ERRATUM TO ‘A NOTE ON THE LARGEST EIGENVALUE OF NON-REGULAR GRAPHS’ *

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Abstract. Let $\lambda_1(G)$ be the largest eigenvalue of the adjacency matrix of graph $G$ with $n$ vertices and maximum degree $\Delta$. Recently, $\Delta - \lambda_1(G) > \frac{\Delta+1}{n+\Delta-8}$ for a non-regular connected graph $G$ was obtained in [B.L. Liu and G. Li, A note on the largest eigenvalue of non-regular graphs, Electron J. Linear Algebra, 17:54–61, 2008]. But unfortunately, a mistake was found in the cited preprint [T. Buykoğlu and J. Leydold, Largest eigenvalues of degree sequences], which led to an incorrect proof of the main result of [B.L. Liu and G. Li]. This paper presents a correct proof of the main result in [B.L. Liu and G. Li], which avoids the incorrect theorem in [T. Buykoğlu and J. Leydold].

Key words. Spectral radius, Non-regular graph, $\lambda_1$-extremal graph, Perron vector.

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1. Introduction. In this paper, we only consider connected, simple and undirected graphs. Let $uv$ be an edge whose end vertices are $u$ and $v$. The symbol $N(u)$ denotes the neighbor set of vertex $u$. Then $d_G(u) = |N(u)|$ is called the degree of $u$. The maximum degree among the vertices of $G$ is denoted by $\Delta$. The sequence $\pi = \pi(G) = (d_1, d_2, \ldots, d_n)$ is called the degree sequence of $G$, where $d_i = d_G(v)$ holds for some $v \in V(G)$. In the entire article, we enumerate the degrees in non-increasing order, i.e., $d_1 \geq d_2 \geq \cdots \geq d_n$.

Let $A(G)$ be the adjacency matrix of $G$. The spectral radius of $G$, denoted by $\lambda_1(G)$, is the largest in modulus eigenvalue of $A(G)$. When $G$ is connected, $A(G)$ is irreducible and by the Perron-Frobenius Theorem (see e.g., [4]), $\lambda_1(G)$ is a simple eigenvalue and has a unique positive unit eigenvector. We refer to such an eigenvector $f$ as the Perron vector of $G$.

Let $G$ be a connected non-regular graph. In [7], $G$ is called $\lambda_1$-extremal if

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\( \lambda_1(G) \geq \lambda_1(G') \) holds for any other connected non-regular graph \( G' \) with the same number of vertices and maximum degree as \( G \). Let \( \mathcal{G}(n, \Delta) \) denote the set of all connected non-regular graphs with \( n \) vertices and maximum degree \( \Delta \).

In [6], the following result was proved.

**Theorem 1.1.** Suppose \( G \in \mathcal{G}(n, \Delta) \). Then

\[
\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}.
\]

But unfortunately, a mistake was found in the cited reference [3], resulting in an incorrect proof of Theorem 1.1. In this paper, we shall give a correct proof of Theorem 1.1, in which we avoid using the wrong theorem in reference [3].

2. Main result.

Let \( V_{<\Delta} = \{ u : d(u) < \Delta \} \). For the characterization of \( \lambda_1 \)-extremal graphs \( G \) of \( \mathcal{G}(n, \Delta) \), we have the following.

**Theorem 2.1.** [6] Suppose \( 2 < \Delta < n - 1 \). If \( G \) is a \( \lambda_1 \)-extremal graph of \( \mathcal{G}(n, \Delta) \), then \( G \) must have one of the following properties:

1. \( |V_{<\Delta}| \geq 2 \), \( V_{<\Delta} \) induces a complete graph.
2. \( |V_{<\Delta}| = 1 \).
3. \( V_{<\Delta} = \{ u, v \} \), \( uv \notin E(G) \) and \( d(u) = d(v) = \Delta - 1 \).

**Definition 2.2.** [6] Suppose \( 2 < \Delta < n - 1 \) and \( G \in \mathcal{G}(n, \Delta) \). Then

\( G \) is called a **type-I graph** if \( G \) has property (1);

\( G \) is called a **type-II graph** if \( G \) has property (2);

\( G \) is called a **type-III graph** if \( G \) has property (3).

By Definition 2.2, it is easy to see the following.

**Proposition 2.3.** If \( G \) is a type-III graph, then

\( \pi(G) = (\Delta, \Delta, ..., \Delta, \Delta - 1, \Delta - 1) \),

where \( 2 < \Delta < n - 1 \).

Suppose \( \pi = (d_1, d_2, ..., d_n) \) and \( \pi' = (d'_1, d'_2, ..., d'_n) \). We write \( \pi \preceq \pi' \) if and only if \( \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i \) and \( \sum_{i=1}^{j} d_i \leq \sum_{i=1}^{j} d'_i \) for all \( j = 1, 2, ..., n \). Let \( C_\pi \) be the class of connected graphs with degree sequence \( \pi \). If \( G \in C_\pi \) and \( \lambda_1(G) \geq \lambda_1(G') \) for any other \( G' \in C_\pi \), then \( G \) is said to **have the greatest maximum eigenvalue** in \( C_\pi \).
The next theorem (i.e., Theorem 2.3 [6]) is a crucial lemma in the proof of Theorem 1.1.

**THEOREM 2.4.** [6] Suppose $2 < \Delta < n - 1$ and $G$ is a $\lambda_1$-extremal graph of $G(n, \Delta)$. Then $G$ must be a type-I or type-II graph.

In [6], the proof of Theorem 2.4 needs the next result which was stated in [3].

**THEOREM 2.5.** [3] Let $\pi$ and $\pi'$ be two distinct degree sequences with $\pi \leq \pi'$. Let $G$ be the graph with greatest maximum eigenvalue in class $C_{\pi}$, and $G'$ in class $C_{\pi'}$, respectively. Then $\lambda_1(G) < \lambda_1(G')$.

By Proposition 2.3, if $G$ is a type-III graph, then $\pi(G) = (\Delta, \Delta, ..., \Delta, \Delta - 1, \Delta - 1)$. Let $G'$ be a graph with $\pi(G') = (\Delta, \Delta, ..., \Delta, \Delta, \Delta - 2)$. It is easy to see that $\pi(G) \leq \pi(G')$. Thus, with the application of Theorem 2.5, one can prove Theorem 2.4. But unfortunately, some counterexamples to Theorem 2.5 have been found; thus the authors of [3] have changed Theorem 2.5 from general graphs to the class of trees (see [2]).

Next we shall give a proof of Theorem 2.4 that does not depend on Theorem 2.5.

Let $G - uv$ be the graph obtained from $G$ by deleting the edge $uv \in E(G)$. Similarly, $G + uv$ denotes the graph obtained from $G$ by adding an edge $uv \notin E(G)$, where $u, v \in V(G)$.

**LEMMA 2.6.** (Shifting [1]) Let $G(V, E)$ be a connected graph with $uv_1 \in E$ and $uv_2 \notin E$. Let $G' = G + uv_2 - uv_1$. Suppose $f$ is the Perron vector of $G$. If $f(v_2) \geq f(v_1)$, then $\lambda_1(G') > \lambda_1(G)$.

**LEMMA 2.7.** (Switching [1], [5]) Let $G(V, E)$ be a connected graph with $u_1v_1 \in E$ and $u_2v_2 \notin E$, but $v_1v_2 \notin E$ and $u_1u_2 \notin E$. Let $G' = G + u_1v_2 + u_2v_1 - u_1v_1 - u_2v_2$. Suppose $f$ is the Perron vector of $G$. If $f(v_1) \geq f(u_2)$ and $f(v_2) \geq f(u_1)$, then $\lambda_1(G') \geq \lambda_1(G)$. The inequality is strict if and only if at least one of the two inequalities is strict.

Let $G_1(n, \Delta)$ denote the set of all connected graphs of type-III.

**LEMMA 2.8.** Let $G$ be a graph in $G_1(n, \Delta)$ with $u_1v_1 \in E(G)$, where $d(u_1) = \Delta$ and $d(v_1) = \Delta - 1$. Suppose $f$ is the Perron vector of $G$. If $f(u_1) \leq f(v_1)$, then $G$ cannot have the greatest maximum eigenvalue in $G_1(n, \Delta)$.

**Proof.** Assume that the contrary holds, i.e., suppose that $G$ has the greatest maximum eigenvalue in $G_1(n, \Delta)$. Without loss of generality, assume $V_{<\Delta}(G) = \{v_1, v_2\}$. By Definition 2.2, we have $d(v_1) = \Delta - 1 = d(v_2)$ and $v_1v_2 \notin E$. We divide the proof into two cases:
Case 1. \(u_1v_2 \in E\). Let \(G' = G + v_1v_2 - u_1v_2\). Thus, \(d_G'(u_1) = \Delta - 1 = d_G'(v_2)\), \(d_G'(v_1) = \Delta\) and \(u_1v_2 \notin E(G')\). Moreover, \(G'\) is also connected. Thus, \(G' \in G_1(n, \Delta)\). By Lemma 2.6, we have \(\lambda_1(G') > \lambda_1(G)\), a contradiction.

Case 2. \(u_1v_2 \notin E\). Since \(d_G(u_1) = \Delta > \Delta - 1 = d_G(v_1)\), then there must exist some \(w \in N(u_1)\) such that \(w \notin N(v_1)\) and \(w \neq v_1\). Let \(G' = G + v_1w - u_1w\). Thus, \(d_G'(u_1) = d_G'(v_2) = \Delta - 1\), \(d_G'(v_1) = \Delta\), and \(u_1v_2 \notin E\). Moreover, \(G'\) is also connected. This implies that \(G' \in G_1(n, \Delta)\). By Lemma 2.6, we have \(\lambda_1(G') > \lambda_1(G)\), a contradiction.

The result follows. \(\Box\)

The following is the proof of Theorem 2.4.

**Proof.** Assume that the contrary holds, i.e., suppose that there is a graph \(G\) of type-III such that \(G\) has the greatest maximum eigenvalue in \(G(n, \Delta)\). (This implies that \(G\) also has the greatest maximum eigenvalue in \(G_1(n, \Delta)\).) Without loss of generality, assume \(V_{\leq \Delta}(G) = \{v_1, v_2\}\). By Definition 2.2, we have \(d(v_1) = \Delta - 1 = d(v_2)\) and \(v_1v_2 \notin E\). Let \(f\) be the Perron vector of \(G\). We consider the next two cases:

Case 1. \(N(v_1) = N(v_2) = \{u_1, \ldots, u_{\Delta - 1}\}\). Since \(2 < \Delta < n - 1\) and \(G\) is connected, there exists \(i, j\) \((1 \leq i < j \leq \Delta - 1)\) such that \(u_iu_j \notin E\) (otherwise, the subgraph of \(G\) induced by \(\{u_1, \ldots, u_{\Delta - 1}\}\) is a complete graph of order \(\Delta - 1\), and it will yield a contradiction to the connection of \(G\) by \(\Delta < n - 1\)).

If \(f(u_i) < f(v_2)\), note that \(d(u_i) = \Delta, d(v_2) = \Delta - 1\) and \(u_iv_2 \in E(G)\), and by Lemma 2.8, \(G\) cannot have the greatest maximum eigenvalue in \(G_1(n, \Delta)\) (also, \(G\) cannot have the greatest maximum eigenvalue in \(G(n, \Delta)\)), a contradiction. Moreover, since \(d(u_j) = \Delta, d(v_1) = \Delta - 1\) and \(u_jv_1 \in E(G)\), it can be proved analogously that \(f(u_j) < f(v_1)\) is also impossible. Thus, \(f(u_i) > f(v_2)\) and \(f(u_j) > f(v_1)\). Let \(G' = G + u_iu_j + v_1v_2 - u_1v_1 - u_jv_2\). Clearly, \(G'\) is also connected and \(G' \in G(n, \Delta)\).

By Lemma 2.7, we can conclude that \(\lambda_1(G') > \lambda_1(G)\), a contradiction.

Case 2. \(N(v_1) \neq N(v_2)\). Without loss of generality, suppose \(f(v_1) \geq f(v_2)\). Two subcases should be considered as follows.

Subcase 1. \(|N(v_1) \cap N(v_2)| \geq 1\). Since \(N(v_1) \neq N(v_2)\), there exists \(u_j\) such that \(u_j \in N(v_2) \setminus N(v_1)\). Let \(G' = G + v_1u_j - v_2u_j\). Note that \(G'\) is also connected and \(G' \in G(n, \Delta)\). By Lemma 2.6, we have \(\lambda_1(G') > \lambda_1(G)\), a contradiction.

Subcase 2. \(|N(v_1) \cap N(v_2)| = 0\). Since \(G\) is connected, there exists a shortest path \(P\) from \(v_2\) to \(v_1\). Note that \(d_G(v_2) = \Delta - 1 \geq 2\) and \(|N(v_1) \cap N(v_2)| = 0\). Then there must exist \(u_j\) such that \(u_j \in N(v_2) \setminus N(v_1)\), but \(u_j \notin P\). Let \(G' = G + v_1u_j - v_2u_j\). Clearly, \(G'\) is also connected and \(G' \in G(n, \Delta)\). By Lemma 2.6, we have \(\lambda_1(G') > \lambda_1(G)\), a contradiction.
This completes the proof. □

With the help of Theorem 2.4, it is not difficult to prove that Theorem 1.1 holds. For details of the proof, one can refer to Section 3 of [6].

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REFERENCES