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CORRIGENDUM TO "DETERMINANTS OF NORMALIZED BOHEMIAN UPPER HESSENBERG MATRICES" [ELECTRON. J. OF LINEAR ALGEBRA 36 (2020) 352–366]*

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Abstract. An amended version of Proposition 3.6 of [Fasi and Negri Porzio, *Electron. J. Linear Algebra* 36:352–366, 2020] is presented. The result shows that the set of possible determinants of upper Hessenberg matrices with ones on the subdiagonal and elements in the upper triangular part drawn from the set $\{-1,1\}$ is $\{2k \mid k \in \langle -2^{n-2}, 2^{n-2}\rangle\}$, instead of $\{2k \mid k \in \langle -n+1, n-1\rangle\}$ as previously stated. This does not affect the main results of the article being corrected and shows that Conjecture 20 in the *Characteristic Polynomial Database* is true.

Key words. Bohemian matrix, Integer matrix, Normalized upper Hessenberg matrix, Determinant.

AMS subject classifications. 11C20, 15A15, 15B36.

Let $\mathcal{H}^n(\{-1,1\})$ denote the set of upper Hessenberg matrices of order n with ones on the subdiagonal and elements drawn from the set $\{-1,1\}$ in the upper triangular part. We rectify [1, Prop. 3.6], which incorrectly enumerated only a subset of the possible determinants of matrices in $\mathcal{H}^n(\{-1,1\})$, and thus provide a proof of [3, Conjecture 20].

PROPOSITION 1. *If $n > 1$, then the set of possible determinants of matrices in the family $\mathcal{H}^n(\{-1,1\})$ is*

$$D_n = \{2k \mid k \in \langle -2^{n-2}, 2^{n-2}\rangle\}. \tag{1.1}$$

Proof. As noted by Ching [2], there are only 2^{n-1} possibly nonzero terms in the determinant expansion of an $n \times n$ Hessenberg matrix. If the matrix is in $\mathcal{H}^n(\{-1,1\})$, then each of these 2^{n-1} monomials evaluates to either $+1$ or -1, which implies that the determinant of any such matrices must be even and cannot be larger than 2^{n-1} in absolute value. Now we explain how to construct a matrix $H \in \mathcal{H}_0^n(\{-1,1\})$ such that $\det H = 2k$, for any $k \in \langle -2^{n-2}, 2^{n-2}\rangle$.

Let $K^{(n)} \in \mathcal{H}_0^n(\{-1,1\})$ be the matrix with entries

$$k_{ij}^{(n)} = \begin{cases} 0, & i > j+1, \\ 1, & i = j+1, \\ -1, & i \le j, \end{cases}$$

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that is, a Hessenberg matrix with 1 on the first subdiagonal and -1 in the upper triangular part. Our proof relies on the identity

$$\det K^{(n)} = (-1)^n \, 2^{n-1}, \tag{1.2}$$

which is a special case of one of the results in [1, Proposition 3.1]. Now consider the matrix

$$H = \left[\begin{array}{ccc|c} & & & b \\ & K^{(n-1)} & & a_0 \\ & & & \vdots \\ & & & a_{n-3} \\ \hline 0 & \dots & 1 & a_{n-2} \end{array}\right],$$

where $b, a_0, \dots, a_{n-2} \in \{-1, 1\}$. Using [1, Lemma 2.1] followed by (1.2), we obtain

$$\det H = (-1)^{n+1}\left(b + \sum_{i=0}^{n-2} (-1)^{i+1} a_i \det K^{(i+1)}\right) = (-1)^{n+1} b + (-1)^{n+1} \sum_{i=0}^{n-2} a_i 2^i, \tag{1.3}$$

and the result can be proven by induction. For the base case $n = 2$, the formula (1.3) reduces to $-(b + a_0)$, which equals 0 if b and a_0 have opposite sign, and 2 or -2 if they are both negative or both positive, respectively. For the inductive step, we write

$$\begin{aligned} \det H &= (-1)^{n+1} b + (-1)^{n+1} \sum_{i=0}^{n-2} a_i 2^i \\ &= (-1)^{n+1} b + (-1)^{n+1} \sum_{i=0}^{n-3} a_i 2^i + (-1)^{n+1} a_{n-2} 2^{n-2}, \end{aligned}$$

and observe that it is enough to show that the coefficients b, a_0, $\dots$, a_{n-2} can be chosen so that $\det H = 2k$ for any $k \in \langle 0, 2^{n-2} \rangle$. A matrix $\widetilde{H}$ such that $\det \widetilde{H} = -2k$ can be obtained by changing the sign of the entries in the last column of H. Let us set a_{n-2} to $(-1)^{n+1}$. By the inductive hypothesis, there exist b, a_0, $\dots$, $a_{n-3} \in \{-1, 1\}$ such that

$$(-1)^n b + (-1)^n \sum_{i=0}^{n-3} a_i 2^i = 2k', \qquad k' \in \langle -2^{n-3}, 2^{n-3} \rangle,$$

which gives

$$\det H = 2k' + 2^{n-2} = 2(k' + 2^{n-3}).$$

Observing that all numbers in $\langle 0, 2^{n-2} \rangle$ can be written in the form $k' + 2^{n-3}$, for $k' \in \langle -2^{n-3}, 2^{n-3} \rangle$, concludes the proof. ∎

Corollary 2. *The matrices in the family $\mathcal{H}^n(\{-1, 1\})$ have $2^{n-1} + 1$ distinct determinants.*

Proof. Since the cardinality of D_n in (1.1) is $2 \cdot 2^{n-2} + 1 = 2^{n-1} + 1$, Proposition 1 proves the claim for $n > 1$. Noting that the only two matrices in $\mathcal{H}^1(\{-1, 1\})$ are -1 and 1 concludes the proof. ∎

Corollary 2 proves [3, Conjecture 20], as it shows that the number of distinct determinants of normalized Bohemian upper Hessenberg matrices with entries from $\{-1, 1\}$ is given by a shift of the OEIS sequence A000051.

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