Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 64-68, January 2009

ELA

ERRATUM TO 'A NOTE ON THE LARGEST EIGENVALUE OF NON-REGULAR GRAPHS' *

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Abstract. Let $\lambda_1(G)$ be the largest eigenvalue of the adjacency matrix of graph G with n vertices and maximum degree Δ. Recently, $\Delta - \lambda_1(G) > \frac{\Delta+1}{n(3n+\Delta-8)}$ for a non-regular connected graph G was obtained in [B.L. Liu and G. Li, A note on the largest eigenvalue of non-regular graphs, *Electron J. Linear Algebra*, 17:54–61, 2008]. But unfortunately, a mistake was found in the cited preprint [T. Bıyıkoğlu and J. Leydold, Largest eigenvalues of degree sequences], which led to an incorrect proof of the main result of [B.L. Liu and G. Li]. This paper presents a correct proof of the main result in [B.L. Liu and G. Li], which avoids the incorrect theorem in [T. Bıyıkoğlu and J. Leydold].

Key words. Spectral radius, Non-regular graph, λ_1-extremal graph, Perron vector.

AMS subject classifications. 05C35, 15A48, 05C50.

1. Introduction. In this paper, we only consider connected, simple and undirected graphs. Let uv be an edge whose end vertices are u and v. The symbol $N(u)$ denotes the neighbor set of vertex u. Then $d_G(u) = |N(u)|$ is called the degree of u. The maximum degree among the vertices of G is denoted by Δ. The sequence $\pi = \pi(G) = (d_1, d_2, ..., d_n)$ is called the *degree sequence* of G, where $d_i = d_G(v)$ holds for some $v \in V(G)$. In the entire article, we enumerate the degrees in non-increasing order, i.e., $d_1 \geq d_2 \geq \cdots \geq d_n$.

Let $A(G)$ be the adjacency matrix of G. The spectral radius of G, denoted by $\lambda_1(G)$, is the largest in modulus eigenvalue of $A(G)$. When G is connected, $A(G)$ is irreducible and by the Perron-Frobenius Theorem (see e.g., [4]), $\lambda_1(G)$ is a simple eigenvalue and has a unique positive unit eigenvector. We refer to such an eigenvector f as the *Perron vector* of G.

Let G be a connected non-regular graph. In [7], G is called λ_1-*extremal* if

*Received by the editors November 20, 2008. Accepted for publication January 06, 2009. Handling Editor: Stephen J. Kirkland.

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A publication of the International Linear Algebra Society
Volume 18, pp. 64-68, January 2009

$\lambda_1(G) \geq \lambda_1(G')$ holds for any other connected non-regular graph G' with the same number of vertices and maximum degree as G. Let $\mathcal{G}(n, \Delta)$ denote the set of all connected non-regular graphs with n vertices and maximum degree Δ.

In [6], the following result was proved.

Theorem 1.1. *Suppose $G \in \mathcal{G}(n, \Delta)$. Then*

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}.$$

But unfortunately, a mistake was found in the cited reference [3], resulting in an incorrect proof of Theorem 1.1. In this paper, we shall give a correct proof of Theorem 1.1, in which we avoid using the wrong theorem in reference [3].

2. Main result.

Let $V_{<\Delta} = \{u : d(u) < \Delta\}$. For the characterization of λ_1-extremal graphs G of $\mathcal{G}(n, \Delta)$, we have the following.

Theorem 2.1. [6] *Suppose $2 < \Delta < n - 1$. If G is a λ_1-extremal graph of $\mathcal{G}(n, \Delta)$, then G must have one of the following properties:*

(1) $|V_{<\Delta}| \geq 2$, *$V_{<\Delta}$ induces a complete graph.*

(2) $|V_{<\Delta}| = 1$.

(3) $V_{<\Delta} = \{u, v\}$, *$uv \notin E(G)$ and $d(u) = d(v) = \Delta - 1$.*

Definition 2.2. [6] Suppose $2 < \Delta < n - 1$ and $G \in \mathcal{G}(n, \Delta)$. Then

G is called a *type-I graph* if G has property (1);

G is called a *type-II graph* if G has property (2);

G is called a *type-III graph* if G has property (3).

By Definition 2.2, it is easy to see the following.

Proposition 2.3. *If G is a type-III graph, then*

$$\pi(G) = (\Delta, \Delta, ..., \Delta, \Delta - 1, \Delta - 1),$$

where $2 < \Delta < n - 1$.

Suppose $\pi = (d_1, d_2, ..., d_n)$ and $\pi' = (d'_1, d'_2, ..., d'_n)$. We write $\pi \trianglelefteq \pi'$ if and only if $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$, and $\sum_{i=1}^{j} d_i \leq \sum_{i=1}^{j} d'_i$ for all $j = 1, 2, ..., n$. Let C_π be the class of connected graphs with degree sequence π. If $G \in C_\pi$ and $\lambda_1(G) \geq \lambda_1(G')$ for any other $G' \in C_\pi$, then G is said to *have the greatest maximum eigenvalue* in C_π.

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The next theorem (i.e., Theorem 2.3 [6]) is a crucial lemma in the proof of Theorem 1.1.

THEOREM 2.4. [6] *Suppose* $2 < \Delta < n-1$ *and* G *is a* λ_1*-extremal graph of* $\mathcal{G}(n,\Delta)$*. Then* G *must be a type-I or type-II graph.*

In [6], the proof of Theorem 2.4 needs the next result which was stated in [3].

THEOREM 2.5. [3] *Let* π *and* π' *be two distinct degree sequences with* $\pi \trianglelefteq \pi'$*. Let* G *be the graph with greatest maximum eigenvalue in class* C_π*, and* G' *in class* $C_{\pi'}$*, respectively. Then* $\lambda_1(G) < \lambda_1(G')$.

By Proposition 2.3, if G is a type-III graph, then $\pi(G) = (\Delta, \Delta, ..., \Delta, \Delta-1, \Delta-1)$. Let G' be a graph with $\pi(G') = (\Delta, \Delta, ..., \Delta, \Delta, \Delta-2)$. It is easy to see that $\pi(G) \trianglelefteq \pi(G')$. Thus, with the application of Theorem 2.5, one can prove Theorem 2.4. But unfortunately, some counterexamples to Theorem 2.5 have been found; thus the authors of [3] have changed Theorem 2.5 from general graphs to the class of trees (see [2]).

Next we shall give a proof of Theorem 2.4 that does not depend on Theorem 2.5.

Let $G - uv$ be the graph obtained from G by deleting the edge $uv \in E(G)$. Similarly, $G + uv$ denotes the graph obtained from G by adding an edge $uv \notin E(G)$, where $u, v \in V(G)$.

LEMMA 2.6. (*Shifting* [1]) *Let* $G(V,E)$ *be a connected graph with* $uv_1 \in E$ *and* $uv_2 \notin E$*. Let* $G' = G + uv_2 - uv_1$*. Suppose* f *is the Perron vector of* G*. If* $f(v_2) \geq f(v_1)$*, then* $\lambda_1(G') > \lambda_1(G)$.

LEMMA 2.7. (*Switching* [1], [5]) *Let* $G(V,E)$ *be a connected graph with* $u_1v_1 \in E$ *and* $u_2v_2 \in E$*, but* $v_1v_2 \notin E$ *and* $u_1u_2 \notin E$*. Let* $G' = G + v_1v_2 + u_1u_2 - u_1v_1 - u_2v_2$*. Suppose* f *is the Perron vector of* G*. If* $f(v_1) \geq f(u_2)$ *and* $f(v_2) \geq f(u_1)$*, then* $\lambda_1(G') \geq \lambda_1(G)$*. The inequality is strict if and only if at least one of the two inequalities is strict.*

Let $\mathcal{G}_1(n,\Delta)$ denote the set of all connected graphs of type-III.

LEMMA 2.8. *Let* G *be a graph in* $\mathcal{G}_1(n,\Delta)$ *with* $u_1v_1 \in E(G)$*, where* $d(u_1) = \Delta$ *and* $d(v_1) = \Delta - 1$*. Suppose* f *is the Perron vector of* G*. If* $f(u_1) \leq f(v_1)$*, then* G *cannot have the greatest maximum eigenvalue in* $\mathcal{G}_1(n,\Delta)$.

Proof. Assume that the contrary holds, i.e., suppose that G has the greatest maximum eigenvalue in $\mathcal{G}_1(n,\Delta)$. Without loss of generality, assume $V_{<\Delta}(G) = \{v_1, v_2\}$. By Definition 2.2, we have $d(v_1) = \Delta - 1 = d(v_2)$ and $v_1v_2 \notin E$. We divide the proof into two cases:

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Case 1. $u_1v_2 \in E$. Let $G' = G + v_1v_2 - u_1v_2$. Thus, $d_{G'}(u_1) = \Delta - 1 = d_{G'}(v_2)$, $d_{G'}(v_1) = \Delta$ and $u_1v_2 \notin E(G')$. Moreover, G' is also connected. Thus, $G' \in \mathcal{G}_1(n, \Delta)$. By Lemma 2.6, we have $\lambda_1(G') > \lambda_1(G)$, a contradiction.

Case 2. $u_1v_2 \notin E$. Since $d_G(u_1) = \Delta > \Delta - 1 = d_G(v_1)$, then there must exist some $w \in N(u_1)$ such that $w \notin N(v_1)$ and $w \neq v_1$. Let $G' = G + v_1w - u_1w$. Thus, $d_{G'}(u_1) = d_{G'}(v_2) = \Delta - 1$, $d_{G'}(v_1) = \Delta$, and $u_1v_2 \notin E$. Moreover, G' is also connected. This implies that $G' \in \mathcal{G}_1(n, \Delta)$. By Lemma 2.6, we have $\lambda_1(G') > \lambda_1(G)$, a contradiction.

The result follows. ∎

The following is the proof of Theorem 2.4.

Proof. Assume that the contrary holds, i.e., suppose that there is a graph G of type-III such that G has the greatest maximum eigenvalue in $\mathcal{G}(n, \Delta)$. (This implies that G also has the greatest maximum eigenvalue in $\mathcal{G}_1(n, \Delta)$.) Without loss of generality, assume $V_{<\Delta}(G) = \{v_1, v_2\}$. By Definition 2.2, we have $d(v_1) = \Delta - 1 = d(v_2)$ and $v_1v_2 \notin E$. Let f be the Perron vector of G. We consider the next two cases:

Case 1. $N(v_1) = N(v_2) = \{u_1, ..., u_{\Delta-1}\}$. Since $2 < \Delta < n-1$ and G is connected, there exists i, j $(1 \leq i < j \leq \Delta - 1)$ such that $u_iu_j \notin E$ (otherwise, the subgraph of G induced by $\{u_1, ..., u_{\Delta-1}\}$ is a complete graph of order $\Delta - 1$, and it will yield a contradiction to the connection of G by $\Delta < n - 1$).

If $f(u_i) \leq f(v_2)$, note that $d(u_i) = \Delta$, $d(v_2) = \Delta - 1$ and $u_iv_2 \in E(G)$, and by Lemma 2.8, G cannot have the greatest maximum eigenvalue in $\mathcal{G}_1(n, \Delta)$ (also, G cannot have the greatest maximum eigenvalue in $\mathcal{G}(n, \Delta)$), a contradiction. Moreover, since $d(u_j) = \Delta$, $d(v_1) = \Delta - 1$ and $u_jv_1 \in E(G)$, it can be proved analogously that $f(u_j) \leq f(v_1)$ is also impossible. Thus, $f(u_i) > f(v_2)$ and $f(u_j) > f(v_1)$. Let $G' = G + u_iu_j + v_1v_2 - u_iv_1 - u_jv_2$. Clearly, G' is also connected and $G' \in \mathcal{G}(n, \Delta)$. By Lemma 2.7, we can conclude that $\lambda_1(G') > \lambda_1(G)$, a contradiction.

Case 2. $N(v_1) \neq N(v_2)$. Without loss of generality, suppose $f(v_1) \geq f(v_2)$. Two subcases should be considered as follows.

Subcase 1. $|N(v_1) \cap N(v_2)| \geq 1$. Since $N(v_1) \neq N(v_2)$, there exists u_j such that $u_j \in N(v_2) \setminus N(v_1)$. Let $G' = G + v_1u_j - v_2u_j$. Note that G' is also connected and $G' \in \mathcal{G}(n, \Delta)$. By Lemma 2.6, we have $\lambda_1(G') > \lambda_1(G)$, a contradiction.

Subcase 2. $|N(v_1) \cap N(v_2)| = 0$. Since G is connected, there exists a shortest path P from v_2 to v_1. Note that $d_G(v_2) = \Delta - 1 \geq 2$ and $|N(v_1) \cap N(v_2)| = 0$. Then there must exist u_j such that $u_j \in N(v_2) \setminus N(v_1)$, but $u_j \notin P$. Let $G' = G + v_1u_j - v_2u_j$. Clearly, G' is also connected and $G' \in \mathcal{G}(n, \Delta)$. By Lemma 2.6, we have $\lambda_1(G') > \lambda_1(G)$, a contradiction.

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This completes the proof. □

With the help of Theorem 2.4, it is not difficult to prove that Theorem 1.1 holds. For details of the proof, one can refer to Section 3 of [6].

Acknowledgment. The authors are grateful to the referees for their valuable comments, corrections and suggestions, which led to an improvement of the original manuscript.

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