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SPECTRA OF WEIGHTED COMPOUND GRAPHS OF GENERALIZED BETHE TREES*

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Abstract. A generalized Bethe tree is a rooted tree in which vertices at the same distance from the root have the same degree. Let $\mathcal{G}_m$ be a connected weighted graph on m vertices. Let $\{\mathcal{B}_i : 1 \leq i \leq m\}$ be a set of trees such that, for $i = 1, 2, \ldots, m$,

(i) $\mathcal{B}_i$ is a generalized Bethe tree of k_i levels,

(ii) the vertices of $\mathcal{B}_i$ at the level j have degree d_{i,k_i-j+1} for $j = 1, 2, \ldots, k_i$, and

(iii) the edges of $\mathcal{B}_i$ joining the vertices at the level j with the vertices at the level $(j+1)$ have weight w_{i,k_i-j} for $j = 1, 2, \ldots, k_i - 1$.

Let $\mathcal{G}_m \{\mathcal{B}_i : 1 \leq i \leq m\}$ be the graph obtained from $\mathcal{G}_m$ and the trees $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$ by identifying the root vertex of $\mathcal{B}_i$ with the ith vertex of $\mathcal{G}_m$. A complete characterization is given of the eigenvalues of the Laplacian and adjacency matrices of $\mathcal{G}_m \{\mathcal{B}_i : 1 \leq i \leq m\}$ together with results about their multiplicities. Finally, these results are applied to the particular case $\mathcal{B}_1 = \mathcal{B}_2 = \cdots = \mathcal{B}_m$.

Key words. Weighted graph, Generalized Bethe tree, Laplacian matrix, Adjacency matrix, Spectral radius, Algebraic connectivity.

AMS subject classifications. 5C50, 15A48.

1. Introduction.

Let $\mathcal{G} = (V, E)$ be a simple graph with vertex set V and edge set E. A weighted graph $\mathcal{G}$ is a graph in which each edge $e \in E$ has a positive weight $w(e)$. Labelling the vertices of $\mathcal{G}$ by $1, 2, \ldots, n$, the Laplacian matrix of $\mathcal{G}$ is the $n \times n$ matrix $L(\mathcal{G}) = (l_{i,j})$ defined by

$$l_{i,j} = \begin{cases} -w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j \\ -\sum_{k \neq i} l_{i,k} & \text{if } i = j \end{cases}$$

and the adjacency matrix of $\mathcal{G}$ is the $n \times n$ matrix $A(\mathcal{G}) = (a_{i,j})$ defined by

$$a_{i,j} = \begin{cases} w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j \\ 0 & \text{if } i = j \end{cases}.$$

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$L(\mathcal{G})$ and $A(\mathcal{G})$ are both real symmetric matrices. From this fact and Geršgorin's Theorem, it follows that the eigenvalues of $L(\mathcal{G})$ are nonnegative real numbers. Moreover, since its rows sum to 0, $(0, \mathbf{e})$ is an eigenpair of $L(\mathcal{G})$, where $\mathbf{e}$ is the all ones vector. Fiedler [1] proved that $\mathcal{G}$ is a connected graph if and only if the second smallest eigenvalue of $L(\mathcal{G})$ is positive. This eigenvalue is called the algebraic connectivity of $\mathcal{G}$.

If $w(e) = 1$ for all $e \in E$, then $\mathcal{G}$ is an unweighted graph. In [4], some of the many results known for the Laplacian matrix of an unweighted graph are given.

A generalized Bethe tree is a rooted tree in which vertices at the same distance from the root have the same degree. In [5], one can find a complete characterization of the spectra of the Laplacian matrix and adjacency matrix of such class of trees.

Let $\{\mathcal{B}_i : 1 \leq i \leq m\}$ be a set of trees such that

(i) $\mathcal{B}_i$ is a generalized Bethe tree of k_i levels,

(ii) the vertices of $\mathcal{B}_i$ at the level j have degree d_{i,k_i-j+1} for $j = 1, 2, \ldots, k_i$, and

(iii) the edges of $\mathcal{B}_i$ joining the vertices at the level j with the vertices at the level $(j+1)$ have weight w_{i,k_i-j} for $j = 1, 2, \ldots, k_i - 1$.

Let $\mathcal{G}_m$ be a connected weighted graph on m vertices $v_1, v_2, \ldots, v_m$. As usual $v_i \sim v_j$ means that v_i and v_j are adjacent. Let $\varepsilon_{i,j} = \varepsilon_{j,i}$ be the weight of the edge $v_i v_j$ if $v_i \sim v_j$ and let $\varepsilon_{i,j} = \varepsilon_{j,i} = 0$ otherwise.

In this paper, we characterize completely the spectra of the Laplacian and adjacency matrices of the graph $\mathcal{G}_m\{\mathcal{B}_i : 1 \leq i \leq m\}$ obtained from $\mathcal{G}_m$ and the trees $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$ by identifying the root vertex of $\mathcal{B}_i$ with v_i. In particular, we apply the results to the case $\mathcal{B}_1 = \mathcal{B}_2 = \cdots = \mathcal{B}_m$.

¿From now on, we write $\mathcal{G}_m\{\mathcal{B}_i\}$ instead of $\mathcal{G}_m\{\mathcal{B}_i : 1 \leq i \leq m\}$.

For $j = 1, 2, 3, \ldots, k_i$, let n_{i,k_i-j+1} be the number of vertices at the level j of $\mathcal{B}_i$. Observe that $n_{i,k_i} = 1$ and $n_{i,k_i-1} = d_{i,k_i}$. We have

$$n_{i,k_i-j} = (d_{i,k_i-j+1} - 1)\, n_{i,k_i-j+1}, \quad 2 \leq j \leq k_i - 1. \tag{1.1}$$

For $i = 1, 2, \ldots, m$, let $d(v_i)$ be the degree of v_i as a vertex of $\mathcal{G}_m$. The total number of vertices in $\mathcal{G}_m\{\mathcal{B}_i\}$ is $n = \sum_{i=1}^{m} \sum_{j=1}^{k_i-1} n_{i,j} + m$.

For $i = 1, 2, \ldots, m$, let

$$\delta_{i,1} = w_{i,1}, \quad \delta_{i,j} = (d_{i,j} - 1)\, w_{i,j-1} + w_{i,j} \ \ (j = 2, 3, \ldots, k_i - 1)$$

$$\delta_{i,k_i} = d_{i,k_i} w_{i,k_i-1} \quad \text{and} \quad \delta_i = \sum_{v_i \sim v_j} \varepsilon_{i,j}.$$

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Observe that if $\mathcal{B}_i$ is an unweighted tree, then $\delta_{i,j} = d_{i,j}$, and if $\mathcal{G}_m$ is an unweighted graph, then $\delta_i = d(v_i)$.

We introduce the following additional notation:

$|A|$ is the determinant of A.

0 and I are the all zeros matrix and the identity matrix of appropriate sizes, respectively.

I_r is the identity matrix of size $r \times r$.

$\mathbf{e}_r$ is the all ones column vector of dimension r.

For $1 \leq i \leq m$ and $1 \leq j \leq k_i - 2$, $m_{i,j} = \frac{n_{i,j}}{n_{i,j+1}}$ and $C_{i,j}$ is the block diagonal matrix defined by

$$C_{i,j} = diag\left\{\mathbf{e}_{m_{i,j}}, \mathbf{e}_{m_{i,j}}, \ldots, \mathbf{e}_{m_{i,j}}\right\}$$

with $n_{i,j+1}$ diagonal blocks. The size of $C_{i,j}$ is $n_{i,j} \times n_{i,j+1}$.

For $1 \leq i \leq m$, let $s_i = \sum_{j=1}^{k_i-2} n_{i,j}$ and E_i be the $s_i \times m$ matrix defined by

$$E_i(p,q) = \begin{cases} 1 & \text{if } q = i \text{ and } s_i + 1 \leq p \leq s_i + n_{i,k_i-1} \\ 0 & \text{elsewhere} \end{cases}.$$

We label the vertices of $\mathcal{G}_m\{\mathcal{B}_i\}$ as follows:

1. Using the labels $1, 2, \ldots, \sum_{j=1}^{k_1-1} n_{1,j}$, we label the vertices of $\mathcal{B}_1$ from the bottom to level 2 and, at each level, in a counterwise sense.

2. Using the labels $\sum_{j=1}^{k_1-1} n_{1,j} + 1, \ldots, \sum_{j=1}^{k_1-1} n_{1,j} + \sum_{j=1}^{k_2-1} n_{2,j}$, we label the vertices of $\mathcal{B}_2$ from the bottom to level 2 and, at each level, in a counterwise sense.

3. We continue labelling the vertices of $\mathcal{B}_3, \mathcal{B}_4, \ldots, \mathcal{B}_m$, in this order, as above.

4. Finally, using the labels $n - m + 1, n - m + 2, \ldots, n$, we label the vertices of $\mathcal{G}_m$.

Thus, the adjacency matrix $A(\mathcal{G}_m\{\mathcal{B}_i\})$ and the matrix $L(\mathcal{G}_m\{\mathcal{B}_i\})$ become

$$(1.2)\quad A(\mathcal{G}_m\{\mathcal{B}_i\}) = \begin{bmatrix} A_1 & 0 & \cdots & 0 & w_{1,k_1-1}E_1 \\ 0 & A_2 & \cdots & 0 & w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m & w_{m,k_m-1}E_m \\ w_{1,k_1-1}E_1^T & w_{2,k_2-1}E_2^T & \cdots & w_{m,k_m-1}E_m^T & A(\mathcal{G}_m) \end{bmatrix}$$

and

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$$L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right) = \begin{bmatrix} L_1 & 0 & \cdots & 0 & -w_{1,k_1-1}E_1 \\ 0 & L_2 & \cdots & 0 & -w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & L_m & -w_{m,k_m-1}E_m \\ -w_{1,k_1-1}E_1^T & -w_{2,k_2-1}E_2^T & \cdots & -w_{m,k_m-1}E_m^T & L_{m+1} \end{bmatrix} \tag{1.3}$$

where, for $i = 1, 2, \ldots, m$, A_i and L_i are the following block tridiagonal matrices:

$$A_i = \begin{bmatrix} 0 & w_{i,1}C_{i,1} & & & \\ w_{i,1}C_{i,1}^T & 0 & w_{i,2}C_{i,2} & & \\ & w_{i,2}C_{i,2}^T & \ddots & \ddots & \\ & & \ddots & 0 & w_{i,k_i-2}C_{i,k_i-2} \\ & & & w_{i,k_i-2}C_{i,k_i-2}^T & 0 \end{bmatrix} \tag{1.4}$$

and

$$L_i = \begin{bmatrix} \delta_{i,1}I_{n_{i,1}} & -w_{i,1}C_{i,1} & & & \\ -w_{i,1}C_{i,1}^T & \delta_{i,2}I_{n_{i,2}} & -w_{i,2}C_{i,2} & & \\ & -w_{i,2}C_{i,2}^T & \ddots & \ddots & \\ & & \ddots & \delta_{i,k_i-2}I_{i,n_{k_i-2}} & -w_{i,k_i-2}C_{i,k_i-2} \\ & & & -w_{i,k_i-2}C_{i,k_i-2}^T & \delta_{i,k_i-1}I_{i,n_{k_i-1}} \end{bmatrix}. \tag{1.5}$$

Moreover,

$$A\left(\mathcal{G}_m\right) = \begin{bmatrix} 0 & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & 0 & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & 0 & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & 0 \end{bmatrix} \tag{1.6}$$

and

$$L_{m+1} = \begin{bmatrix} \delta_{1,k_1}+\delta_1 & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \cdots & -\varepsilon_{1,m} \\ -\varepsilon_{1,2} & \delta_{2,k_2}+\delta_2 & -\varepsilon_{2,3} & \cdots & -\varepsilon_{2,m} \\ -\varepsilon_{1,3} & -\varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \delta_{m-1,k_{m-1}}+\delta_{m-1} & -\varepsilon_{m-1,m} \\ -\varepsilon_{1,m} & -\varepsilon_{2,m} & \cdots & -\varepsilon_{m-1,m} & \delta_{m,k_m}+\delta_m \end{bmatrix}. \tag{1.7}$$

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2. Preliminaries.

Lemma 2.1. *Let*

$$X = \begin{bmatrix} X_1 & 0 & \cdots & 0 & \pm w_{1,k_1-1}E_1 \\ 0 & X_2 & \cdots & 0 & \pm w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X_m & \pm w_{m,k_m-1}E_m \\ \pm w_{1,k_1-1}E_1^T & \pm w_{2,k_2-1}E_2^T & \cdots & \pm w_{m,k_m-1}E_m^T & X_{m+1} \end{bmatrix}$$

where, for $i = 1, 2, \ldots, m$, X_i *is the block tridiagonal matrix*

$$X_i = \begin{bmatrix} \alpha_{i,1}I_{n_{i,1}} & \pm w_{i,1}C_{i,1} & & & \\ \pm w_{i,1}C_{i,1}^T & \alpha_{i,2}I_{n_{i,2}} & \pm w_{i,2}C_{i,2} & & \\ & \pm w_{i,2}C_{i,2}^T & \ddots & \ddots & \\ & & \ddots & \alpha_{i,k_i-2}I_{i,n_{k_i-2}} & \pm w_{i,k_i-2}C_{i,k_i-2} \\ & & & \pm w_{i,k_i-2}C_{i,k_i-2}^T & \alpha_{i,k_i-1}I_{i,n_{k_i-1}} \end{bmatrix},$$

and

$$X_{m+1} = \begin{bmatrix} \alpha_1 & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & \alpha_2 & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \alpha_{m-1} & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & \alpha_m \end{bmatrix} \tag{2.1}$$

or

$$X_{m+1} = \begin{bmatrix} \alpha_1 & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \cdots & -\varepsilon_{1,m} \\ -\varepsilon_{1,2} & \alpha_2 & -\varepsilon_{2,3} & \cdots & -\varepsilon_{2,m} \\ -\varepsilon_{1,3} & -\varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \alpha_{m-1} & -\varepsilon_{m-1,m} \\ -\varepsilon_{1,m} & -\varepsilon_{2,m} & \cdots & -\varepsilon_{m-1,m} & \alpha_m \end{bmatrix}, \tag{2.2}$$

respectively. For $i = 1, 2, \ldots, m$, *let*

$$\begin{aligned} \beta_{i,1} &= \alpha_{i,1}, \\ \beta_{i,j} &= \alpha_{i,j} - \frac{n_{i,j-1}}{n_{i,j}}\frac{w_{i,j-1}^2}{\beta_{i,j-1}}, \quad j = 2, 3, \ldots, k_i - 1, \\ \beta_i &= \alpha_i - n_{i,k_i-1}\frac{w_{i,k_i-1}^2}{\beta_{i,k_i-1}}. \end{aligned}$$

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If $\beta_{i,j} \neq 0$ *for all* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, k_i - 1$*, then*

$$|X| = \left(\prod_{i=1}^{m} \prod_{j=1}^{k_i-1} \beta_{i,j}^{n_{i,j}} \right) |Y_{m+1}| , \tag{2.3}$$

where

$$Y_{m+1} = \begin{bmatrix} \beta_1 & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & \beta_2 & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \beta_{m-1} & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & \beta_m \end{bmatrix} \tag{2.4}$$

or

$$Y_{m+1} = \begin{bmatrix} \beta_1 & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \cdots & -\varepsilon_{1,m} \\ -\varepsilon_{1,2} & \beta_2 & -\varepsilon_{2,3} & \cdots & -\varepsilon_{2,m} \\ -\varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \beta_{m-1} & -\varepsilon_{m-1,m} \\ -\varepsilon_{1,m} & -\varepsilon_{2,m} & \cdots & -\varepsilon_{m-1,m} & \beta_m \end{bmatrix}, \tag{2.5}$$

respectively.

Proof. We give a proof for X_{m+1} in (2.1). Suppose $\beta_{1,j} \neq 0$ for all $j = 1, 2, \ldots, k_1 - 1$. After some steps of the Gaussian elimination procedure, without row interchanges, we reduce X to the intermediate matrix

$$\begin{bmatrix} R_1 & 0 & \cdots & 0 & \pm w_{1,k_1-1} E_1 \\ 0 & X_2 & \cdots & 0 & \pm w_{2,k_2-1} E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X_m & \pm w_{m,k_m-1} E_m \\ 0 & \pm w_{2,k_2-1} E_2^T & \cdots & \pm w_{m,k_m-1} E_m^T & X_{m+1}^{(1)} \end{bmatrix},$$

where R_1 is the block bidiagonal matrix

$$R_1 = \begin{bmatrix} \beta_{1,1} I_{n_{1,1}} & \pm w_{1,1} C_{1,1} & & \\ & \beta_{1,2} I_{n_{1,2}} & \ddots & \\ & & \ddots & \pm w_{1,k_1-2} C_{1,k_1-2} \\ & & & \beta_{1,k_1-1} I_{n_{1,k_1-1}} \end{bmatrix}$$

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and $X_{m+1}^{(1)}$ is the matrix

$$X_{m+1}^{(1)} = \begin{bmatrix} \beta_1 & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & \alpha_2 & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \alpha_{m-1} & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & \alpha_m \end{bmatrix}.$$

Suppose, in addition, that $\beta_{i,j} \neq 0$ for all $i = 2,3,\ldots,m$ and $j = 1,2,\ldots,k_i - 1$. We continue the Gaussian elimination procedure to finally obtain the upper triangular matrix

$$\begin{bmatrix} R_1 & 0 & \cdots & 0 & \pm w_{1,k_1-1}E_1 \\ 0 & R_2 & \cdots & 0 & \pm w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & R_m & \pm w_{m,k_m-1}E_m \\ 0 & 0 & \cdots & 0 & Y_{m+1} \end{bmatrix},$$

where, for $i = 2,3,\ldots,m$,

$$R_i = \begin{bmatrix} \beta_{i,1}I_{n_{i,1}} & \pm w_{i,1}C_{i,1} & & \\ & \beta_{i,2}I_{n_{i,2}} & \ddots & \\ & & \ddots & \pm w_{i,k_i-2}C_{i,k_i-2} \\ & & & \beta_{i,k_i-1}I_{i,n_{k_i-1}} \end{bmatrix}$$

and Y_{m+1} is as in (2.4). A similar proof yields to Y_{m+1} in (2.5) whenever X_{m+1} is in form (2.2). Thus, (2.3) is proved. $\square$

¿From now on, we denote by $\widetilde{A}$ the submatrix obtained from A by deleting its last row and its last column. Moreover, for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,m$, let $F_{i,j}$ be the $k_i \times k_j$ matrix with $F_{i,j}(k_i, k_j) = 1$ and zeroes elsewhere.

In the proof of the following lemma, we will use the facts $\begin{vmatrix} A & 0 \\ C & B \end{vmatrix} = \begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = |A|\,|B|$ and $\begin{vmatrix} A & 0 & B \\ C & D & E \\ F & 0 & G \end{vmatrix} = \begin{vmatrix} A & B \\ F & G \end{vmatrix} |D| = \begin{vmatrix} A & C & B \\ 0 & D & 0 \\ F & E & G \end{vmatrix}$ for matrices of appropriate sizes.

Lemma 2.2. *For $i = 1,2,\ldots,m$, let B_i be a matrix of size $k_i \times k_i$ and $\mu_{i,j}$ be*

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arbitrary scalars. Then it holds that

$$
\begin{vmatrix}
B_1 & \mu_{1,2}F_{1,2} & \cdots & \mu_{1,m-1}F_{1,m-1} & \mu_{1,m}F_{1,m} \\
\mu_{2,1}F_{1,2}^T & B_2 & \cdots & \cdots & \mu_{2,m}F_{2,m} \\
\mu_{3,1}F_{1,3}^T & \mu_{3,2}F_{2,3}^T & \ddots & & \vdots \\
\vdots & \vdots & & B_{m-1} & \mu_{m-1,m}F_{m-1,m} \\
\mu_{m,1}F_{1,m}^T & \mu_{m,2}F_{2,m}^T & \cdots & \mu_{m,m-1}F_{m-1,m}^T & B_m
\end{vmatrix} \tag{2.6}
$$

$$
= \begin{vmatrix}
|B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| & \mu_{1,m}\left|\widetilde{B_m}\right| \\
\mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| & \cdots & \cdots & \mu_{2,m}\left|\widetilde{B_m}\right| \\
\mu_{3,1}\left|\widetilde{B_1}\right| & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & & \vdots \\
\vdots & \vdots & & |B_{m-1}| & \mu_{m-1,m}\left|\widetilde{B_m}\right| \\
\mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| & |B_m|
\end{vmatrix}.
$$

Proof. Let us write $B_i = \begin{bmatrix} \widetilde{B_i} & \mathbf{b}_{i,1} \\ \mathbf{b}_{i,2}^T & b_i \end{bmatrix}$. We use induction on m. For $m = 2$, we have

$$
\begin{aligned}
\begin{vmatrix} B_1 & \mu_{1,2}F_{1,2} \\ \mu_{2,1}F_{1,2}^T & B_2 \end{vmatrix}
&= \begin{vmatrix}
\widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 \\
\mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} \\
0 & 0 & \widetilde{B_2} & \mathbf{b}_{2,1} \\
0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2
\end{vmatrix} \\
&= \begin{vmatrix}
\widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 \\
\mathbf{b}_{1,2}^T & b_1 & 0 & 0 \\
0 & 0 & \widetilde{B_2} & \mathbf{b}_{2,1} \\
0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2
\end{vmatrix}
+ \begin{vmatrix}
\widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 \\
\mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} \\
0 & 0 & \widetilde{B_2} & 0 \\
0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & 0
\end{vmatrix} \\
&= \begin{vmatrix} \widetilde{B_1} & \mathbf{b}_{1,1} \\ \mathbf{b}_{1,2}^T & b_1 \end{vmatrix}
\begin{vmatrix} \widetilde{B_2} & \mathbf{b}_{2,1} \\ \mathbf{b}_{2,2}^T & b_2 \end{vmatrix}
+ \begin{vmatrix}
\widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 \\
\mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} \\
0 & 0 & \widetilde{B_2} & 0 \\
0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & 0
\end{vmatrix} \\
&= \begin{vmatrix} |B_1| & 0 \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| \end{vmatrix}
+ \left|\widetilde{B_2}\right|
\begin{vmatrix}
\widetilde{B_1} & \mathbf{b}_{1,1} & 0 \\
\mathbf{b}_{1,2}^T & b_1 & \mu_{1,2} \\
0 & \mu_{2,1} & 0
\end{vmatrix} \\
&= \begin{vmatrix} |B_1| & 0 \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| \end{vmatrix}
- \left|\widetilde{B_2}\right| \mu_{1,2}
\begin{vmatrix} \widetilde{B_1} & \mathbf{b}_{1,1} \\ 0 & \mu_{2,1} \end{vmatrix}.
\end{aligned}
$$

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Clearly,

$$\begin{vmatrix} 0 & \mu_{1,2}\left|\widetilde{B_2}\right| \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| \end{vmatrix} = -\left|\widetilde{B_2}\right|\mu_{1,2}\begin{vmatrix} \widetilde{B_1} & \mathbf{b}_{1,1} \\ 0 & \mu_{2,1} \end{vmatrix}.$$

Thus,

$$\begin{aligned} \begin{vmatrix} B_1 & \mu_{1,2}F_{1,2} \\ \mu_{2,1}F_{1,2}^T & B_2 \end{vmatrix} &= \begin{vmatrix} |B_1| & 0 \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| \end{vmatrix} + \begin{vmatrix} 0 & \mu_{1,2}\left|\widetilde{B_2}\right| \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| \end{vmatrix} \\ &= \begin{vmatrix} |B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| \end{vmatrix}. \end{aligned}$$

We have proved (2.6) for $m = 2$. Let $m \geq 3$, and assume that (2.6) is true for $m-1$. The meaning of the scalars t, t_1 and t_2 below will be clear from the context. By linearity on the last column,

$$t := \begin{vmatrix} \widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} & \cdots & 0 & \mu_{1,m-1} & 0 & \mu_{1,m} \\ 0 & 0 & \widetilde{B_2} & \mathbf{b}_{2,1} & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & \cdots & \cdots & 0 & \mu_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & & \widetilde{B_{m-1}} & \mathbf{b}_{m-1,1} & 0 & 0 \\ 0 & \mu_{m-1,1} & \vdots & \vdots & & \mathbf{b}_{m-1,2}^T & b_{m-1} & 0 & \mu_{m-1,m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \widetilde{B_m} & \mathbf{b}_{m,1} \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} & \mathbf{b}_{m,2}^T & b_m \end{vmatrix}$$

$$= \begin{vmatrix} \widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} & \cdots & 0 & \mu_{1,m-1} & 0 & 0 \\ 0 & 0 & \widetilde{B_1} & \mathbf{b}_{2,1} & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & & \widetilde{B_{m-1}} & \mathbf{b}_{m-1,1} & 0 & 0 \\ 0 & \mu_{m-1,1} & \vdots & \vdots & & \mathbf{b}_{m-1,2}^T & b_{m-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \widetilde{B_m} & \mathbf{b}_{m,1} \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} & \mathbf{b}_{m,2}^T & b_m \end{vmatrix}$$

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$$+\begin{vmatrix} \widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} & \cdots & 0 & \mu_{1,m-1} & 0 & \mu_{1,m} \\ 0 & 0 & \widetilde{B_2} & \mathbf{b}_{2,1} & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & \cdots & \cdots & 0 & \mu_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & & \widetilde{B_{m-1}} & \mathbf{b}_{m-1,1} & 0 & 0 \\ 0 & \mu_{m-1,1} & \vdots & \vdots & & \mathbf{b}_{m-1,2}^T & b_{m-1} & 0 & \mu_{m-1,m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \widetilde{B_m} & 0 \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} & \mathbf{b}_{m,2}^T & 0 \end{vmatrix} := t_1 + t_2.$$

By definition,

$$t_1 = \begin{vmatrix} B_1 & \mu_{1,2}F_{1,2} & \cdots & \cdots & \mu_{1,m-1}F_{1,m-1} & 0 \\ \mu_{2,1}F_{1,2}^T & B_2 & \mu_{2,3}F_{2,3} & \cdots & \mu_{2,m-1}F_{2,m-1} & 0 \\ \vdots & \mu_{3,2}F_{2,3}^T & \ddots & & \vdots & \vdots \\ \mu_{m-2,1}F_{1,m-2}^T & \vdots & & \ddots & \mu_{m-2,m-1}F_{m-2,m-1} & \vdots \\ \mu_{m-1,1}F_{1,m-1}^T & \mu_{m-1,2}F_{2,m-1}^T & \cdots & \cdots & B_{m-1} & 0 \\ \mu_{m,1}F_{1,m}^T & \mu_{m,2}F_{2,m}^T & \cdots & \cdots & \mu_{m,m-1}F_{m-1,m}^T & B_m \end{vmatrix},$$

and hence,

$$t_1 = \begin{vmatrix} B_1 & \mu_{1,2}F_{1,2} & \cdots & \cdots & \mu_{1,m-1}F_{1,m-1} \\ \mu_{2,1}F_{1,2}^T & B_2 & \mu_{2,3}F_{2,3} & \cdots & \mu_{2,m-1}F_{2,m-1} \\ \vdots & \mu_{3,2}F_{2,3}^T & \ddots & & \vdots \\ \mu_{m-2,1}F_{1,m-2}^T & \vdots & & \ddots & \mu_{m-2,m-1}F_{m-2,m-1} \\ \mu_{m-1,1}F_{1,m-1}^T & \mu_{m-1,2}F_{2,m-1}^T & \cdots & \cdots & B_{m-1} \end{vmatrix} |B_m|.$$

We apply the induction hypothesis on the first factor obtaining that

$$t_1 = \begin{vmatrix} |B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| & \mu_{2,3}\left|\widetilde{B_3}\right| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| \\ \vdots & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & & \vdots \\ \mu_{m-2,1}\left|\widetilde{B_1}\right| & \vdots & & \ddots & \mu_{m-2,m-1}\left|\widetilde{B_{m-1}}\right| \\ \mu_{m-1,1}\left|\widetilde{B_1}\right| & \mu_{m-1,2}\left|\widetilde{B_2}\right| & \cdots & \cdots & |B_{m-1}| \end{vmatrix} |B_m|.$$

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Now it is easy to see that

$$
(2.7)\ t_1 = \begin{vmatrix}
|B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \\
\mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| & \mu_{2,3}\left|\widetilde{B_3}\right| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \\
\vdots & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & & \vdots & \vdots \\
\mu_{m-2,1}\left|\widetilde{B_1}\right| & \vdots & & \ddots & \mu_{m-2,m-1}\left|\widetilde{B_{m-1}}\right| & \vdots \\
\mu_{m-1,1}\left|\widetilde{B_1}\right| & \mu_{m-1,2}\left|\widetilde{B_2}\right| & \cdots & \cdots & |B_{m-1}| & 0 \\
\mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| & |B_m|
\end{vmatrix}.
$$

We have

$$
t_2 = \begin{vmatrix}
\widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
\mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} & \cdots & 0 & \mu_{1,m-1} & \mu_{1,m} \\
0 & 0 & \widetilde{B_2} & \mathbf{b}_{2,1} & \cdots & \cdots & \cdots & 0 \\
0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & \cdots & \cdots & \mu_{2,m} \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots \\
0 & 0 & \vdots & \vdots & & \widetilde{B_{m-1}} & \mathbf{b}_{m-1,1} & 0 \\
0 & \mu_{m-1,1} & \vdots & \vdots & & \mathbf{b}_{m-1,2}^T & b_{m-1} & \mu_{m-1,m} \\
0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} & 0
\end{vmatrix} \left|\widetilde{B_m}\right|.
$$

Expanding along the last column, we get

$$
t_2 = (s_1 + s_2 + \cdots + s_{m-1}) \left|\widetilde{B_m}\right|
$$

where, for $i = 1, 2, \ldots, m-1$, the summand s_i is the cofactor of the entry $\mu_{i,m}$. In particular,

$$
s_1 = (-1)^{k_2+\cdots+k_{m-1}+1}\,\mu_{1,m} \begin{vmatrix}
\widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \widetilde{B_2} & \mathbf{b}_{2,1} & \cdots & 0 & 0 \\
0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & 0 & \mu_{2,m-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \vdots & \vdots & & \widetilde{B_{m-1}} & \mathbf{b}_{m-1,1} \\
0 & \mu_{m-1,1} & \vdots & \vdots & & \mathbf{b}_{m-1,2}^T & b_{m-1} \\
0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1}
\end{vmatrix}.
$$

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After $k_{m-1} + k_{m-2} + \cdots + k_2$ row interchanges, we obtain

$$s_1 = -\mu_{1,m} \begin{vmatrix} \widetilde{B_1} & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} \\ 0 & 0 & \widetilde{B_2} & \mathbf{b}_{2,1} & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & 0 & \mu_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \widetilde{B_{m-1}} & \mathbf{b}_{m-1,1} \\ 0 & \mu_{m-1,1} & 0 & \mu_{m-1,2} & \cdots & \mathbf{b}_{m-1,2}^T & b_{m-1} \end{vmatrix}.$$

We apply the induction hypothesis to get that

$$s_1 = -\mu_{1,m} \begin{vmatrix} \mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| \\ \mu_{2,1}\left|\widetilde{B_1}\right| & \left|B_2\right| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| \\ \vdots & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & \vdots \\ \vdots & \vdots & & \mu_{m-1,m-1}\left|\widetilde{B_{m-1}}\right| \\ \mu_{m-1,1}\left|\widetilde{B_1}\right| & \mu_{m-1,2}\left|\widetilde{B_2}\right| & \cdots & \left|B_{m-1}\right| \end{vmatrix}.$$

After $m-2$ row interchanges, we have

$$s_1 = (-1)^{m-1}\mu_{1,m} \begin{vmatrix} \mu_{2,1}\left|\widetilde{B_1}\right| & \left|B_2\right| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| \\ \mu_{3,1}\left|\widetilde{B_1}\right| & \mu_{3,2}\left|\widetilde{B_2}\right| & \cdots & \vdots \\ \vdots & \vdots & \ddots & \mu_{m-1,m-1}\left|\widetilde{B_{m-1}}\right| \\ \mu_{m-1,1}\left|\widetilde{B_1}\right| & \mu_{m-1,2}\left|\widetilde{B_2}\right| & \cdots & \left|B_{m-1}\right| \\ \mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| \end{vmatrix}. \tag{2.8}$$

We may write

$$s_1 = \begin{vmatrix} \left|B_1\right| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| & \mu_{1,m} \\ \mu_{2,1}\left|\widetilde{B_1}\right| & \left|B_2\right| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \\ \mu_{3,1}\left|\widetilde{B_1}\right| & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \left|B_{m-1}\right| & 0 \\ \mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \end{vmatrix}. \tag{2.9}$$

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In fact, expanding (2.9) along the last row we obtain (2.8). Similarly,

$$s_2 = \begin{vmatrix} |B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| & \mu_{2,m} \\ \mu_{3,1}\left|\widetilde{B_1}\right| & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & \vdots & 0 \\ \vdots & \vdots & & |B_{m-1}| & \vdots \\ \mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \end{vmatrix}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$s_{m-1} = \begin{vmatrix} |B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| & \vdots \\ \mu_{3,1}\left|\widetilde{B_1}\right| & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & \vdots & 0 \\ \vdots & \vdots & & |B_{m-1}| & \mu_{m-1,m} \\ \mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \end{vmatrix}.$$

Therefore,

$$t_2 = \begin{vmatrix} |B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| & \mu_{1,m}\left|\widetilde{B_m}\right| \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| & \mu_{2,m}\left|\widetilde{B_m}\right| \\ \mu_{3,1}\left|\widetilde{B_1}\right| & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & \vdots & \vdots \\ \vdots & \vdots & & |B_{m-1}| & \mu_{m-1,m}\left|\widetilde{B_m}\right| \\ \mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| & 0 \end{vmatrix}. \tag{2.10}$$

By (2.7) and (2.10), we have

$$t = t_1 + t_2 = \begin{vmatrix} |B_1| & \mu_{1,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{1,m-1}\left|\widetilde{B_{m-1}}\right| & \mu_{1,m}\left|\widetilde{B_m}\right| \\ \mu_{2,1}\left|\widetilde{B_1}\right| & |B_2| & \cdots & \mu_{2,m-1}\left|\widetilde{B_{m-1}}\right| & \mu_{2,m}\left|\widetilde{B_m}\right| \\ \mu_{3,1}\left|\widetilde{B_1}\right| & \mu_{3,2}\left|\widetilde{B_2}\right| & \ddots & \vdots & \vdots \\ \vdots & \vdots & & |B_{m-1}| & \mu_{m-1,m}\left|\widetilde{B_m}\right| \\ \mu_{m,1}\left|\widetilde{B_1}\right| & \mu_{m,2}\left|\widetilde{B_2}\right| & \cdots & \mu_{m,m-1}\left|\widetilde{B_{m-1}}\right| & |B_m| \end{vmatrix}.$$

This completes the proof. ▯

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3. The spectrum of the Laplacian matrix.

Definition 3.1. For $i = 1, 2, \ldots, m$, let

$$P_{i,0}(\lambda) = 1 \quad \text{and} \quad P_{i,1}(\lambda) = \lambda - \delta_{i,1}$$

and, for $j = 2, 3, \ldots, k_i - 1$,

$$P_{i,j}(\lambda) = (\lambda - \delta_{i,j}) P_{i,j-1}(\lambda) - \frac{n_{i,j-1}}{n_{i,j}} w_{i,j-1}^2 P_{i,j-2}(\lambda). \tag{3.1}$$

Moreover, for $i = 1, 2, \ldots, m$, let

$$P_i(\lambda) = (\lambda - \delta_{i,k_i} - \delta_i) P_{i,k_i-1}(\lambda) - n_{i,k_i-1} w_{i,k_i-1}^2 P_{i,k_i-2}(\lambda)$$

and

$$\Omega_i = \{j : 1 \le j \le k_i - 1 : n_{i,j} > n_{i,j+1}\}.$$

Theorem 3.2. *The following hold:*

(a)

$$|\lambda I - L(\mathcal{G}_m\{\mathcal{B}_i\})| = P(\lambda) \prod_{i=1}^{m} \prod_{j \in \Omega_i} P_{i,j}^{n_{i,j} - n_{i,j+1}}(\lambda), \tag{3.2}$$

where

$$P(\lambda) = \begin{vmatrix} P_1(\lambda) & \varepsilon_{1,2} P_{2,k_2-1}(\lambda) & \cdots & \varepsilon_{1,m} P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,2} P_{1,k_1-1}(\lambda) & P_2(\lambda) & \cdots & \varepsilon_{2,m} P_{m,k_m-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \varepsilon_{m-1,m} P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,m} P_{1,k_1-1}(\lambda) & \varepsilon_{2,m} P_{2,k_2-1}(\lambda) & \cdots & P_m(\lambda) \end{vmatrix}.$$

(b) *The set of eigenvalues of* $L(\mathcal{G}_m\{\mathcal{B}_i\})$ *is*

$$\sigma(L(\mathcal{G}_m\{\mathcal{B}_i\})) = \left(\cup_{i=1}^{m} \cup_{j \in \Omega_i} \{\lambda : P_{i,j}(\lambda) = 0\}\right) \cup \{\lambda : P(\lambda) = 0\}.$$

Proof. (a) $L(\mathcal{G}_m\{\mathcal{B}_i\})$ is given by $(1.3), (1.5)$ and (1.7). We apply Lemma 2.1 to the matrix $X = \lambda I - L(\mathcal{G}_m\{\mathcal{B}_i\})$. For this matrix,

$$\alpha_{i,j} = \lambda - \delta_{i,j} \;\; (1 \le i \le m, 1 \le j \le k_i - 1) \quad \text{and} \quad \alpha_i = \lambda - \delta_{i,k_i} - \delta_i \;\; (1 \le i \le m).$$

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Let $\beta_{i,j},\ \beta_i$ be as in Lemma 2.1. We first suppose that $\lambda \in \mathbb{R}$ is such that $P_{i,j}(\lambda) \neq 0$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i - 1$. For brevity, we write $P_{i,j}(\lambda) = P_{i,j}$ and $P(\lambda) = P$. Then we have

$$\begin{aligned}
\beta_{i,1} &= \lambda - \delta_{i,1} = \frac{P_{i,1}}{P_{i,0}} \neq 0, \\
\beta_{i,2} &= (\lambda - \delta_{i,2}) - \frac{n_{i,1}}{n_{i,2}} \frac{w_{i,1}^2}{\beta_{i,1}} = (\lambda - \delta_{i,2}) - \frac{n_{i,1}}{n_{i,2}} w_{i,1}^2 \frac{P_{i,0}}{P_{i,1}} \\
&= \frac{(\lambda - \delta_{i,2}) P_{i,1} - \frac{n_{i,1}}{n_{i,2}} w_{i,1}^2 P_{i,0}}{P_{i,1}} = \frac{P_{i,2}}{P_{i,1}} \neq 0, \\
&\vdots \qquad\qquad \vdots \\
\beta_{i,k_i-1} &= (\lambda - \delta_{i,k_i-1}) - \frac{n_{k_i-2}}{n_{k_i-1}} \frac{w_{i,k_i-2}^2}{\beta_{i,k_i-2}} = (\lambda - \delta_{i,k_i-1}) - \frac{n_{i,k_i-2}}{n_{i,k_i-1}} w_{i,k_i-2}^2 \frac{P_{i,k_i-3}}{P_{i,k_i-2}} \\
&= \frac{(\lambda - \delta_{i,k_i-1}) P_{i,k_i-2} - \frac{n_{k_i-2}}{n_{i,k_i-1}} w_{i,k_i-2}^2 P_{i,k_i-3}}{P_{i,k_i-2}} = \frac{P_{i,k_i-1}}{P_{i,k_i-2}} \neq 0.
\end{aligned}$$

Moreover, for $i = 1, \ldots, m$,

$$\begin{aligned}
\beta_i &= \lambda - \delta_{i,k_i} - \delta_i - n_{i,k_i-1} \frac{w_{i,k_i-1}^2}{\beta_{i,k_i-1}} = \lambda - \delta_{i,k_i} - \delta_i - n_{i,k_i-1} w_{i,k_i-1}^2 \frac{P_{i,k_i-2}}{P_{i,k_i-1}} \\
&= \frac{(\lambda - \delta_{i,k_i} - \delta_i) P_{i,k_i-1} - n_{i,k_i-1} w_{i,k_i-1}^2 P_{i,k_i-2}}{P_{i,k_i-1}} = \frac{P_i}{P_{i,k_i-1}}.
\end{aligned}$$

By (2.3), it follows that

$$\begin{aligned}
&\det\left(\lambda I - L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)\right) \\
&= \left(\prod_{i=1}^{m} \prod_{j=1}^{k_i-1} \beta_{i,j}^{n_{i,j}}\right) |C_{m+1}| \\
&= \left(\prod_{i=1}^{m} \frac{P_{i,1}^{n_{i,1}}}{P_{i,0}^{n_{i,1}}} \frac{P_{i,2}^{n_{i,2}}}{P_{i,1}^{n_{i,2}}} \frac{P_{i,3}^{n_{i,3}}}{P_{i,2}^{n_{i,3}}} \cdots \frac{P_{i,k_i-2}^{n_{i,k_i-2}}}{P_{i,k_i-3}^{n_{i,k_i-2}}} \frac{P_{i,k_i-1}^{n_{i,k_i-1}}}{P_{i,k_i-2}^{n_{i,k_i-1}}}\right) |C_{m+1}| \\
&= \left(\prod_{i=1}^{m} P_{i,1}^{n_{i,1}-n_{i,2}} P_{i,2}^{n_{i,2}-n_{i,3}} \cdots P_{i,k_i-2}^{n_{i,k_i-2}-n_{i,k_i-1}} P_{i,k_i-1}^{n_{i,k_i-1}}\right) |C_{m+1}|,
\end{aligned}$$

where

$$|C_{m+1}| = \begin{vmatrix}
\frac{P_1}{P_{1,k_1-1}} & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\
\varepsilon_{1,2} & \frac{P_2}{P_{2,k_2-1}} & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\
\varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\
\vdots & \vdots & & \frac{P_{m-1}}{P_{m-1,k_{m-1}-1}} & \varepsilon_{m-1,m} \\
\varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & \frac{P_m}{P_{m,k_m-1}}
\end{vmatrix}$$

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$$= \frac{1}{\prod_{i=1}^{m} P_{i,k_i-1}} \begin{vmatrix} P_1 & \varepsilon_{1,2}P_{2,k_2-1} & \cdots & \cdots & \varepsilon_{1,m}P_{m,k_m-1} \\ \varepsilon_{1,2}P_{1,k_1-1} & P_2 & \cdots & \cdots & \varepsilon_{2,m}P_{m,k_m-1} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \varepsilon_{m-1,m}P_{m,k_m-1} \\ \varepsilon_{1,m}P_{1,k_1-1} & \varepsilon_{2,m}P_{2,k_2-1} & \cdots & \cdots & P_m \end{vmatrix}$$
$$= \frac{1}{\prod_{i=1}^{m} P_{i,k_i-1}} P(\lambda).$$

Hence,

$$\begin{aligned}
&|\lambda I - L(\mathcal{G}_m\{\mathcal{B}_i\})| \\
&= \left(\prod_{i=1}^{m} P_{i,1}^{n_{i,1}-n_{i,2}} P_{i,2}^{n_{i,2}-n_{i,3}} \cdots P_{i,k_i-2}^{n_{i,k_i-2}-n_{i,k_i-1}} P_{i,k_i-1}^{n_{i,k_i-1}}\right) \frac{1}{\prod_{i=1}^{m} P_{i,k_i-1}} P(\lambda) \\
&= \left(\prod_{i=1}^{m} P_{i,1}^{n_{i,1}-n_{i,2}} P_{i,2}^{n_{i,2}-n_{i,3}} \cdots P_{i,k_i-2}^{n_{i,k_i-2}-n_{i,k_i-1}} P_{i,k_i-1}^{n_{i,k_i-1}-n_{i,k_i}}\right) P(\lambda) \\
&= P(\lambda) \prod_{i=1}^{m} \prod_{j=1}^{k_i-1} P_{i,j}^{n_{i,j}-n_{i,j+1}} \\
&= P(\lambda) \prod_{i=1}^{m} \prod_{j\in\Omega_i} P_{i,j}^{n_{i,j}-n_{i,j+1}}(\lambda).
\end{aligned}$$

We have used the fact that $n_{i,k_i} = 1$. Thus, (3.2) is proved for all $\lambda \in \mathbb{R}$ such that $P_{i,j}(\lambda) \neq 0$, for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i - 1$. Now, we consider $\lambda_0 \in \mathbb{R}$ such that $P_{l,s}(\lambda_0) = 0$ for some $1 \leq l \leq m$ and $1 \leq s \leq k_l - 1$. Since the zeros of any nonzero polynomial are isolated, there exists a neighborhood $N(\lambda_0)$ of λ_0 such that $P_{i,j}(\lambda) \neq 0$ for all $\lambda \in N(\lambda_0) - \{\lambda_0\}$, and for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i - 1$. Hence,

$$|\lambda I - L(\mathcal{G}_m\{\mathcal{B}_i\})| = P(\lambda) \prod_{i=1}^{m} \prod_{j\in\Omega_i} P_{i,j}^{n_{i,j}-n_{i,j+1}}(\lambda)$$

for all $\lambda \in N(\lambda_0) - \{\lambda_0\}$. By continuity, taking the limit as λ tends to λ_0, we obtain

$$|\lambda_0 I - L(\mathcal{G}_m\{\mathcal{B}_i\})| = P(\lambda_0) \prod_{i=1}^{m} \prod_{j\in\Omega_i} P_{i,j}^{n_{i,j}-n_{i,j+1}}(\lambda_0).$$

Therefore, (3.2) holds for all $\lambda \in \mathbb{R}$.

(b) It is an immediate consequence of part (a). $\square$

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Definition 3.3. For $i = 1, 2, 3, \ldots, m$, let T_i be the $k_i \times k_i$ symmetric matrix defined by

$$T_i = \begin{bmatrix} \delta_{i,1} & w_{i,1}\sqrt{d_{i,2}-1} & & & \\ w_{i,1}\sqrt{d_{i,2}-1} & \delta_{i,2} & \ddots & & \\ & \ddots & \ddots & w_{i,k_i-2}\sqrt{d_{i,k_i-1}-1} & \\ & & \ddots & \delta_{i,k_i-1} & w_{i,k_i-1}\sqrt{d_{i,k_i}} \\ & & & w_{i,k_i-1}\sqrt{d_{i,k_i}} & \delta_{i,k_i}+\delta_i \end{bmatrix}.$$

Moreover, for $i = 1, 2, \ldots, m$ and for $j = 1, 2, 3, \ldots, k_i - 1$, let $T_{i,j}$ be the $j \times j$ leading principal submatrix of T_i.

Lemma 3.4. *For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i - 1$, we have*

$$|\lambda I - T_{i,j}| = P_{i,j}(\lambda). \tag{3.3}$$

Moreover, for $i = 1, 2, 3, \ldots, m$,

$$|\lambda I - T_i| = P_i(\lambda). \tag{3.4}$$

Proof. It is well known [6, page 229] that the characteristic polynomial, Q_j, of the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$\begin{bmatrix} c_1 & b_1 & & & \\ b_1 & c_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{k-2} & c_{k-1} & b_{k-1} \\ & & & b_{k-1} & c_k \end{bmatrix},$$

satisfies the three-term recursion formula

$$Q_j(\lambda) = (\lambda - c_j) Q_{j-1}(\lambda) - b_{j-1}^2 Q_{j-2}(\lambda) \tag{3.5}$$

with

$$Q_0(\lambda) = 1 \quad \text{and} \quad Q_1(\lambda) = \lambda - c_1.$$

We recall that, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i$, the polynomials $P_{i,j}$ are defined by formula (3.1). Let $1 \le i \le m$ be fixed. By (1.1), $\sqrt{\frac{n_{i,j}}{n_{i,j+1}}} = \sqrt{d_{i,j+1} - 1}$ for $j = 1, 2, \ldots, k_i - 2$. For the matrix T_{i,k_i-1}, we have $c_j = \delta_{i,j}$ for $j = 1, 2, \ldots, k_i - 1$ and $b_j = w_{i,j}\sqrt{d_{i,j+1} - 1} = w_{i,j}\sqrt{\frac{n_{i,j}}{n_{i,j+1}}}$ for $j = 1, 2, \ldots, k_i - 2$. Replacing in (3.5), we get the polynomials $P_{i,j}$, $j = 1, 2, \ldots, k_i - 1$. Thus, (3.3) is proved. The proof of (3.4) is similar. ∎

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LEMMA 3.5. *Let $r = \sum_{i=1}^{m} k_i$, and let G be the $r \times r$ symmetric matrix defined by*

$$G = \begin{bmatrix} T_1 & -\varepsilon_{1,2}F_{1,2} & -\varepsilon_{1,3}F_{1,3} & \cdots & -\varepsilon_{1,m}F_{1,m} \\ -\varepsilon_{1,2}F_{1,2}^T & T_2 & -\varepsilon_{2,3}F_{2,3} & \cdots & -\varepsilon_{2,m}F_{2,m} \\ -\varepsilon_{1,3}F_{1,3}^T & -\varepsilon_{2,3}F_{2,3}^T & \ddots & & \vdots \\ \vdots & \vdots & & T_{m-1} & -\varepsilon_{m-1,m}F_{m-1,m} \\ -\varepsilon_{1,m}F_{1,m}^T & -\varepsilon_{2,m}F_{2,m}^T & \cdots & \cdots & T_m \end{bmatrix}.$$

Then

$$|\lambda I - G| = P(\lambda).$$

Proof. We have

$$\lambda I - G = \begin{bmatrix} \lambda I - T_1 & \varepsilon_{1,2}F_{1,2} & \varepsilon_{1,3}F_{1,3} & \cdots & \varepsilon_{1,m}F_{1,m} \\ \varepsilon_{1,2}F_{1,2}^T & \lambda I - T_2 & \varepsilon_{2,3}F_{2,3} & \cdots & \varepsilon_{2,m}F_{2,m} \\ \varepsilon_{1,3}F_{1,3}^T & \varepsilon_{2,3}F_{2,3}^T & \ddots & & \vdots \\ \vdots & \vdots & & \lambda I - T_{m-1} & \varepsilon_{m-1,m}F_{m-1,m} \\ \varepsilon_{1,m}F_{1,m}^T & \varepsilon_{2,m}F_{2,m}^T & \cdots & \varepsilon_{m-1,m}F_{m-1,m}^T & \lambda I - T_m \end{bmatrix}$$

We apply Lemma 2.2 to $\lambda I - G$ to get

$$|\lambda I - G|$$
$$= \begin{vmatrix} |\lambda I - T_1| & \varepsilon_{1,2}\left|\widetilde{\lambda I - T_2}\right| & \cdots & \cdots & \varepsilon_{1,m}\left|\widetilde{\lambda I - T_m}\right| \\ \varepsilon_{1,2}\left|\widetilde{\lambda I - T_1}\right| & |\lambda I - T_2| & \cdots & \cdots & \varepsilon_{2,m}\left|\widetilde{\lambda I - T_m}\right| \\ \vdots & \vdots & \ddots & & \vdots \\ \varepsilon_{1,m-1}\left|\widetilde{\lambda I - T_1}\right| & \vdots & & |\lambda I - T_{m-1}| & \varepsilon_{m-1,m}\left|\widetilde{\lambda I - T_m}\right| \\ \varepsilon_{1,m}\left|\widetilde{\lambda I - T_1}\right| & \varepsilon_{2,m}\left|\widetilde{\lambda I - T_2}\right| & \cdots & \varepsilon_{m-1,m}\left|\widetilde{\lambda I - T_{m-1}}\right| & |\lambda I - T_m| \end{vmatrix}.$$

We observe that $\widetilde{\lambda I - T_i} = \lambda I - T_{i,k_i-1}$. We now use Lemma 3.4 to obtain

$$|\lambda I - G| = \begin{vmatrix} P_1(\lambda) & \varepsilon_{1,2}P_{2,k_2-1}(\lambda) & \cdots & \cdots & \varepsilon_{1,m}P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,2}P_{1,k_1-1}(\lambda) & P_2(\lambda) & \cdots & \cdots & \varepsilon_{2,m}P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,3}P_{1,k_1-1}(\lambda) & \varepsilon_{2,3}P_{2,k_2-1}(\lambda) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \varepsilon_{m-1,m}P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,m}P_{1,k_1-1}(\lambda) & \varepsilon_{2,m}P_{2,k_2-1}(\lambda) & \cdots & \cdots & P_m(\lambda) \end{vmatrix}$$
$$= P(\lambda).$$

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The proof is complete. □

THEOREM 3.6.

(a) $\sigma\left(L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)\right)=\left(\cup_{i=1}^m \cup_{j\in\Omega_i}\sigma\left(T_{i,j}\right)\right)\cup\sigma\left(G\right)$.

(b) *The multiplicity of each eigenvalue of* $T_{i,j}$, *as an eigenvalue of* $L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)$, *is* $n_{i,j}-n_{i,j+1}$ *for* $j\in\Omega_i$.

(c) *The matrix* G *is singular.*

Proof. It is known that the eigenvalues of a symmetric tridiagonal matrix with nonzero codiagonal entries are simple [2]. This fact, Theorem 3.2, Lemma 3.4 and Lemma 3.5 yield (a) and (b). One can easily check that $|T_{i,j}|=\omega_{i,1}\omega_{i,2}\cdots\omega_{i,j}>0$ for $1\le i\le m$ and $1\le j\le k_i-1$. This fact and part (a) imply that 0 is an eigenvalue of G. Hence, G is a singular matrix. □

THEOREM 3.7. *The spectral radius of* $L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)$ *is the largest eigenvalue of the matrix* G.

Proof. Since $L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)$ is a positive semidefinite matrix, its spectral radius is its largest eigenvalue. By Theorem 3.6, the eigenvalues of $L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)$ are the eigenvalues of the matrices $T_{i,j}$ for $i\in\Omega_i$ and $1\le j\le k_i-1$ together with the eigenvalues of G. Since the eigenvalues of the matrices $T_{i,j}$ interlace the eigenvalues of G, we conclude that the spectral radius of $L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)$ is the largest eigenvalue of G. □

Next, we summarize the above results for the particular case of unweighted trees $\mathcal{B}_i$ and unweighted graph $\mathcal{G}_m$.

THEOREM 3.8. *If each* $\mathcal{B}_i$ *is an unweighted generalized Bethe tree and* $\mathcal{G}_m$ *is an unweighted graph, then the following hold:*

(a)

$$\sigma\left(L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)\right)=\left(\cup_{i=1}^m \cup_{j\in\Omega_i}\sigma\left(T_{i,j}\right)\right)\cup\sigma\left(G\right),$$

where G *is the matrix defined in Lemma 3.5 with*

$$T_i=\begin{bmatrix} 1 & \sqrt{d_{i,2}-1} & & & \\ \sqrt{d_{i,2}-1} & d_{i,2} & \ddots & & \\ & \ddots & \ddots & \sqrt{d_{i,k_i-1}-1} & \\ & & \sqrt{d_{i,k_i-1}-1} & d_{i,k_i-1} & \sqrt{d_{i,k_i}} \\ & & & \sqrt{d_{i,k_i}} & d_{i,k_i}+d\left(v_i\right) \end{bmatrix}$$

of size $k_i\times k_i$ *and* $T_{i,j}$ *is the* $j\times j$ *leading principal submatrix of* T_i.

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(*b*) *The multiplicity of each eigenvalue of* $T_{i,j}$, *as an eigenvalue of* $L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)$, *is* $n_{i,j}-n_{i,j+1}$ *for* $j\in\Omega_i$.

(*c*) *The matrix* G *is singular.*

(*d*) *The spectral radius of* $L\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)$ *is the largest eigenvalue of the matrix* G.

4. The spectrum of the adjacency matrix. By (1.2), we have

$$A\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)=\begin{bmatrix} A_1 & 0 & \cdots & 0 & w_{1,k_1-1}E_1 \\ 0 & A_2 & \cdots & 0 & w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m & w_{m,k_m-1}E_m \\ w_{1,k_1-1}E_1^T & w_{2,k_2-1}E_2^T & \cdots & w_{m,k_m-1}E_m^T & A\left(\mathcal{G}_m\right) \end{bmatrix},$$

where the diagonal blocks A_i $(1\leq i\leq m)$ are given by (1.4) and $A\left(\mathcal{G}_m\right)$ is given by (1.6).

We may apply Lemma 2.1 to $X=\lambda I-A\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)$. For this matrix, $\alpha_{i,j}=\lambda$ for $1\leq i\leq m$ and $1\leq j\leq k_i-1$, and $\alpha_i=\lambda$ for $1\leq i\leq m$.

Definition 4.1. For $i=1,2,\ldots,m$, let

$$Q_{i,0}\left(\lambda\right)=1,\quad Q_{i,1}\left(\lambda\right)=\lambda$$

and, for $j=2,3,\ldots,k_i-1$, let

$$Q_{i,j}\left(\lambda\right)=\lambda Q_{i,j-1}\left(\lambda\right)-\frac{n_{i,j-1}}{n_{i,j}}w_{i,j-1}^2Q_{i,j-2}\left(\lambda\right).$$

Moreover, for $i=1,2,\ldots,m$, let

$$Q_i\left(\lambda\right)=\lambda Q_{i,k_i-1}\left(\lambda\right)-n_{i,k_i-1}w_{i,j-1}^2Q_{i,k_i-2}\left(\lambda\right).$$

Theorem 4.2. *The following hold:*

(*a*)

$$\left|\lambda I-A\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)\right|=Q\left(\lambda\right)\prod_{i=1}^{m}\prod_{j\in\Omega_i}Q_{i,j}^{n_{i,j}-n_{i,j+1}}\left(\lambda\right).$$

(*b*)

$$\sigma\left(A\left(\mathcal{G}_m\left\{\mathcal{B}_i\right\}\right)\right)=\left(\cup_{i=1}^m\cup_{j\in\Omega_i}\left\{\lambda:Q_{i,j}\left(\lambda\right)=0\right\}\right)\cup\left\{\lambda:Q\left(\lambda\right)=0\right\},$$

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where

$$Q(\lambda) = \left| \begin{array}{cccc} Q_1(\lambda) & -\varepsilon_{1,2} Q_{2,k_2-1}(\lambda) & \cdots & -\varepsilon_{1,m} Q_{m,k_m-1}(\lambda) \\ -\varepsilon_{1,2} Q_{1,k_1-1}(\lambda) & Q_2(\lambda) & \cdots & -\varepsilon_{2,m} Q_{m,k_m-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_{1,m} Q_{1,k_1-1}(\lambda) & -\varepsilon_{2,m} Q_{2,k_2-1}(\lambda) & \cdots & Q_m(\lambda) \end{array} \right|.$$

Proof. Similar to the proof of Theorem 3.2. □

DEFINITION 4.3. For $i = 1, 2, \ldots, m$, let S_i be the $k_i \times k_i$ symmetric matrix defined by

$$S_i = \left[\begin{array}{ccccc} 0 & w_{i,1}\sqrt{d_{i,2}-1} & & & \\ w_{i,1}\sqrt{d_{i,2}-1} & 0 & \ddots & & \\ & \ddots & \ddots & w_{i,k_i-2}\sqrt{d_{i,k_i-1}-1} & \\ & & \ddots & 0 & w_{i,k_i-1}\sqrt{d_{i,k_i}} \\ & & & w_{i,k_i-1}\sqrt{d_{i,k_i}} & 0 \end{array} \right].$$

Moreover, for $i = 1, 2, \ldots, m$ and for $j = 1, 2, \ldots, k_i - 1$, let $S_{i,j}$ be the $j \times j$ leading principal submatrix of S_i.

LEMMA 4.4. *For $i = 1, 2, \ldots, m$ and for $j = 1, 2, \ldots, k_i - 1$, we have*

$$|\lambda I - S_{i,j}| = Q_{i,j}(\lambda).$$

Moreover, for $i = 1, 2, \ldots, m$,

$$|\lambda I - S_i| = Q_i(\lambda).$$

Proof. Similar to the proof of Lemma 3.4. □

LEMMA 4.5. *Let $r = \sum_{i=1}^{m} k_i$ and H be the $r \times r$ symmetric matrix defined by*

$$H = \left[\begin{array}{ccccc} S_1 & \varepsilon_{1,2}F_{1,2} & \varepsilon_{1,3}F_{1,3} & \cdots & \varepsilon_{1,m}F_{1,m} \\ \varepsilon_{1,2}F_{1,2}^T & S_2 & \varepsilon_{2,3}F_{2,3} & \cdots & \varepsilon_{2,m}F_{2,m} \\ \varepsilon_{1,3}F_{1,3}^T & \varepsilon_{2,3}F_{2,3}^T & \ddots & & \vdots \\ \vdots & \vdots & & S_{m-1} & \varepsilon_{m-1,m}F_{m-1,m} \\ \varepsilon_{1,m}F_{1,m}^T & \varepsilon_{2,m}F_{2,m}^T & \cdots & \varepsilon_{m-1,m}F_{m-1,m}^T & S_m \end{array} \right].$$

Then

$$|\lambda I - H| = Q(\lambda).$$

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Proof. We have

$$\lambda I - H = \begin{bmatrix} \lambda I - S_1 & -\varepsilon_{1,2}F_{1,2} & \cdots & \cdots & -\varepsilon_{1,m}F_{1,m} \\ -\varepsilon_{1,2}F_{1,2}^T & \lambda I - S_2 & \cdots & \cdots & -\varepsilon_{2,m}F_{2,m} \\ -\varepsilon_{1,3}F_{1,3}^T & -\varepsilon_{2,3}F_{2,3}^T & \ddots & & \vdots \\ \vdots & \vdots & & \lambda I - S_{m-1} & -\varepsilon_{m-1,m}F_{m-1,m} \\ -\varepsilon_{1,m}F_{1,m}^T & -\varepsilon_{2,m}F_{2,m}^T & \cdots & -\varepsilon_{m-1,m}F_{m-1,m}^T & \lambda I - S_m \end{bmatrix}.$$

We apply Lemma 2.2 to $\lambda I - H$ to obtain that

$$|\lambda I - H| = \begin{vmatrix} |\lambda I - S_1| & -\varepsilon_{1,2}\left|\widetilde{\lambda I - S_2}\right| & \cdots & \cdots & -\varepsilon_{1,m}\left|\widetilde{\lambda I - S_m}\right| \\ -\varepsilon_{1,2}\left|\widetilde{\lambda I - S_1}\right| & |\lambda I - S_2| & \cdots & \cdots & -\varepsilon_{2,m}\left|\widetilde{\lambda I - S_m}\right| \\ \vdots & \vdots & \ddots & & \vdots \\ -\varepsilon_{1,m-1}\left|\widetilde{\lambda I - S_1}\right| & \vdots & & |\lambda I - S_{m-1}| & -\varepsilon_{m-1,m}\left|\widetilde{\lambda I - S_m}\right| \\ -\varepsilon_{1,m}\left|\widetilde{\lambda I - S_1}\right| & -\varepsilon_{2,m}\left|\widetilde{\lambda I - S_2}\right| & \cdots & -\varepsilon_{m-1,m}\left|\widetilde{\lambda I - S_{m-1}}\right| & |\lambda I - S_m| \end{vmatrix}.$$

We observe that $\widetilde{\lambda I - S_i} = \lambda I - S_{i,k_i-1}$. We now use Lemma 4.4 to obtain $|\lambda I - H| = Q(\lambda)$. ☐

Theorem 4.6.

(*a*) $\sigma(A(\mathcal{G}_m\{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j\in\Omega_i} \sigma(S_{i,j})) \cup \sigma(H)$.

(*b*) *The multiplicity of each eigenvalue of the matrix $S_{i,j}$, as an eigenvalue of $A(\mathcal{G}_m\{\mathcal{B}_i\})$, is $n_{i,j} - n_{i,j+1}$ for $j \in \Omega_i$.*

(*c*) *The largest eigenvalue of H is the spectral radius of $A(\mathcal{G}_m\{\mathcal{B}_i\})$.*

Proof. The proofs of (*a*) and (*b*) are similar to the proof of Theorem 3.6. Finally, (*c*) follows from part (*a*) and the interlacing property of the eigenvalues. ☐

The following theorem summarizes the above results for the case of unweighted trees $\mathcal{B}_i$ and unweighted graph $\mathcal{G}_m$.

Theorem 4.7. *If each $\mathcal{B}_i$ is an unweighted generalized Bethe tree and $\mathcal{G}_m$ is an unweighted graph, then we have:*

(*a*) $\sigma(A(\mathcal{G}_m\{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j\in\Omega_i} \sigma(S_{i,j})) \cup \sigma(H)$, *where H is the matrix defined*

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in Lemma 4.5 with

$$S_i = \begin{bmatrix} 0 & \sqrt{d_{i,2}-1} & & & \\ \sqrt{d_{i,2}-1} & 0 & \ddots & & \\ & \ddots & \ddots & \sqrt{d_{i,k_i-1}-1} & \\ & & \sqrt{d_{i,k_i-1}-1} & 0 & \sqrt{d_{i,k_i}} \\ & & & \sqrt{d_{i,k_i}} & 0 \end{bmatrix}$$

of size $k_i \times k_i$ *and* $S_{i,j}$ *is the* $j \times j$ *leading principal submatrix of* S_i*.*

(b) *The multiplicity of each eigenvalue of the matrix* $S_{i,j}$*, as an eigenvalue of* $A(\mathcal{G}_m\{\mathcal{B}_i\})$*, is* $n_{i,j} - n_{i,j+1}$ *for* $j \in \Omega_i$*.*

(c) *The spectral radius of* $A(\mathcal{G}_m\{\mathcal{B}_i\})$ *is the largest eigenvalue of the matrix* H*.*

5. Unweighted compound graphs of copies of a generalized Bethe tree. In this section, we assume that $\mathcal{G}_m$ is any connected unweighted graph and that $\mathcal{B}_1 = \mathcal{B}_2 = \cdots = \mathcal{B}_m = \mathcal{B}$, where $\mathcal{B}$ is an unweighted generalized Bethe of k levels in which d_{k-j+1} and n_{k-j+1} are the degree of the vertices and the number of them at the level j. Then

$$\begin{aligned} &k_1 = k_2 = \cdots = k_m = k, \\ &\Omega_1 = \cdots = \Omega_m = \Omega = \{j : 1 \le j \le k-1, n_{j+1} > n_j\}, \\ &F_{i,j} = F, \end{aligned}$$

where F is a $k \times k$ matrix whose entries are 0 except $F(k,k) = 1$. Moreover, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k-1$, the matrix $T_{i,j}$ is the $j \times j$ leading principal submatrix of

$$T_i = \begin{bmatrix} 1 & \sqrt{d_2-1} & & & \\ \sqrt{d_2-1} & d_2 & \ddots & & \\ & \ddots & \ddots & \sqrt{d_{k-1}-1} & \\ & & \sqrt{d_{k-1}-1} & d_{k-1} & \sqrt{d_k} \\ & & & \sqrt{d_k} & d_k + d(v_i) \end{bmatrix},$$

and

$$G = \begin{bmatrix} T_1 & -\varepsilon_{1,2}F & -\varepsilon_{1,3}F & \cdots & -\varepsilon_{1,m}F \\ -\varepsilon_{1,2}F & T_2 & -\varepsilon_{2,3}F & \cdots & -\varepsilon_{2,m}F \\ -\varepsilon_{1,3}F & -F & \ddots & & \vdots \\ \vdots & \vdots & & T_{m-1} & -\varepsilon_{m-1,m}F \\ -\varepsilon_{1,m}F & -\varepsilon_{2,m}F & \cdots & -\varepsilon_{m-1,m}F & T_m \end{bmatrix}.$$

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We recall that the Kronecker product [8] of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of sizes $m \times m$ and $n \times n$, respectively, is defined to be the $(mn) \times (mn)$ matrix $A \otimes B = (a_{i,j}B)$. For matrices A, B, C and D of appropriate sizes,

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

We write $\mathcal{G}_m\{\mathcal{B}\}$ instead of $\mathcal{G}_m\{\mathcal{B}_i\}$.

¿From now on

$$B = \begin{bmatrix} 1 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & d_2 & \ddots & & \\ & \ddots & \ddots & \sqrt{d_{k-1} - 1} & \\ & & \sqrt{d_{k-1} - 1} & d_{k-1} & \sqrt{d_k} \\ & & & \sqrt{d_k} & d_k \end{bmatrix}.$$

Theorem 5.1. *If $\mathcal{B}$ is an unweighted generalized Bethe tree and $\mathcal{G}_m$ is an unweighted graph, then*

$$\sigma(L(\mathcal{G}_m\{\mathcal{B}\})) = (\cup_{j\in\Omega}\sigma(B_j)) \cup (\cup_{s=1}^{m}\sigma(B + \gamma_s F))$$

where, for $j = 1, 2, \ldots, k-1$, B_j is the $j \times j$ leading principal submatrix of

$$B + \gamma_s F = \begin{bmatrix} 1 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & d_2 & \ddots & & \\ & \ddots & \ddots & \sqrt{d_{k-1} - 1} & \\ & & \sqrt{d_{k-1} - 1} & d_{k-1} & \sqrt{d_k} \\ & & & \sqrt{d_k} & d_k + \gamma_s \end{bmatrix},$$

and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{m-1} > \gamma_m = 0$ are the Laplacian eigenvalues of $\mathcal{G}_m$.

Proof. By Theorem 3.8,

$$\sigma(L(\mathcal{G}_m\{\mathcal{B}_i\})) = (\cup_{i=1}^{m} \cup_{j\in\Omega_i} \sigma(T_{i,j})) \cup \sigma(G).$$

For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k-1$, $T_{i,j} = B_j$. As a consequence,

$$\sigma(L(\mathcal{G}_m\{\mathcal{B}\})) = (\cup_{j\in\Omega}\sigma(B_j)) \cup \sigma(G).$$

It remains to prove that $\sigma(G) = \cup_{s=1}^{m}\sigma(L(B + \gamma_s F))$. We may write

$$G = I_m \otimes B + L(\mathcal{G}_m) \otimes F,$$

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where

$$L(\mathcal{G}_m) = \begin{bmatrix} d(v_1) & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \cdots & -\varepsilon_{1,m} \\ -\varepsilon_{1,2} & d(v_2) & -\varepsilon_{2,3} & \cdots & -\varepsilon_{2,m} \\ -\varepsilon_{1,3} & -\varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & -\varepsilon_{m-1,m} \\ -\varepsilon_{1,m} & -\varepsilon_{2,m} & \cdots & -\varepsilon_{m-1,m} & d(v_m) \end{bmatrix}$$

is the Laplacian matrix of $\mathcal{G}_m$. Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be the eigenvalues $L(\mathcal{G}_m)$, and let

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{m-1} & \mathbf{v}_m \end{bmatrix}$$

be an orthogonal matrix whose columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are eigenvectors corresponding to the eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_m$. Therefore,

$$\begin{aligned} (V \otimes I_k) G \left(V^T \otimes I_k\right) &= (V \otimes I_k)(I_m \otimes B + L(\mathcal{G}_m) \otimes F)\left(V^T \otimes I_k\right) \\ &= I_m \otimes B + \left(V L(\mathcal{G}_m) V^T\right) \otimes F. \end{aligned}$$

We have

$$\begin{aligned} \left(V L(\mathcal{G}_m) V^T\right) \otimes F &= \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \gamma_{m-1} & \\ & & & \gamma_m \end{bmatrix} \otimes F \\ &= \begin{bmatrix} \gamma_1 F & & & & \\ & \gamma_2 F & & & \\ & & \ddots & & \\ & & & \gamma_{m-1} F & \\ & & & & \gamma_m F \end{bmatrix}, \end{aligned}$$

and hence,

$$(V \otimes I_k) G \left(V^T \otimes I_k\right) = \begin{bmatrix} B + \gamma_1 F & & & & \\ & B + \gamma_2 F & & & \\ & & \ddots & & \\ & & & B + \gamma_{m-1} F & \\ & & & & B + \gamma_m F \end{bmatrix}.$$

Since G and $(V \otimes I_k) G \left(V^T \otimes I_k\right)$ are similar matrices, we conclude that

$$\sigma(G) = \cup_{s=1}^{m} \sigma(B + \gamma_s F).$$

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The proof is complete. $\square$

Corollary 5.2.

(a) *For $j \in \Omega$, the multiplicity of each eigenvalue of B_j, as an eigenvalue of $L(\mathcal{G}_m\{\mathcal{B}\})$, is $m(n_j - n_{j+1})$.*

(b) $\det(B + \gamma_s F) = \gamma_s$. *In particular, B is a singular matrix.*

(c) *The spectral radius of $L(\mathcal{G}_m\{\mathcal{B}\})$ is the largest eigenvalue of $B + \gamma_1 F$.*

(d) *The algebraic connectivity of $L(\mathcal{G}_m\{\mathcal{B}\})$ is the smallest eigenvalue of $B + \gamma_{m-1}F$.*

Proof. (a) We have $T_{i,j} = B_j$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k-1$. Then the result is an immediate consequence of part (b) of Theorem 3.8.

(b) It follows easily applying the Gaussian elimination procedure.

$(c), (d)$ The eigenvalues of $L(\mathcal{G}_m\{\mathcal{B}\})$ are the eigenvalues of the matrices B_j for $j \in \Omega$ together with the eigenvalues of the matrices $B + \gamma_s F$. The eigenvalues of each B_j interlace the eigenvalues of any $B + \gamma_s F$. Then the spectral radius of $L(\mathcal{G}_m\{\mathcal{B}\})$ is the maximum of the spectral radii of the matrices $B + \gamma_s F$ and the algebraic connectivity of $L(\mathcal{G}_m\{\mathcal{B}\})$ is the minimum eigenvalue in $\cup_{s=1}^{m-1}\sigma(B + \gamma_s F)$. For $s = 1, 2, \ldots, m$,

$$B + \gamma_1 F = (B + \gamma_s F) + (\gamma_1 - \gamma_s) F.$$

We use the fact that the spectral radius of an irreducible nonnegative matrix increases when any of its entries increases [7], to conclude part (c). For $s = 1, 2, \ldots, m-1$,

$$B + \gamma_s F = (B + \gamma_{m-1} F) + (\gamma_s - \gamma_{m-1}) F.$$

We use now the fact that the eigenvalues of a Hermitian matrix do not decrease if a positive semidefinite matrix is added to it [3], to conclude part (d). $\square$

We consider now the adjacency matrix of $\mathcal{G}_m\{\mathcal{B}\}$. For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k-1$, the matrices $S_{i,j}$, S_i and H in Theorem 4.7 become

$$S_{i,j} = U_j = \begin{bmatrix} 0 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & 0 & \ddots & & \\ & \ddots & \ddots & \sqrt{d_{j-1} - 1} & \\ & & \sqrt{d_{j-1} - 1} & 0 & \sqrt{d_j - 1} \\ & & & \sqrt{d_j - 1} & 0 \end{bmatrix},$$

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$$S_1 = S_2 = \cdots = S_m = S = \begin{bmatrix} 0 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & 0 & \ddots & & \\ & \ddots & \ddots & \sqrt{d_{k-1} - 1} & \\ & & \sqrt{d_{k-1} - 1} & 0 & \sqrt{d_k} \\ & & & \sqrt{d_k} & 0 \end{bmatrix}$$

and

$$H = \begin{bmatrix} S & \varepsilon_{1,2}F & \varepsilon_{1,3}F & \cdots & \varepsilon_{1,m}F \\ \varepsilon_{1,2}F & S & \varepsilon_{2,3}F & \cdots & \varepsilon_{2,m}F \\ \varepsilon_{1,3}F & F & \ddots & & \vdots \\ \vdots & \vdots & & S & \varepsilon_{m-1,m}F \\ \varepsilon_{1,m}F & \varepsilon_{2,m}F & \cdots & \varepsilon_{m-1,m}F & S \end{bmatrix}.$$

Theorem 5.3. *If $\mathcal{B}$ is an unweighted generalized Bethe tree and $\mathcal{G}_m$ is an unweighted graph, then*

$$\sigma\left(A\left(\mathcal{G}_m\left\{\mathcal{B}\right\}\right)\right) = \left(\cup_{j\in\Omega}\sigma\left(U_j\right)\right) \cup \left(\cup_{s=1}^{m}\sigma\left(S + \gamma_s F\right)\right),$$

where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{m-1} \geq \gamma_m$ are the eigenvalues of the adjacency matrix of $\mathcal{G}_m$.

Proof. The proof uses Theorem 4.7 and is similar to the proof of Theorem 5.1. □

Corollary 5.4.

(*a*) *For $j \in \Omega$, the multiplicity of each eigenvalue of U_j, as an eigenvalue of $A\left(\mathcal{G}_m\left\{\mathcal{B}\right\}\right)$, is $m\left(n_j - n_{j+1}\right)$.*

(*b*) *The spectral radius of $A\left(\mathcal{G}_m\left\{\mathcal{B}\right\}\right)$ is the largest eigenvalue of $S + \gamma_1 F$.*

Proof. Similar to the proof of Corollary 5.2. □

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