2.2),

$$P[A/\alpha]P^T = PA(\alpha')P^T - PA(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha')P^T. \tag{4.5}$$

Direct calculation gives

$$\widetilde{A}_{ii} = A_{ii} - A_{i0}[A(\alpha)]^{-1}A_{0i}, \quad i = 1, 2, \cdots, s \ (s \geq 2), \tag{4.6}$$

which is the Schur complement of the matrix

$$A(\alpha \cup \alpha_i) = \left[\begin{array}{cc} A(\alpha) & A(\alpha, \alpha_i) \\ A(\alpha_i, \alpha) & A(\alpha_i) \end{array}\right] = \left[\begin{array}{cc} A(\alpha) & A_{0i} \\ A_{i0} & A_{ii} \end{array}\right]$$

with respect to $A(\alpha_i) = A_{ii}$. Since $\widetilde{A}_{ii}$ is irreducible, it follows from Lemma 3.14 that $\widetilde{A}_{ii} = [A(\alpha \cup \alpha_i)]/\alpha = [A(\beta_i)]/\eta_i$, where $\beta_i \subseteq \alpha \cup \alpha_i$, $\eta_i = \beta_i - \alpha_i \supseteq \emptyset$ and $A(\beta_i)$ is an irreducible principal submatrix, defined in (3.15), of the matrix $A(\alpha \cup \alpha_i)$. Since $\bigcup_{j=1}^{s} \alpha_j = \alpha'$ and $s \geq 2$, $\alpha_i \subset \alpha'$ and $\beta_i \subseteq \alpha \cup \alpha_i \subset N$. Thus, $A(\beta_i)$ is also an irreducible principal submatrix of the matrix A. As a result, one must have $A(\beta_i) \in H_{|\beta_i|}$. Otherwise, $A(\beta_i) \notin H_{|\beta_i|}$. Since $A \in DE_n \subset D_n$ yields $A(\beta_i) \in D_{|\beta_i|}$ and $A(\beta_i) \notin H_{|\beta_i|}$ is irreducible, Lemma 3.5 indicates that $A(\beta_i) \in DE_{|\beta_i|}$ which shows that the irreducible matrix A has a diagonally equipotent principal submatrix $A(\beta_i)$. It then follows from Lemma 2.3 that the matrix A is reducible, which contradicts the irreducibility of A. Therefore, $A(\beta_i) \in ID_{|\beta_i|}$. From Lemma 2.4, we have $A(\beta_i) \in H_{|\beta_i|}$. Then, Lemma 3.11 gives $\widetilde{A}_{ii} = [A(\alpha \cup \alpha_i)]/\alpha = [A(\beta_i)]/\eta \in H_{|\alpha_i|}$. In the end, we conclude that $\widetilde{A}_{ii} \in H_{|\alpha_i|}$ for all $i = 1, 2, \cdots, s$. It then follows form (4.1) that $A/\alpha \in H_{n-|\alpha|}$. This completes sufficiency.

Now, we prove necessity by contradiction. Assume that the nonsingular matrix $A \in DE_n$ is reducible. Then, there exists an $n \times n$ permutation matrix Q such that

$$QAQ^T = \left[\begin{array}{cc} A(\alpha_1) & A(\alpha_1, \alpha_2) \\ 0 & A(\alpha_2) \end{array}\right], \tag{4.7}$$

where $\alpha_1 \subset N$ and $A(\alpha_2) \in DE_{|\alpha_2|}$ is irreducible. Set $\alpha = \alpha_1$, then

$$A/\alpha = A(\alpha_2) \in DE_{|\alpha_2|}$$

for $A(\alpha_1)$ is nonsingular. Therefore, $A/\alpha \notin H_{n-|\alpha|}$ in which $\alpha = \alpha_1 \subset N$, which contradicts $A/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$. Thus, sufficiency is true. This completes the proof. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

Example 4.1 Consider the irreducible matrix

$$A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ 1 & 2 & 0 & -1 \\ 2 & 1 & 5 & -2 \\ 1 & 2 & 3 & 6 \end{bmatrix} \in DE_4. \tag{4.8}$$

Since the comparison matrix of A

$$\mu(A) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -2 & -1 & 5 & -2 \\ -1 & -2 & -3 & 6 \end{bmatrix} \in Z_4$$

can be verified to be singular, $A \notin H_4$. However, setting $\alpha = \{1, 2\}$, it is easy to see that the Schur complement matrix of A,

$$A/\alpha = \begin{bmatrix} 5.4286 & -0.8571 \\ 3.0000 & 7.0000 \end{bmatrix} \in H_2.$$

Furthermore, we can verify $A/\alpha \in H_{4-|\alpha|}$ for each $\alpha \subset \{1, 2, 3, 4\}$, which shows that the sufficiency of Theorem 4.1 is true.

Example 4.2 Consider the reducible matrix

$$B = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ 1 & 6 & -2 & -1 & -1 & -1 \\ 2 & 1 & 10 & -2 & -3 & -2 \\ 0 & 0 & 0 & 6 & -2 & -4 \\ 0 & 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 3 & 1 & 4 \end{bmatrix} \in DE_6. \tag{4.9}$$

It follows from Lemma 3.1 that B is nonsingular . For $\alpha = \{1, 2, 3\}$, we have

$$B/\alpha = \begin{bmatrix} 6 & -2 & -4 \\ 2 & 8 & -6 \\ 3 & 1 & 4 \end{bmatrix} \notin H_3.$$

This illustrates that for a nonsingular matrix $A \in DE_n$, the irreducibility of A guarantees that $A/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$.

THEOREM 4.2. *If an irreducible matrix $A \in D_n$ is nonsingular, then $A/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$.*

Proof. If $A \in D_n \cap H_n$, it follows from Lemma 3.11 that $A/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$. If $A \in D_n$ but $A \notin H_n$, the irreducibility of A and Lemma 3.5 yield that

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

$A \in DE_n$. Again, A is nonsingular. Thus, $A \in DE_n$ is nonsingular. Since A is irreducible, it follows from the sufficiency of Theorem 4.1 that $A/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$. This completes the proof. □

In the following we consider the case in which the matrix $A \in D_n$ is reducible.

THEOREM 4.3. *Given a reducible matrix $A \in D_n$ and a set $\alpha = N - \alpha' \subset N$, if $A \notin H_n$ is nonsingular, then $A/\alpha \in H_{n-|\alpha|}$ if and only if $A(\alpha') \in H_{n-|\alpha|}$.*

Proof. The sufficiency can be proved by the following two cases.

(i) Assume $A(\alpha') \in H_{n-|\alpha|}$. If A/α is irreducible, then it follows from Lemma 3.14 that A/α is the Schur complement of some irreducible principal submatrix $A(\alpha_i)$ (defined in (3.15)) of A with respect to the principal submatrix $A(\eta)$, where $\eta = \alpha_i - \alpha'$. If $\eta = \emptyset$, then $A/\alpha = [A(\alpha_i)]/\emptyset = A(\alpha_i) = A(\alpha') \in H_{n-|\alpha|}$. If $\eta \neq \emptyset$ and $A(\alpha_i) \in H_{|\alpha_i|}$, Lemma 3.11 yields $A/\alpha \in H_{n-|\alpha|}$. If $\eta \neq \emptyset$ and $A(\alpha_i) \notin H_{|\alpha_i|}$, the irreducibility of $A(\alpha_i) \in D_{|\alpha_i|}$ and Lemma 3.5 give $A(\alpha_i) \in DE_{|\alpha_i|}$. Since $A \in D_n$ is nonsingular, it follows from Lemma 3.2 that $A(\alpha_i)$ is nonsingular. Thus, $A(\alpha_i) \in DE_{|\alpha_i|}$ is nonsingular. Again, since $A(\alpha_i)$ is irreducible, Theorem 4.1 shows $A/\alpha \in H_{n-|\alpha|}$.

(ii) If A/α is reducible, the proof is similar to the proof of Case (ii) in Theorem 4.1. This completes the proof of the sufficiency.

Now, we prove the necessity by contradiction. Assume that $A(\alpha') \notin H_{n-|\alpha|}$. Since $A \in D_n$, $A(\alpha') \in D_{n-|\alpha|}$. If $A(\alpha')$ is irreducible, Lemma 3.5 indicates $A(\alpha') \in DE_{n-|\alpha|}$. Remark 2.2 gives $A(\alpha', \alpha) = 0$. Then (2.2) yields $A/\alpha = A(\alpha') \notin H_{n-|\alpha|}$ which contradicts $A/\alpha \in H_{n-|\alpha|}$. Thus, $A(\alpha') \in H_{n-|\alpha|}$.
If $A(\alpha') \notin H_{n-|\alpha|}$ is reducible, Lemma 3.5 indicates that $A(\alpha')$ has the diagonally equipotent principal submatrix of largest order, say $A(\theta)$ for $\theta = \alpha' - \theta'$. Remark 2.2 gives that $A(\theta', \theta) = 0$ and $A(\alpha, \theta) = 0$. Therefore, there exists an $n \times n$ permutation matrix Q such that

$$QAQ^T = \begin{bmatrix} A(\alpha) & A(\alpha, \theta') & A(\alpha, \theta) \\ A(\theta', \alpha) & A(\theta') & A(\theta', \theta) \\ 0 & 0 & A(\theta) \end{bmatrix}. \tag{4.10}$$

and hence

$$\begin{aligned} A/\alpha &= A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha') \\ &= \begin{bmatrix} A(\theta') & A(\theta', \theta) \\ 0 & A(\theta) \end{bmatrix} - \begin{bmatrix} A(\theta', \alpha) \\ 0 \end{bmatrix} [A(\alpha)]^{-1} \begin{bmatrix} A(\alpha, \theta') & A(\alpha, \theta) \end{bmatrix} \\ &= \begin{bmatrix} [A(\alpha \cup \theta')]/\alpha & [A(\alpha \cup \theta', \alpha \cup \theta)]/\alpha \\ 0 & A(\theta) \end{bmatrix}. \end{aligned} \tag{4.11}$$

(4.11) shows that A/α has a diagonally equipotent principal submatrix $A(\theta)$. There-

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

fore, Lemma 3.5 gives $A/\alpha = A(\alpha') \notin H_{n-|\alpha|}$, which also contradicts $A/\alpha \in H_{n-|\alpha|}$. As a result, $A(\alpha') \in H_{n-|\alpha|}$. This completes the proof. □

Example 4.3 Let B be the matrix given in (4.9). It then follows from Lemma 3.5 and Lemma 3.1 that $B \notin H_6$ and is nonsingular. Set $\alpha = \{3, 4\}$ and $\alpha' = \{1, 2, 5, 6\}$ such that

$$B(\alpha') = \begin{bmatrix} 5 & -1 & -1 & -1 \\ 1 & 6 & -1 & -1 \\ 0 & 0 & 8 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \in H_4.$$

Direct computation yields that the Schur complement matrix of B,

$$B/\alpha = \begin{bmatrix} 5.2000 & -0.9000 & -1.7000 & -1.6000 \\ 1.4000 & 6.2000 & -2.0667 & -1.5333 \\ 0 & 0 & 8.6667 & 7.3333 \\ 0 & 0 & 2.0000 & 6.0000 \end{bmatrix} \in SD_4 \subset H_4,$$

which shows that the sufficiency of Theorem 4.1 is true.

If $B(\alpha') \in H_{|\alpha'|}$ can not be satisfied, then $B/\alpha' \in H_{n-|\alpha'|}$ will not be obtained. For example, setting $\alpha = \{1, 2, 3\}$ and $\alpha' = \{4, 5, 6\}$ such that $B(\alpha') \notin H_{|\alpha'|}$, Example 4.2 shows $B/\alpha' \notin H_{n-|\alpha'|}$. This illustrates that for any nonsingular matrix $B \in D_n$, the condition $B(\alpha') \in H_{|\alpha'|}$ for $\alpha' = N - \alpha \subset N$ guarantees that $B/\alpha \in H_{n-|\alpha|}$.

Using Lemma 3.12 and Theorem 4.3, we can obtain the following theorem.

THEOREM 4.4. *Given a matrix $A \in D_n$ and a set $\alpha = N - \alpha' \subset N$, if A is nonsingular and $A(\alpha') \in H_{n-|\alpha|}$, then $A/\alpha \in H_{n-|\alpha|}$.*

In the rest of this section, we will propose some theorems to decide whether a matrix is an irreducible or not.

THEOREM 4.5. *Given a matrix $A = (a_{ij}) \in C^{n\times n}$ with the matrix $\pi(A)$ defined by (2.5), then A is irreducible if and only if $[\pi(A)]/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$.*

Proof. According to Lemma 2.8, we need only prove that $\pi(A)$ is irreducible if and only if $[\pi(A)]/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$.

The sufficiency will be proved firstly. Assume that $[\pi(A)]/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$. Then, Lemma 2.4 gives that $[\pi(A)]/\alpha$ is nonsingular for each $\alpha \subset N$. Again, the existence of $[\pi(A)]/\alpha$ shows that $\pi(\alpha) = [\pi(A)](\alpha)$ is nonsingular for each $\alpha \subset N$. It then follows from Lemma 3.13 that $\pi(A)$ is also nonsingular. According to the definition of $\pi(A)$ and (2.5), we have $\pi(A) \in DE_n$. Therefore, it follows from the necessity of Theorem 4.1 that $\pi(A)$ is irreducible, which completes sufficiency.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

Now, we prove necessity. Assume that $\pi(A)$ is irreducible. The definition of $\pi(A) = (m_{ij})$ and (2.5) indicate $\pi(A) \in DE_n$. Thus, $m_{ii} \neq 0$ for all $i \in N$. Since $\pi(A)$ does not satisfy the sufficiency of Lemma 3.1, i.e., there does not exists a $n \times n$ unitary diagonal matrix U such that

$$U^{-1}D_\pi^{-1}\pi(A)U = \mu(D_\pi^{-1}A_k), \tag{4.12}$$

where $D_\pi = diag(m_{11}, m_{22}, \cdots, m_{nn})$, it follows from Lemma 3.1 that $\pi(A)$ is nonsingular. Hence, according to the sufficiency of Theorem 4.1, we have $[\pi(A)]/\alpha \in H_{n-|\alpha|}$ for each $\alpha \subset N$. This completes the proof. □

THEOREM 4.6. *Given a matrix $A = (a_{ij}) \in C^{n\times n}$ with the matrix $\pi(A)$ defined by (2.5), then A is irreducible if and only if $[\pi(A)]/i \in H_{n-1}$ for each $i \in N$, where $[\pi(A)]/i = [\pi(A)]/\alpha_i$ and $\alpha_i = \{i\}$ for $i \in N$.*

Proof. Necessity can be obtained immediately from the necessity of Theorem 4.5. The following will prove sufficiency by contradiction. Assume that A is reducible. It then follows from Lemma 2.8 that $\pi(A)$ is also reducible. Thus, there exists an $n \times n$ permutation matrix Q such that

$$Q[\pi(A)]Q^T = Q\pi Q^T = \begin{bmatrix} \pi(\alpha) & \pi(\alpha, \alpha') \\ 0 & \pi(\alpha') \end{bmatrix}, \tag{4.13}$$

where $\alpha' = N - \alpha \neq \emptyset$ and $\pi(\alpha') \in DE_{|\alpha_2|}$. Set $i_0 \in \alpha$, then

$$[\pi(A)]/i_0 = \pi(\alpha') \in DE_{|\alpha'|}.$$

Therefore, $[\pi(A)]/i_0/\alpha \notin H_{n-1}$ for some $i_0 \in N$, which contradicts $[\pi(A)]/i \in H_{n-1}$ for each $i \in N$. Thus, sufficiency holds. This completes the proof. □

Example 4.4 Let A be the matrix given in (4.8) and compute $\pi(A) = A$. Since $[\pi(A)]/i \in H_3$ for each $i \in \{1, 2, 3, 4\}$, A is irreducible, which shows the sufficiency of Theorem 4.5 is valid.

On the other hand, let B be the matrix given in (4.9) and compute $\pi(B) = B$. Set $i = 1$, then direct computation gives

$$B/1 = \begin{bmatrix} 6.2 & -1.8 & -1 & -1 & -1 \\ 1.4 & 10.4 & -2 & -3 & -2 \\ 0 & 0 & 6 & -2 & -4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 3 & 1 & 4 \end{bmatrix}.$$

Since $B(\{4, 5, 6\}) = \begin{bmatrix} 6 & -2 & -4 \\ 2 & 8 & -6 \\ 3 & 1 & 4 \end{bmatrix}$ is an irreducibly diagonally equipotent principal submatrix of $B/1$, Lemma 3.5 yields $[B/1] \notin H_5$. But, it can be verified $[B/5] \in H_5$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 69-87, January 2009

Therefore, B is reducible since there exists some $i \in \{1,2,3,4,5,6\}$ such that $[B/i] \notin H_5$. This demonstrates validity of the necessity of Theorem 4.5.

5. Conclusions. This paper studies the Schur complements of generally diagonally dominant matrices and a criterion for irreducibility of matrices. Some results are proven resulting in new conditions on the nonsingular matrix $A \in D_n$ and the subset $\alpha \subset N$ so that the Schur complement matrix A/α is an $H-$matrix. Subsequently, a criterion for irreducibility of matrices is presented to show that the matrix A is irreducible if and only if $[\pi(A)]/i \in H_{n-1}$ for each $i \in N$.

Acknowledgment. The authors would like to thank the anonymous referees for their valuable comments and suggestions, which actually stimulated this work.

REFERENCES

[1] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York, 1979.
[2] D. Carlson and T. Markham. Schur complements of diagonally dominant matrices. *Czech. Math. J.,* 29(104): 246–251, 1979.
[3] G.H. Golub and C.F. Van Loan. *Matrix Computations*, third ed. Johns Hopkins University Press, Baltimore, 1996.
[4] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, 1985.
[5] C.R. Johnson. Inverse $M-$matrices. *Linear Algebra Appl.,* 47:195–216, 1982.
[6] R. Kress. *Numerical Analysis*. Springer, New York, 1998.
[7] T.G. Lei, C.W. Woo, J.Z. Liu, and F. Zhang. On the Schur Complements of Diagonally Dominant Matrices. *Proceedings of the SIAM Conference on Applied Linear Algebra*, 2003.
[8] Jianzhou Liu and Yungqing Huang. Some properties on Schur complements of $H-$matrices and diagonally dominant matrices. *Linear Algebra Appl.,* 389:365–380, 2004.
[9] Jianzhou Liu, Yungqing Huang, and Fuzhen Zhang. The Schur complements of generalized doubly diagonally dominant matrices. *Linear Algebra Appl.,* 378:231–244, 2004.
[10] Ljiljana Cvetković, Vladimir Kostić, Maja Kovačević, and Tomasz Szulć. Further results on H-matrices and their Schur complements. *Appl. Math. Comput.,* 198(2):506–510, 2008.
[11] A.W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and its Applications*. Academic Press, New York, 1979.
[12] R.S. Varga. *Matrix Iterative Analysis*, second ed. Spriger-Verlag, Berlin, Heidelberg, 2000.
[13] B. Wendroff. *Theoretical Numerical Analysis*. Academic Press, New York, 1966.
[14] J.H. Wilkinson. Error analysis of direct methods of matrix inversion. *J. Assoc. Comput. Mach.,* 8:281–330, 1961.
[15] Cheng-yi Zhang, Chengxian Xu, and Yao-tang Li. The eigenvalue distribution on Schur complements of $H-$matrices. *Linear Algebra Appl.,* 422:250–264, 2007.
[16] Fuzhen Zhang. *The Schur Complement and Its Applications*. Springer, New York, 2005.