Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 58-63, January 2009

ELA

THE NUMERICAL RADIUS OF A WEIGHTED SHIFT OPERATOR WITH GEOMETRIC WEIGHTS*

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Abstract. Let T be a weighted shift operator T on the Hilbert space $\ell^2(\mathbf{N})$ with geometric weights. Then the numerical range of T is a closed disk about the origin, and its numerical radius is determined in terms of the reciprocal of the minimum positive root of a hypergeometric function. This function is related to two Rogers-Ramanujan identities.

Key words. Numerical radius, Weighted shift operator, Rogers-Ramanujan identities.

AMS subject classifications. 47A12, 47B37, 33D15.

1. Introduction. Let T be an operator on a separable Hilbert space. The numerical range of T is defined to be the set

$$W(T) = \{< Tx, x >: \|x\| = 1\}.$$

The numerical range is always nonempty, bounded and convex. The numerical radius $w(T)$ is the supremum of the modulus of $W(T)$. We consider a weighted shift operator T on the Hilbert space $\ell^2(\mathbf{N})$ defined by

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ a_1 & 0 & 0 & 0 & \dots \\ 0 & a_2 & 0 & 0 & \dots \\ 0 & 0 & a_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where $\{a_n\}$ is a bounded sequence. It is known (cf. [1]) that $W(T)$ is a circular disk about the origin. Stout [3] shows that $W(T)$ is an open disk if the weights are periodic and nonzero. For example, when $a_n = 1$ for all n, $W(T)$ is the open unit disk. Clearly, $w(T)$ is the maximal eigenvalue of the selfadjoint operator $(T+T^*)/2$. Stout [3] gives a formula for the numerical radius of a weighted shift operator T, by introducing an

*Received by the editors August 10, 2008. Accepted for publication January 3, 2009. Handling Editor: Shmuel Friedland.

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Supported in part by Taiwan National Science Council.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 58-63, January 2009

object which is the determinant of the operator $I - z\,(T+T^*)/2$ in the sense of limit process of the determinants of finite-dimensional weighted shift matrices.

Let T be the weighted shift operator defined in (1.1) with square summable weights. Denote by $F_T(z)$ the determinant of $I - z(T+T^*)$. It is given by

$$F_T(z) = 1 + \sum_{n=1}^{\infty} (-1)^n c_n z^{2n}, \tag{1.2}$$

where

$$c_n = \sum a_{i_1}^2 a_{i_2}^2 \cdots a_{i_n}^2, \tag{1.3}$$

the sum is taken over

$$1 \le i_1 < i_2 < \cdots < i_n < \infty, \;\; i_2 - i_1 \ge 2, i_3 - i_2 \ge 2, \;\; \ldots, \;\;\; i_n - i_{n-1} \ge 2.$$

Stout [3] proves that $w(T+T^*) = 1/\lambda$, where λ is the minimum positive root of $F_T(z)$.

In this paper, we follow the method of Stout [3] to compute the numerical radius of a geometrically weighted shift operator in terms of the minimum positive root of a hypergeometric function. This function is related to two Rogers-Ramanujan identities.

2. Geometric weights. Let T be an operator on a Hilbert space, and let $T = UP$ be the polar decomposition of T. The Aluthge transformation $\Delta(T)$ of T is defined by

$$\Delta(T) = P^{\frac{1}{2}} U P^{\frac{1}{2}}.$$

In [4], the numerical range of the Aluthge transformation of an operator is treated. The authors have learned from T. Yamazaki that an upper and a lower bounds for the numerical radius of a geometrically weighted shift operator can be obtained from the numerical range of the Aluthge transformation in the following way:

THEOREM 2.1. *Let T be a weighted shift operator with geometric weights $\{q^{n-1}, n \in \mathbf{N}\}$, $0 < q < 1$. Then $W(T)$ is a closed disk about the origin, and*

$$\frac{1}{4q^{3/2}} \sqrt{54 - 36q - 2q^2 - 2\sqrt{(1-q)(9-q)^3}} \le w(T) \le 1/(2-\sqrt{q}).$$

Proof. Since $\sum_{n=1}^{\infty} a_n^2 = 1/(1-q^2) < \infty$, T is Hilbert-Schmidt, and thus compact. Then by [3, Corollary 8], $W(T)$ is closed.

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Let T be the geometrically weighted shift with weights $\{q^{n-1}\}$. Then the positive semidefinite part P of the polar decomposition of T is

$$P = \operatorname{diag}\left\{1, q, q^2, q^3, \ldots, q^{n-1}, \ldots\right\},$$

and we obtain that

$$\Delta(T) = \sqrt{q}\ T.$$

By [5], the inequality

$$w(T) \leq ||T||/2 + w(\Delta(T))/2$$

holds. Thus, we have

$$w(T) \leq ||T||/2 + \sqrt{q}\ w(T)/2.$$

Since the operator norm $||T|| = 1$, it follows that

$$w(T) \leq 1/(2 - \sqrt{q}).$$

For the lower bound, we consider the unit vector $x \in \ell^2(\mathbf{N})$ with coordinates $x_n = (1-\alpha)^{1/2}\alpha^{(n-1)/2}$, where $0 < \alpha < 1$. Then

$$< Tx, x >= x_1x_2 + qx_2x_3 + \cdots + q^{n-1}x_nx_{n+1} + \cdots = \frac{\sqrt{\alpha}(1-\alpha)}{1-q\alpha}.$$

Hence,

$$w(T) \geq \sup_{0<\alpha<1} \frac{\sqrt{\alpha}(1-\alpha)}{1-q\alpha} = \frac{1}{4q^{3/2}}\sqrt{54 - 36q - 2q^2 - 2\sqrt{(1-q)(9-q)^3}},$$

and the proof is complete. □

Suppose that q is a positive real number with $0 < q < 1$. We consider a weighted shift operator with geometric weights, $a_n = q^{n-1}$ $(n = 1, 2, \ldots)$. In this case, we denote by $F_q(z)$ the function $F_T(z)$ in (1.2).

Theorem 2.2. *Let T be a weighted shift operator with geometric weights $\{q^{n-1}$, $n \in \mathbf{N}\}$, $0 < q < 1$. Then $W(T)$ is a closed disk about the origin, and*

$$F_q(z) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n(n-1)}}{(1-q^2)(1-q^4)(1-q^6)\cdots(1-q^{2n})} z^{2n}. \tag{2.1}$$

Proof. By using the geometric series formula

$$\sum_{n=0} q^n = \frac{1}{1-q},$$

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we prove formula (2.1) by induction on the coefficients c_n of (1.3). It is obvious that

$$c_1 = \sum_{i=1}^{\infty} a_i^2 = \sum_{i=0}^{\infty} q^{2i} = \frac{1}{1-q^2}.$$

We assume that $c_1, c_2, \ldots, c_n$ are of the desired form for the formula (2.1). Then

$$\begin{aligned} c_{n+1} &= 1 \times \sum_{2\le j_1, j_1+2\le j_2, \ldots, j_{n-1}+2\le j_n} q^{2j_1} q^{2j_2} \cdots q^{2j_n} \\ &\quad + q^2 \sum_{3\le j_1, j_1+2\le j_2, \ldots, j_{n-1}+2\le j_n} q^{2j_1} q^{2j_2} \cdots q^{2j_n} + \cdots \\ &= 1 \times q^{4n}(1 \times q^4 q^8 \cdots q^{4(n-1)} + \cdots) \\ &\quad + q^2 q^{6n}(1 \times q^4 q^8 \cdots q^{4(n-1)} + \cdots) + \cdots \\ &= (q^{4n} + q^{6n+2} + q^{8n+4} + \cdots) c_n \\ &= \frac{q^{4n} q^{2n(n-1)}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})(1-q^{2n+2})} \\ &= \frac{q^{2n(n+1)}}{(1-q^2)(1-q^4)\cdots(1-q^{2(n+1)})}. \quad \square \end{aligned}$$

We give an example for $q = 0.2$. In this case, the upper and lower bounds of Theorem 2.1 are estimated by

$$0.414 \approx \frac{1}{4q^{3/2}} \sqrt{54 - 36q - 2q^2 - 2\sqrt{(1-q)(9-q)^3}} \le w(T) \le 1/(2-\sqrt{q}) \approx 0.644.$$

On the other hand, the minimum positive root of $F_{0.2}(z)$ in (2.1) is estimated by 0.980552. Thus, an approximate value of the maximum spectrum of $T + T^*$ is given by $1.01983 = 1/0.980552$. Therefore, $w(T) \approx 1.01983/2 = 0.50991$.

We consider an approximating sequence $F_n(z:q)$ of $F_q(z)$ given by

$$\begin{aligned} F_1(z:q) &= 1 - \frac{1}{(1-q^2)} z^2, \\ F_2(z:q) &= 1 - \frac{1}{(1-q^2)} z^2 + \frac{q^4}{(1-q^2)(1-q^4)} z^4, \\ F_3(z:q) &= 1 - \frac{1}{(1-q^2)} z^2 + \frac{q^4}{(1-q^2)(1-q^4)} z^4 - \frac{q^{12}}{(1-q^2)(1-q^4)(1-q^6)} z^6, \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{aligned}$$

The positive solution of $F_1(z:q) = 0$ in z satisfies

$$\frac{1}{z} = \frac{1}{\sqrt{1-q^2}} = 1 + \frac{q^2}{2} + \frac{3q^4}{8} + \cdots .$$

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The minimum positive solution of $F_2(z:q)=0$ in z satisfies

$$\begin{aligned}\frac{1}{z} &= \frac{\sqrt{2}q^2}{\sqrt{1-q^4-(1-6q^4+4q^6+5q^8-4q^{10})^{1/2}}}\\ &= 1+\frac{q^2}{2}-\frac{q^4}{8}+\cdots .\end{aligned}$$

The minimum positive root of $F_q(z)$ is assumed to be the limit of the minimum positive solutions of $F_n(z:q)=0$. By successive usage of indefinite coefficients method, we find that

$$\frac{1}{z_0} = 1+\frac{q^2}{2}-\frac{q^4}{8}+\frac{9q^6}{16}-\frac{101q^8}{128}+\frac{375q^{10}}{256}-\frac{2549q^{12}}{1024}+\frac{9977q^{14}}{2048}-\cdots . \tag{2.2}$$

Notice that the numerical radius $1/(2z_0)$ of (2.2) is a sharper estimate than the bound obtained in Theorem 2.1 near $q=0$. Indeed, we have the series for the bound

$$1/(2-\sqrt{q}) = \frac{1}{2}\left(1+\frac{q^{1/2}}{2}+\frac{q}{4}+\frac{q^{3/2}}{8}+\frac{q^2}{16}+\cdots\right).$$

3. Hypergeometric q-series. There is an interesting phenomenon in the hypergeometric function (2.1). By replacing z^2 by z and q^2 by q in (2.1), we set

$$H_q(z) = 1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n-1)}}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^n)}z^n. \tag{3.1}$$

The minimum positive solution of the equation $H_q(z)=0$ can be found in a series of q,

$$z = 1-q+q^2-2q^3+4q^4-8q^5+16q^6-33q^7+70q^8-\cdots .$$

An analogous function of $H_q(z)$ is known as the Euler equation. Recall the q-series identity

$$\prod_{n=1}^{\infty}(1+q^n x) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}x^n. \tag{3.2}$$

The series (3.2) is called the Euler function in q when $x=-1$. If the term $q^{n(n+1)/2}$ in (3.2) is replaced by $q^{n(n+1)}$, then it is related to the function (3.1) corresponding to the minimum positive eigenvalue.

The function $F_q(z)$ is closely related to Rogers-Ramanujan identities. Substituting $z=iq$ into $F_q(z)$, we have

$$F_q(iq) = 1+\sum_{n=1}^{\infty}\frac{q^{2n^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}. \tag{3.3}$$

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Substituting $z = iq^2$ into $F_q(z)$, we have

$$F_q(iq^2) = 1 + \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}. \tag{3.4}$$

Replacing q^2 by q in (3.3) and (3.4), we have respectively, in basic hypergeometric q-series (cf. [2]), the following two Rogers-Ramanujan identities

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

and

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

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