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THE ANTI-SYMMETRIC ORTHO-SYMMETRIC SOLUTIONS OF THE MATRIX EQUATION $A^T XA = D$*

QING-FENG XIAO†, XI-YAN HU†, AND LEI ZHANG†

Abstract. In this paper, the following problems are discussed.

Problem I. Given matrices $A \in R^{n\times m}$ and $D \in R^{m\times m}$, find $X \in ASR_P^n$ such that $A^T XA = D$, where

$$ASR_P^n = \{X \in ASR^{n\times n} | PX \in SR^{n\times n} \text{ for given } P \in OR^{n\times n} \text{ satisfying } P^T = P\}.$$

Problem II. Given a matrix $\tilde{X} \in R^{n\times n}$, find $\hat{X} \in S_E$ such that

$$\|\tilde{X} - \hat{X}\| = \inf_{X \in S_E} \|\tilde{X} - X\|,$$

where $\|\cdot\|$ is the Frobenius norm, and S_E is the solution set of Problem I.

Expressions for the general solution of Problem I are derived. Necessary and sufficient conditions for the solvability of Problem I are provided. For Problem II, an expression for the solution is given as well.

Key words. Anti-symmetric ortho-symmetric matrix, Matrix equation, Matrix nearness problem, Optimal approximation, Least-square solutions.

AMS subject classifications. 65F15, 65F20.

1. Introduction. Let $R^{n\times m}$ denote the set of all $n \times m$ real matrices, and let $OR^{n\times n}$, $SR^{n\times n}$, $ASR^{n\times n}$ denote the set of all $n \times n$ orthogonal matrices, the set of all $n \times n$ real symmetric matrices, the set of all $n \times n$ real skew-symmetric matrices, respectively. The symbol I_k will stand for the identity matrix of order k, A^+ for the Moore-Penrose generalized inverse of a matrix A, and rank(A) for the rank of matrix A. For matrices $A, B \in R^{n\times m}$, the expression $A * B$ will be the Hadamard product of A and B; also $\|\cdot\|$ will denote the Frobenius norm. Defining the inner product $(A, B) = \mathrm{tr}(B^T A)$ for matrices $A, B \in R^{n\times m}$, $R^{n\times m}$ becomes a Hilbert space. The norm of a matrix generated by this inner product is the Frobenius norm. If $A = (a_{ij}) \in R^{n\times n}$, let $L_A = (l_{ij}) \in R^{n\times n}$ be defined as follows: $l_{ij} = a_{ij}$ whenever $i > j$ and $l_{ij} = 0$ otherwise $(i, j = 1, 2, \ldots, n)$. Let e_i be the i-th column of the identity matrix $I_n (i = 1, 2, \ldots, n)$ and set $S_n = (e_n, e_{n-1}, \ldots, e_1)$. It is easy to see that

$$S_n^T = S_n, \quad S_n^T S_n = I_n.$$

An inverse problem [2-6] arising in the structural modification of the dynamic behaviour of a structure calls for the solution of the matrix equation

$$A^T XA = D, \tag{1.1}$$

where $A \in R^{n\times m}$, $D \in R^{m\times m}$, and the unknown X is required to be real and symmetric, and positive semidefinite or possibly definite. No assumption is made

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†College of Mathematics and Econometrics, Hunan University, Changsha 410082, P.R. of China (qfxiao@hnu.cn). Research supported by National Natural Science Foundation of China (under Grant 10571047) and the Doctorate Foundation of the Ministry of Education of China (under Grant 20060532014).

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about the relative sizes of m and n, and it is assumed throughout that $A \neq 0$ and $D \neq 0$.

Equation (1.1) is a special case of the matrix equation

$$AXB = C. \tag{1.2}$$

Consistency conditions for equation (1.2) were given by Penrose [7] (see also [1]). When the equation is consistent, a solution can be obtained using generalized inverses. Khatri and Mitra [8] gave necessary and sufficient conditions for the existence of symmetric and positive semidefinite solutions as well as explicit formulae using generalized inverses. In [9,10] solvability conditions for symmetric and positive definite solutions and general solutions of Equation (1.2) were obtained through the use of the generalized singular value decomposition [11-13].

For important results on the inverse problem $A^TXA = D$ associated with several kinds of different sets S, for instance, symmetric matrices, symmetric nonnegative definite matrices, bisymmetric (same as persymmetric) matrices, bisymmetric nonnegative definite matrices and so on, we refer the reader to [14-17].

For the case the unknown A is anti-symmetric ortho-symmetric, [18] has discussed the inverse problem $AX = B$. However, for this case, the inverse problem $A^TXA = D$ has not been dealt with yet. This problem will be considered here.

Definition 1.1. A matrix $P \in R^{n\times n}$ is said to be a *symmetric orthogonal matrix* if $P^T = P, \quad P^TP = I_n$.

In this paper, without special statement, we assume that P is a given symmetric orthogonal matrix.

Definition 1.2. A matrix $X \in R^{n\times n}$ is said to be a *anti-symmetric ortho-symmetric matrix* if $X^T = -X, \quad (PX)^T = PX$. We denote the set of all $n \times n$ *anti-symmetric ortho-symmetric matrices* by ASR^n_P.

The problem studied in this paper can now be described as follows.

Problem I. Given matrices $A \in R^{n\times m}$ and $D \in R^{m\times m}$, find an anti-symmetric ortho-symmetric matrix X such that

$$A^TXA = D.$$

In this paper, we discuss the solvability of this problem and an expression for its solution is presented.

The optimal approximation problem of a matrix with the above-given matrix restriction comes up in the processes of test or recovery of a linear system due to incomplete data or revising given data. A preliminary estimate $\tilde{X}$ of the unknown matrix X can be obtained by the experimental observation values and the information of statistical distribution. The optimal estimate of X is a matrix $\hat{X}$ that satisfies the given matrix restriction for X and is the best approximation of $\tilde{X}$, see [19-21].

In this paper, we will also consider the so-called optimal approximation problem associated with $A^TXA = D$. It reads as follows.

Problem II. Given matrix $\tilde{X} \in R^{n\times n}$, find $\hat{X} \in S_E$ such that

$$\|\tilde{X} - \hat{X}\| = \inf_{X\in S_E} \|\tilde{X} - X\|,$$

where S_E is the solution set of Problem I.

We point out that if Problem I is solvable, then Problem II has a unique solution, and in this case an expression for the solution can be derived.

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The paper is organized as follows. In Section 2, we obtain the general form of S_E and the sufficient and necessary conditions under which Problem I is solvable mainly by using the structure of ASR_P^n and orthogonal projection matrices. In Section 3, the expression for the solution of the matrix nearness problem II will be provided.

2. The expression of the general solution of problem I. In this section we first discuss some structure properties of symmetric orthogonal matrices. Then, given such a matrix P, we consider structural properties of the subset ASR_P^n of $R^{n\times n}$. Finally, we present necessary and sufficient conditions for the existence of and the expressions for the anti-symmetric ortho-symmetric (with respect to the given P) solutions of problem I.

LEMMA 2.1. *Assume P is a symmetric orthogonal matrix of size n, and let*

$$P_1 = \frac{1}{2}(I_n + P), \quad P_2 = \frac{1}{2}(I_n - P). \tag{2.1}$$

Then P_1 and P_2 are orthogonal projection matrices satisfying $P_1+P_2 = I_n,\ \ P_1P_2 = 0$.

Proof. By direct computation. □

LEMMA 2.2. *Assume P_1 and P_2 are defined as* (2.1) *and* $\operatorname{rank}(P_1) = r$. *Then* $\operatorname{rank}(P_2) = n - r$, *and there exist unit column orthogonal matrices $U_1 \in R^{n\times r}$ and $U_2 \in R^{n\times(n-r)}$ such that*

$$P_1 = U_1U_1^T, \quad P_2 = U_2U_2^T, \quad P = U_1U_1^T - U_2U_2^T, \quad U_1^TU_2 = 0.$$

Proof. Since P_1 and P_2 are orthogonal projection matrices satisfying $P_1 + P_2 = I_n,\ \ P_1P_2 = 0$, the column space $R(P_2)$ of the matrix P_2 is the orthogonal complement of the column space $R(P_1)$ of the matrix P_1, in other words, $R^n = R(P_1) \oplus R(P_2)$. Hence, if $\operatorname{rank}(P_1) = r$, then $\operatorname{rank}(P_2) = n - r$. On the other hand, $\operatorname{rank}(P_1) = r$, $\operatorname{rank}(P_2) = n - r$, and P_1 , P_2 are orthogonal projection matrices. Thus there exist unit column orthogonal matrices $U_1 \in R^{n\times r}$ and $U_2 \in R^{n\times(n-r)}$ such that $P_1 = U_1U_1^T,\ \ P_2 = U_2U_2^T$. Using $R^n = R(P_1) \oplus R(P_2)$, we have $U_1^TU_2 = 0$. Substituting $P_1 = U_1U_1^T,\ \ P_2 = U_2U_2^T$ into (2.1), we have $P = U_1U_1^T - U_2U_2^T$. □

Elaborating on Lemma 2.2 and its proof, we note that $U = (U_1, U_2)$ is an orthogonal matrix and that the symmetric orthogonal matrix P can be expressed as

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^T. \tag{2.2}$$

LEMMA 2.3. *The matrix $X \in ASR_P^n$ if and only if X can be expressed as*

$$X = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T, \tag{2.3}$$

where $F \in R^{r\times(n-r)}$ and U is the same as (2.2).

Proof. Assume $X \in ASR_P^n$. By Lemma 2.2 and the definition of ASR_P^n, we have

$$P_1XP_1 = \frac{I+P}{2}X\frac{I+P}{2} = \frac{1}{4}(X + PX + XP + PXP) = \frac{1}{4}(PX + XP),$$

$$P_2XP_2 = \frac{I-P}{2}X\frac{I-P}{2} = \frac{1}{4}(X - PX - XP + PXP) = -\frac{1}{4}(PX + XP).$$

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Hence,

$$X = (P_1 + P_2)X(P_1 + P_2) = P_1XP_1 + P_1XP_2 + P_2XP_1 + P_2XP_2$$

$$= P_1XP_2 + P_2XP_1 = U_1U_1^TXU_2U_2^T + U_2U_2^TXU_1U_1^T.$$

Let $F = U_1^TXU_2,\quad G = U_2^TXU_1$, it is easy to verify that $F^T = -G$. Then we have

$$X = U_1FU_2^T + U_2GU_1^T = U\begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix}U^T = U\begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix}U^T$$

Conversely, for any $F \in R^{r\times(n-r)}$, let

$$X = U\begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix}U^T.$$

It is easy to verify that $X^T = -X$. Using (2.2), we have

$$PXP = PU\begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix}U^TP$$

$$= U\begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}U^TU\begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix}U^TU\begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}U^T$$
$$= U\begin{pmatrix} 0 & -F \\ F^T & 0 \end{pmatrix}U^T = -X.$$

This implies that $X = U\begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix}U^T \in ASR_P^n$. □

LEMMA 2.4. *Let $A \in R^{n\times n}$, $D \in ASR^{n\times n}$ and assume $A - A^T = D$. Then there is precisely one $G \in SR^{n\times n}$ such that*

$$A = L_D + G,$$

and $G = \frac{1}{2}(A + A^T) - \frac{1}{2}(L_D + L_D^T)$.

Proof. For given $A \in R^{n\times n}$, $D \in ASR^{n\times n}$ and $A - A^T = D$, it is easy to verify that there exist unique $G = \frac{1}{2}(A + A^T) - \frac{1}{2}(L_D + L_D^T) \in SR^{n\times n}$, and we have $A = \frac{1}{2}(A - A^T) + \frac{1}{2}(A + A^T) = \frac{1}{2}(L_D - L_D^T) + \frac{1}{2}(A + A^T) = L_D + \frac{1}{2}(A + A^T) - \frac{1}{2}(L_D + L_D^T) = L_D + G$. □

Let $A \in R^{n\times m}$ and $D \in R^{m\times m}$, U defined in (2.2). Set

$$U^TA = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},\quad A_1 \in R^{r\times m},\quad A_2 \in R^{(n-r)\times m}. \tag{2.4}$$

The generalized singular value decomposition (see [11,12,13]) of the matrix pair $[A_1^T, A_2^T]$ is

$$A_1^T = M\Sigma_{A_1}W^T,\quad A_2^T = M\Sigma_{A_2}V^T, \tag{2.5}$$

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where $W \in C^{m\times m}$ is a nonsingular matrix, $W \in OR^{r\times r}$, $V \in OR^{(n-r)\times(n-r)}$ and

$$\Sigma_{A_1} = \begin{array}{c} \left(\begin{array}{cccc} I_k & & & \\ & S_1 & & \\ & & & O_1 \\ \hline & O & & \end{array}\right) \end{array} \begin{array}{c} k \\ s \\ t-k-s \\ m-t \end{array}, \tag{2.6}$$

$$\Sigma_{A_2} = \begin{array}{c} \left(\begin{array}{cccc} O_2 & & & \\ & S_2 & & \\ & & & I_{t-k-s} \\ \hline & O & & \end{array}\right) \end{array} \begin{array}{c} k \\ s \\ t-k-s \\ m-t \end{array}, \tag{2.7}$$

where

$$t = \operatorname{rank}(A_1^T, A_2^T), \quad k = t - rank(A_2^T),$$

$$s = \operatorname{rank}(A_1^T) + \operatorname{rank}(A_2^T) - t$$

$$S_1 = \operatorname{diag}(\alpha_1, \cdots, \alpha_s), \quad S_2 = \operatorname{diag}(\beta_1, \cdots, \beta_s),$$

with $1 > \alpha_1 \geq \cdots \geq \alpha_s > 0, \quad 0 < \beta_1 \leq \cdots \leq \beta_s < 1$, and $\alpha_i^2 + \beta_i^2 = 1, \quad i = 1, \cdots, s$.

O, O_1 and O_2 are corresponding zero submatrices.

Then we can immediately obtain the following theorem about the general solution to Problem I.

THEOREM 2.5. *Given $A \in R^{n\times m}$ and $D \in R^{m\times m}$, U defined in (2.2), and U^TA has the partition form of (2.4), the generalized singular value decomposition of the matrix pair $[A_1^T, A_2^T]$ as (2.5). Partition the matrix $M^{-1}DM^{-T}$ as*

$$M^{-1}DM^{-T} = \begin{array}{c} \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix} \\ \begin{array}{cccc} k & s & t-k-s & m-t \end{array} \end{array} \begin{array}{c} k \\ s \\ t-k-s \\ m-t \\ \ \end{array},$$

then the problem I has a solution $X \in ASR_P^n$ if and only if

$$D^T = -D, D_{11} = O, D_{33} = O, D_{41} = O, D_{42} = O, D_{43} = O, D_{44} = O. \tag{2.8}$$

In that case it has the general solution

$$X = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T, \tag{2.9}$$

where

$$F = W \begin{pmatrix} X_{11} & D_{12}S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1}(L_{D_{22}} + G)S_2^{-1} & S_1^{-1}D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} V^T, \tag{2.10}$$

with $X_{11} \in R^{r\times(n-r+k-t)}$, $X_{21} \in R^{s\times(n-r+k-t)}$, $X_{31} \in R^{(r-k-s)\times(n-r+k-t)}$, $X_{32} \in R^{(r-k-s)\times s}$, $X_{33} \in R^{(r-k-s)\times(t-k-s)}$ and $G \in SR^{s\times s}$ are arbitrary matrices.

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Proof. The necessity. Assume Eq.(1.1) has a solution $X \in ASR_P^n$. By the definition of ASR_P^n, it is easy to verify that $D^T = -D$, and we have from Lemma 2.3 that X can be expressed as

$$X = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T. \tag{2.11}$$

where $F \in R^{r\times(n-r)}$.

Note that U is an orthogonal matrix, and the definition of $A_i(i = 1, 2)$, Eq.(1.1) is equivalent to

$$A_1^T F A_2 - A_2^T F A_1 = D. \tag{2.12}$$

Substituting (2.5) into (2.12), then we have

$$\Sigma_{A_1}(W^T F V)\Sigma_{A_2}^T - \Sigma_{A_2}(W^T F V)^T \Sigma_{A_1}^T = M^{-1} D M^{-T}, \tag{2.13}$$

Partition the matrix $W^T F V$ as

$$W^T F V = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}, \tag{2.14}$$

where $X_{11} \in R^{r\times(n-r+k-t)}$, $X_{22} \in R^{s\times s}$, $X_{33} \in R^{(r-k-s)\times(t-k-s)}$.

Taking $W^T F V$ and $M^{-1} D M^{-T}$ into (2.13), we have

$$\begin{pmatrix} 0 & X_{12}S_2 & X_{13} & 0 \\ -S_2 X_{21}^T & S_1 X_{22} S_2 - (S_1 X_{22} S_2)^T & S_1 X_{23} & 0 \\ -X_{13}^T & -X_{23}^T S_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix}. \tag{2.15}$$

Therefore (2.15) holds if and only if (2.8) holds and

$$X_{12} = D_{12}S_2^{-1}, \quad X_{13} = D_{13}, \quad X_{23} = S_1^{-1} D_{23}$$

and

$$S_1 X_{22} S_2 - (S_1 X_{22} S_2)^T = D_{22}.$$

It follows from Lemma 2.4 that $X_{22} = S_1^{-1}(L_{D_{22}} + G)S_2^{-1}$, where $G \in SR^{s\times s}$ is arbitrary matrix. Substituting the above into (2.14), (2.11), thus we have formulation (2.9) and (2.10).

The sufficiency. Let

$$F_G = W \begin{pmatrix} X_{11} & D_{12}S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1}(L_{D_{22}} + G)S_2^{-1} & S_1^{-1} D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} V^T.$$

Obviously, $F_G \in R^{r\times(n-r)}$. By Lemma 2.3 and

$$X_0 = U \begin{pmatrix} 0 & F_G \\ -F_G^T & 0 \end{pmatrix} U^T,$$

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we have $X_0 \in ASR_P^n$. Hence

$$A^T X_0 A = A^T UU^T X_0 UU^T A = \begin{pmatrix} A_1^T & A_2^T \end{pmatrix} \begin{pmatrix} 0 & F_G \\ -F_G & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$
$$= \begin{pmatrix} A_2^T(-F_G) & A_1^T F_G \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = A_2^T(-F_G)A_1 + A_1^T F_G A_2$$
$$= M \begin{pmatrix} 0 & D_{12}S_2^T & D_{13} & 0 \\ -S_2 D_{12}^T & L_{D_{22}} - L_{D_{22}}^T & D_{23} & 0 \\ -D_{13}^T & -D_{23}^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} M^T = D.$$

This implies that $X_0 = U \begin{pmatrix} 0 & F_G \\ -F_G^T & 0 \end{pmatrix} U^T \in ASR_P^n$ is the anti-symmetric ortho-symmetric solution of Eq. (1.1). The proof is completed. □

3. The expression of the solution of Problem II. To prepare for an explicit expression for the solution of the matrix nearness problem II, we first verify the following lemma.

LEMMA 3.1. *Suppose that $E, F \in R^{s\times s}$, and let $S_a = diag(a_1, \cdots, a_s) > 0$, $S_b = diag(b_1, \cdots, b_s) > 0$. Then there exist a unique $S_s \in SR^{s\times s}$ and a unique $S_r \in ASR^{s\times s}$ such that*

$$\|S_a S S_b - E\|^2 + \|S_a S S_b - F\|^2 = \min \tag{3.1}$$

and

$$S_s = \Phi * [S_a(E + F)S_b + S_b(E + F)^T S_a], \tag{3.2}$$

$$S_r = \Phi * [S_a(E + F)S_b - S_b(E + F)^T S_a], \tag{3.3}$$

where

$$\Phi = (\psi_{ij}) \in SR^{s\times s}, \psi_{ij} = \frac{1}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, 1 \le i, j \le s. \tag{3.4}$$

Proof. We prove only the existence of S_r and (3.3). For any $S = (s_{ij}) \in ASR^{s\times s}$, $E = (e_{ij}), F = (f_{ij}) \in R^{s\times s}$, since $s_{ii} = 0$, $s_{ij} = -s_{ji}$,

$$\|S_a S S_b - E\|^2 + \|S_a S S_b - F\|^2 = \sum_{1\le i,j\le s} [(a_i b_j s_{ij} - e_{ij})^2 + (a_i b_j s_{ij} - f_{ij})^2]$$

$$= \sum_{1\le i<j\le s} [(a_i b_j s_{ij} - e_{ij})^2 + (-a_j b_i s_{ij} - e_{ji})^2 + (a_i b_j s_{ij} - f_{ij})^2 + (-a_j b_i s_{ij} - f_{ji})^2] +$$

$$+ \sum_{1\le i\le s} (e_{ij}^2 + e_{ij}^2).$$

Hence, there exists a unique solution $S_r = (\hat{s_{ij}}) \in ASR^{s\times s}$ for (3.1) such that

$$\hat{s_{ij}} = \frac{a_i b_j (e_{ij} + f_{ij}) - a_j b_i (e_{ji} + f_{ji})}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, \quad 1 \le i, j \le s.$$

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This amounts to the same as (3.3). □

THEOREM 3.2. *Let $\tilde{X} \in R^{n\times n}$, the generalized singular value decomposition of the matrix pair $[A_1^T, A_2^T]$ as (2.5), let*

$$U^T\tilde{X}U = \begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{pmatrix}, \tag{3.5}$$

$$W^T Z_{12}^* V = \begin{pmatrix} X_{11}^* & X_{12}^* & X_{13}^* \\ X_{21}^* & X_{22}^* & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix}, \quad W^T Z_{21}^{*T} V = \begin{pmatrix} Y_{11}^* & Y_{12}^* & Y_{13}^* \\ Y_{21}^* & Y_{22}^* & Y_{23}^* \\ Y_{31}^* & Y_{32}^* & Y_{33}^* \end{pmatrix}, \tag{3.6}$$

if Problem I is solvable, then Problem II has a unique solution $\hat{X}$, which can be expressed as

$$\hat{X} = U \begin{pmatrix} 0 & \tilde{F} \\ -\tilde{F}^T & 0 \end{pmatrix} U^T, \tag{3.7}$$

where

$$\tilde{F} = W \begin{pmatrix} \frac{1}{2}(X_{11}^* - Y_{11}^*) & D_{12}S_2^{-1} & D_{13} \\ \frac{1}{2}(X_{21}^* - Y_{21}^*) & S_1^{-1}(L_{D_{22}} + \tilde{G})S_2^{-1} & S_1^{-1}D_{23} \\ \frac{1}{2}(X_{31}^* - Y_{31}^*) & \frac{1}{2}(X_{32}^* - Y_{32}^*) & \frac{1}{2}(X_{33}^* - Y_{33}^*) \end{pmatrix} V^T,$$

$$\tilde{G} = \Phi * [S_1^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})S_2^{-1} + S_2^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})^T S_1^{-1}],$$

with

$$\Phi = (\psi_{ij}) \in SR^{s\times s}, \quad \psi_{ij} = \frac{a_i^2 a_j^2 b_i^2 b_j^2}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, \quad 1 \le i, j \le s.$$

Proof. Using the invariance of the Frobenius norm under unitary transformations, from (2.9), (3.5) and (3.6) we have

$$||X - \tilde{X}||^2 = ||Z_{11}^*||^2 + ||F - Z_{12}^*||^2 + ||-F^T - Z_{21}^*||^2 + ||Z_{22}^*||^2$$

$$= \left\| \begin{pmatrix} X_{11} & D_{12}S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1}(L_{D_{22}} + G)S_2^{-1} & S_1^{-1}D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} - W^T Z_{12}^* V \right\|^2 + ||Z_{11}^*||^2$$
$$+ \left\| \begin{pmatrix} X_{11} & D_{12}S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1}(L_{D_{22}} + G)S_2^{-1} & S_1^{-1}D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} + W^T Z_{21}^{*T} V \right\|^2 + ||Z_{22}^*||^2.$$

Thus

$$\|\tilde{X} - \hat{X}\| = \inf_{X \in S_E} \|\tilde{X} - X\|$$

is equivalent to

$$\| X_{11} - X_{11}^* \|^2 + \| X_{11} + Y_{11}^* \|^2 = \min, \quad \| X_{21} - X_{21}^* \|^2 + \| X_{21} + Y_{21}^* \|^2 = \min,$$

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$$\| X_{31} - X_{31}^* \|^2 + \| X_{31} + Y_{31}^* \|^2 = \min, \quad \| X_{32} - X_{32}^* \|^2 + \| X_{32} + Y_{32}^* \|^2 = \min,$$

$$\| X_{33} - X_{33}^* \|^2 + \| X_{33} + Y_{33}^* \|^2 = \min,$$

$$||S_1^{-1}GS_2^{-1} - (X_{22}^* - S_1^{-1}L_{D_{22}}S_2^{-1})||^2 + ||S_1^{-1}GS_2^{-1} + (Y_{22}^* + S_1^{-1}L_{D_{22}}S_2^{-1})||^2 = \min.$$

From Lemma 3.1 we have $X_{11} = \frac{1}{2}(X_{11}^* - Y_{11}^*),\ X_{21} = \frac{1}{2}(X_{21}^* - Y_{21}^*),\ X_{31} = \frac{1}{2}(X_{31}^* - Y_{31}^*),\ X_{32} = \frac{1}{2}(X_{32}^* - Y_{32}^*),\ X_{33} = \frac{1}{2}(X_{33}^* - Y_{33}^*)$ and

$$G = \Phi * [S_1^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})S_2^{-1} + S_2^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})^T S_1^{-1}].$$

Taking $X_{11}, X_{21}, X_{31}, X_{32}, X_{33}$ and G into (2.9), (2.10), we obtain that the solution of (the matrix nearness) Problem II can be expressed as (3.7). $\square$

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