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AN UPPER BOUND ON THE REACHABILITY INDEX FOR A SPECIAL CLASS OF POSITIVE 2-D SYSTEMS*

ESTEBAN BAILO†, JOSEP GELONCH†, AND SERGIO ROMERO-VIVÓ‡

Abstract. The smallest number of steps needed to reach all nonnegative local states of a positive two-dimensional (2-D) system is the local reachability index of the system. The study of such a number is still an open problem which seems to be a hard task. In this paper, an expression depending on the dimension n as well as an upper bound on the local reachability index of a special class of systems are derived. Moreover, this reachability index is greater than any other bound proposed in previous literature.

Key words. Hurwitz products, Influence digraph, Local reachability index, Nonnegative matrices, Positive two-dimensional (2-D) systems, Reachability.

AMS subject classifications. 93C55, 15A48, 93B03.

1. Introduction.

Positive two-dimensional systems have received considerable attention in the last decade, as they naturally arise in different physical problems such as the pollutant diffusion in a river (see [16]), gas absorption and water stream heating (see [25]), among others. The structural properties of these 2-D state-space models have been recently analyzed in [18], [21] and [22].

One of the most frequent representations of positive 2-D systems is the Fornasini-Marchesini state-space model (see [17] and [18]) which is as follows:

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \tag{1.1}$$

with local states $x_{\cdot,\cdot} \in \mathbb{R}^n_+$, inputs $u_{\cdot,\cdot} \in \mathbb{R}^m_+$, state matrices $A_1, A_2 \in \mathbb{R}^{n\times n}_+$, input matrices $B_1, B_2 \in \mathbb{R}^{n\times m}_+$ and initial global state $\chi_0 := \{x_{h,k} \ : \ (h,k) \in \mathcal{C}_0\}$ being the separation set $\mathcal{C}_0 := \{(h,k) \ : \ h,k \in \mathbb{Z}, h+k=0\}$. Let us denote this system by (A_1, A_2, B_1, B_2).

If any nonnegative local state is achieved from the zero initial global state choosing a nonnegative input sequence, then the system is said to be locally reachable. Local

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†Departament de Matemàtica, Universitat de Lleida, Av. Alcalde Rovira Roure 191, 25198 Lleida, Spain (ebailo@matematica.udl.es, jgelonch@matematica.udl.es).

‡Institut de Matemàtica Multidisciplinar. Departament de Matemàtica Aplicada, Universitat Politècnica de València, València, Spain (sromero@imm.upv.es). Supported by Spanish DGI grant number MTM2007-64477 and DPI2007-66728-C02-01.

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reachability is equivalent to the possibility, in a finite number of steps, of attaining each vector of the standard basis of $\mathbb{R}^n$ or alternatively a positive multiple of the aforementioned vectors (see [18]).

The smallest number of steps needed to reach all local states of the system is the local reachability index of that system. This index plays an important role because it enables us to determine in a finite number of steps whether a system is locally reachable or not. Moreover, once this index is obtained, several new open problems could be addressed such as the study of reachability algorithms (see [4], [11] and [24]), of a canonical form by means of an appropriated positive similarity ([5] and [7]), of the definition of the pertaining complete sequence of invariants (see [5], [8] and [10]) as well as the analysis of this property under feedback ([6] and [20]) and so on.

The reachability index for positive 1-D systems has been studied in [5], [9], [14], [15] and [26] among others, and it is always bounded by n (see [12] and [13]). However, despite the numerous research efforts, an upper bound on such a number for a positive 2-D system is still an unanswered question.

With regard to this subject, different studies such as [18] and [23] have provided the first results, which have been revised in [2] and [3]. Lately, in [1], the local reachability index has been characterized for a particular class of positive 2-D systems, which are a generalization of the systems presented in [2] and [3], and an upper bound on this index has been derived which even turns to be $\frac{(n+1)^2}{4}$ in suitable conditions. However, it is worth mentioning that the obtained upper bound is valid for this class of chosen systems but not in general.

This work has been organized as follows: Section II presents some notations and basic definitions. Finally, in Section III, a special class of positive 2-D systems is shown to be always (positively) locally reachable. Moreover, an expression depending on the dimension n for the corresponding indices is deduced as well as its associated tight upper bound.

2. Notations and Preliminary Definitions. We denote by $\lfloor z \rfloor$ the lower integer-part of $z \in \mathbb{R}$ and by $col_j(A)$ the jth column of the matrix A.

DEFINITION 2.1. *The Hurwitz products* of the $n \times n$ matrices A_1 and A_2 are defined as follows:

- $A_1{}^i \sqcup\!\sqcup^j A_2 = 0$, when either i or j is negative,
- $A_1{}^i \sqcup\!\sqcup^0 A_2 = A_1^i$, if $i \geq 0$, $A_1{}^0 \sqcup\!\sqcup^j A_2 = A_2^j$, if $j \geq 0$,
- $A_1{}^i \sqcup\!\sqcup^j A_2 = A_1(A_1{}^{i-1} \sqcup\!\sqcup^j A_2) + A_2(A_1{}^i \sqcup\!\sqcup^{j-1} A_2)$, if $i, j > 0$.

Note that $\displaystyle\sum_{i+j=\ell} A_1{}^i \sqcup\!\sqcup^j A_2 = (A_1 + A_2)^\ell$.

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Definition 2.2. (see [18]) A 2-D state-space model (1.1) is *(positively) locally reachable* if, upon assuming $\chi_0 = 0$, for every state $x^* \in \mathbb{R}^n_+$, there exists $(h, k) \in \mathbb{Z}\times\mathbb{Z}$, $h + k > 0$, and a nonnegative input sequence $u_{\cdot,\cdot}$ such that $x_{h,k} = x^*$. When so, the state is said to be *(positively) reachable in $h + k$ steps*. The smallest number of steps allowing to reach all nonnegative local states represents *the local reachability index* I_{LR} of such a system.

Characterizations of the local reachability of the positive 2-D systems (1.1) can be established in terms of the reachability matrix of the quadruple (A_1, A_2, B_1, B_2). The reachability matrix in k-steps is given by

$$\begin{aligned}\mathcal{R}_k &= [\, B_1 \;\; B_2 \;\; A_1B_1 \;\; A_1B_2 + A_2B_1 \;\; A_2B_2 \;\; A_1^2B_1 \;\; \cdots \;\; A_2^{k-1}B_2 \,] \\ &= [\, (A_1{}^{i-1} \sqcup\!\sqcup^{j} A_2)B_1 + (A_1{}^{i} \sqcup\!\sqcup^{j-1} A_2)B_2 \,]_{i,j\geq 0,\ 0<i+j\leq k}\end{aligned}$$

where k belongs to $\mathbb{N}$. It is known (see [18]) that the local reachability property holds if and only if there are n pairs $(h_i, k_i) \in \mathbb{N}\times\mathbb{N}$, $i = 1, \dots, n$, and n indices $j = j(i) \in \{1, 2, \dots, m\}$ such that $(A_1{}^{h_i-1} \sqcup\!\sqcup^{k_i} A_2)col_j(B_1) + (A_1{}^{h_i} \sqcup\!\sqcup^{k_i-1} A_2)col_j(B_2)$ is a positive ith monomial vector, that is, there exists $k \in \mathbb{N}$ such that $\mathcal{R}_k$ contains an $n\times n$ monomial matrix. We recall that a positive ith monomial vector (or simply i-monomial vector throughout this paper) is a positive multiple of the ith unit vector of $\mathbb{R}^n$. In the same way, a monomial matrix is a nonsingular matrix having a unique positive entry in each row and column.

To study the properties of the local reachability index we use digraph theory. Namely, we consider a family of coloured digraphs constructed from the matrices of the system (A_1, A_2, B_1, B_2) as follows:

Definition 2.3. *(see [18])* Associated with system (1.1), a directed digraph called *2-D influence digraph* is defined. It is denoted by $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ and it is given by $(S, V, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2)$, where $S = \{s_1, s_2, \dots, s_m\}$ is the set of *sources*, $V = \{v_1, v_2, \dots, v_n\}$ is the set of *vertices*, $\mathcal{A}_1$ and $\mathcal{A}_2$ are subsets of $V\times V$ whose elements are named $\mathcal{A}_1$*-arcs* and $\mathcal{A}_2$*-arcs* (or simply 1-arcs and 2-arcs) respectively, while $\mathcal{B}_1$ and $\mathcal{B}_2$ are subsets of $S\times V$ whose elements are tagged $\mathcal{B}_1$*-arcs* and $\mathcal{B}_2$*-arcs* (or simply 1-arcs and 2-arcs) respectively. There is an $\mathcal{A}_1$-arc ($\mathcal{A}_2$-arc) from v_j to v_i if and only if the (i, j)th entry of A_1 (A_2) is nonzero. There is a $\mathcal{B}_1$-arc ($\mathcal{B}_2$-arc) from s_ℓ to v_i if and only if the (i, ℓ)th entry of B_1 (B_2) is nonzero.

Definition 2.4. A *path* in $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ from v_{i_1} to v_{i_p} is an alternating sequence of vertices and arcs $\{v_{i_1}, (v_{i_1}, v_{i_2}), v_{i_2}, \dots, (v_{i_{p-1}}, v_{i_p}), v_{i_p}\}$ such that $(v_{i_k}, v_{i_{k+1}}) \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ for all $k = 1, 2, \dots, p-1$. A path is termed *closed* if the initial and final vertices coincide. In accordance with reference [18], an s_ℓ*-path* is a path where $v_{i_1} = s_\ell$. The *path length* is defined to be equal to the number of arcs it contains, that is, $p + q$ being p (q) the number of 1-arcs (2-arcs) occurring in

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Volume 18, pp. 1-12, January 2009

the path $\mathcal{P}$. Furthermore, the pair (p, q) is called the *composition* of $\mathcal{P}$. Finally, for abbreviation, a *circuit* is a closed path and if each vertex appears exactly once as the first vertex of an arc, then the circuit is said to be a *cycle*.

Remark 2.5. Let $\mathcal{P}$ and $\mathcal{Q}$ be two paths $\{v_{i_1}, (v_{i_1}, v_{i_2}), v_{i_2}, \ldots, (v_{i_{p-1}}, v_{i_p}), v_{i_p}\}$ and $\{v_{j_1}, (v_{j_1}, v_{j_2}), v_{j_2}, \ldots, (v_{j_q-1}, v_{j_q}), v_{i_q}\}$ such that $v_{i_p} = v_{j_1}$. To shorten, we denote the path from v_{i_1} to v_{j_q} consisting of the suitable adjacent joint of $\mathcal{P}$ and $\mathcal{Q}$ briefly by $\mathcal{P} \sqcup \mathcal{Q}$. Besides that, if $\mathcal{C}$ is a cycle, from now on, $\eta\mathcal{C}$ stands for the circuit resulting of doing η laps around the cycle, η being a positive integer.

Remark 2.6. In the sequel, to avoid ambiguities, we will write an 1-arc (2-arc) connecting two consecutive vertices v_k and v_{k+1} simply as $v_k \longrightarrow v_{k+1}$ ($v_k \dashrightarrow v_{k+1}$), that is, using continuous arrows (dashed arrows).

Definition 2.7*(see [18] and [19])* If there exists an s_ℓ-path in $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ from the source s_ℓ to the vertex v_i, then v_i is said to be *reachable* from s_ℓ. Besides that, if for any $\ell \in \{1, \ldots, m\}$, all s_ℓ-paths of composition (p, q) end in the same vertex $v \in V$, then v is said to be *deterministically reachable* from the source s_ℓ with composition (p, q).

The shortest length of the s_ℓ-paths deterministically reaching v is called the *ℓ-index* of v, i.e.

$$\mathcal{I}_\ell(v) =$$

$$\min\{p + q \,|\, (p, q)_\ell \text{ is the composition of an } \mathrm{s}_\ell - \text{path deterministically reaching v}\}.$$

The *determination index* of v is $I_D(v) = \min\limits_{\ell=1,\ldots,m}\{\mathcal{I}_\ell(v)\}$.

We observe that $I_D(v)$ stands for the minimum number of steps where a positive multiple of the vth vector of the standard basis of $\mathbb{R}^n$ is deterministically reachable. Therefore, taking into account that the system is locally reachable when each one of the vth vectors for any $v \in V$ is deterministically reachable (see [18]), then, obviously, the local reachability index I_{LR} of a positive 2-D system is the maximum of the determination indices v for all $v \in V$, that is, $I_{LR} = \max\limits_{v \in V}\{I_D(v)\}$.

3. Local Reachability Index for a new family of systems. In [3], the authors showed with the following example that the local reachability index can be greater than $\frac{(n+1)^2}{4}$:

Example 3.1. *Let (A_1, A_2, B_1, B_2) be a positive 2-D system given by*

$$(A_1, A_2, B_1, B_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

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ELA

with a 2-D influence digraph corresponding to Fig. 3.1 where continuous arrows (dashed arrows) represent 1-arcs (2-arcs). Thus, the system is locally reachable with $I_{LR} = 6$, *while* $\frac{(n+1)^2}{4} = 4$ *for* $n = 3$.

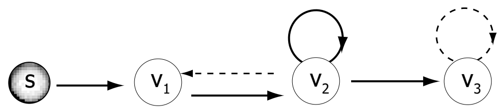

FIG. 3.1. *Digraph for the system in example 3.1*

On the basis of this example, we have obtained a new family of systems whose local reachability index largely exceeds the prior upper bound.

Let us consider an nth order positive 2-D system (A_1, A_2, B_1, B_2) with $n \geq n_1 \geq 3$, $B_2 = O$, where O is the zero matrix of an appropriate size, $B_1 = [\,\alpha e_1\,]$ with $\alpha > 0$ (that is, a 1-monomial column vector) and 2-D influence digraph given in Fig. 3.2 where the vertices $v_1, v_2, \ldots, v_n$ have been relabeled as the subindices $1, 2, \ldots, n$ to simplify. The preceding replacement is also carried out throughout this paper.

Therefore, the matrices of the system (A_1, A_2, B_1, B_2) are given as follows:

$$(3.1)\; A_1 = \left[\begin{array}{c|c} \hat{A}'_{11} & \hat{A}'_{12} \\ \hline O & O \end{array}\right], \quad A_2 = \left[\begin{array}{c|c} \hat{A}''_{11} & O \\ \hline \hat{A}''_{21} & \hat{A}''_{22} \end{array}\right], \; B_1 = [\,\alpha e_1\,] \text{ and } B_2 = [\,O\,]$$

where

$$\hat{A}'_{11} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ + & 0 & \cdots & 0 & 0 & 0 \\ 0 & + & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & + & + & 0 \\ 0 & 0 & \cdots & 0 & + & 0 \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \; \hat{A}'_{12} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & + \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_1 \times (n-n_1)},$$

$$\hat{A}''_{11} = \begin{bmatrix} 0 & + & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & + \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \hat{A}''_{21} = \begin{bmatrix} + & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(n-n_1) \times (n_1)},$$

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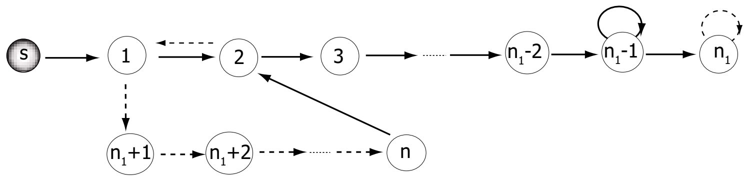

FIG. 3.2. *Digraph for the new family of systems.*

and

$$\hat{A}''_{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ + & 0 & \cdots & 0 & 0 \\ 0 & + & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & + & 0 \end{bmatrix} \in \mathbb{R}^{(n-n_1)\times(n-n_1)},$$

$+$ denoting a strictly positive entry.

Notice that we may consider for the sake of simplicity all strictly positive entries equal to one. Furthermore, it is worth pointing out that we may assume that the digraph for this family of systems is reduced to be the digraph given in Fig. 3.3 in the case $n = n_1$. Then, we observe that the matrices of the system given in example 3.1 are structured as those seen above for $n = 3$ and $n_1 = 3$. It is convenient to establish now two preliminary technical lemmas in order to get an expression of the local reachability index for this class of systems.

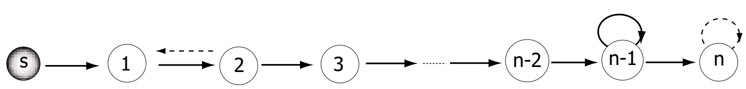

FIG. 3.3. *Digraph for the new family of systems in the case $n = n_1$.*

Preliminarily, to simplify notation, we define the paths (see Fig. 3.2) $\mathcal{P}_0 \equiv \{s \longrightarrow 1\}$, $\mathcal{P}_1^k \equiv \{1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow k\}$ for $2 \leq k \leq n_1 - 1$, $\mathcal{P}_1 = \mathcal{P}_1^{n_1-1}$, $\mathcal{P}_2^j \equiv \{1 \dashrightarrow (n_1{+}1) \dashrightarrow \cdots \dashrightarrow j\}$ for $n_1{+}1 \leq j \leq n$, $\mathcal{P}_2 = \mathcal{P}_2^n$, $\mathcal{P}_3 \equiv \{n \longrightarrow 2 \dashrightarrow 1\}$ and $\mathcal{P}_4 \equiv \{(n_1 - 1) \longrightarrow n_1\}$ whose compositions are $(1, 0)$, $(k - 1, 0)$, $(n_1 - 2, 0)$, $(0, j - n_1)$, $(0, n - n_1)$, $(1, 1)$ and $(1, 0)$ respectively. In the same way, we define the cycles $\mathcal{C}_1 \equiv \{1 \longrightarrow 2 \dashrightarrow 1\}$, $\mathcal{C}_2 \equiv \{(n_1 - 1) \longrightarrow (n_1 - 1)\}$, $\mathcal{C}_3 \equiv \{n_1 \dashrightarrow n_1\}$ and $\mathcal{C}_4 = \mathcal{P}_2 \sqcup \mathcal{P}_3$ with respective compositions $(1, 1)$, $(1, 0)$, $(0, 1)$ and $(1, n - n_1 + 1)$.

LEMMA 3.2. *If there exists an s-path of composition (p, q) ending in vertex 1,*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 1-12, January 2009

then there is at least an s*-path of composition* $(p+\lambda, q+\delta)$ *terminating at a vertex belonging to* $V \setminus \{n_1\}$ *for every* λ *and* $\delta \in \mathbb{Z}_+$ *such that* $0 \leq \delta \leq n - n_1 + \lambda$.

Proof. Firstly, assuming that we have an s-path $\mathcal{P}$ of composition (p, q) ending in vertex 1, then the cycle $\mathcal{C}_1$ allows us to affirm that for all $\eta \in \mathbb{Z}_+$ there is also an s-path of composition $(p+\eta, q+\eta)$ terminating at vertex 1 since this cycle adds (η, η) to the composition of the initial path after doing η laps.

Combining these s-paths $\mathcal{P} \sqcup \eta\mathcal{C}_1$ with both the paths $\mathcal{P}_1^k$ for $2 \leq k \leq n_1 - 1$ and the path $\mathcal{P}_1$ and the cycle $\mathcal{C}_2$, it is immediately seen that for all $\alpha \in \mathbb{Z}_+$ we can find s-paths of composition $(p+\eta+\alpha, q+\eta)$ ending in a vertex belonging to $V_1 = \{1, 2, 3, \ldots, n_1 - 1\}$. Thus, taking $\lambda = \eta + \alpha$ and $\delta = \eta$, it is inferred that there are s-paths of composition $(p+\lambda, q+\delta)$ terminating at a vertex belonging to V_1 for every $\lambda \in \mathbb{Z}_+$ and $0 \leq \delta \leq \lambda$.

Consequently, if $n_1 = n \geq 3$, then the assertion of this theorem is established because, in this case, the inequality $0 \leq \delta \leq n - n_1 + \lambda$ is reduced to $0 \leq \delta \leq \lambda$ and $V \setminus \{n_1\} = V_1$.

Secondly, if $n > n_1 \geq 3$, the paths $\mathcal{P}_2^j$, $n_1 + 1 \leq j \leq n$, allow us to claim that for all $\beta \in \mathbb{Z}_+$ satisfying $1 \leq \beta \leq n - n_1$ there is an s-path of composition $(p+\eta, q+\eta+\beta)$ terminating at a vertex belonging to $V_2 = \{n_1+1, n_1+2, \ldots, n\}$. Choosing $\lambda = \eta$ and $\delta = \eta+\beta$, we have an s-path of composition $(p+\lambda, q+\delta)$ ending in a vertex belonging to V_2 for all λ, $\delta \in \mathbb{Z}_+$ with $\lambda + 1 \leq \delta \leq n - n_1 + \lambda$. Hence, since $V \setminus \{n_1\} = V_1 \cup V_2$ the statement of this lemma is easily derived from. $\square$

REMARK 3.3. In the event that $n > n_1$, we stress that for a fixed value $p \in \mathbb{N}$ the s-paths consisting exactly of p 1-arcs which terminate at vertex 1 are only those equal to $\mathcal{P}_0 \sqcup \alpha\mathcal{C}_1 \sqcup \beta\mathcal{C}_4$ with $\alpha + \beta + 1 = p$ (except for some other rearrangements of the same cycles, for instance, $\mathcal{P}_0 \sqcup \mathcal{C}_4 \sqcup \mathcal{C}_1 \sqcup (\beta - 1)\mathcal{C}_4 \sqcup (\alpha - 1)\mathcal{C}_1$). With respect to their lengths, the longest path is clearly $\mathcal{P}_0 \sqcup (p-1)\mathcal{C}_4$. Then, $\mathcal{P}_0 \sqcup (p-1)\mathcal{C}_4 \sqcup \mathcal{P}_2$ is the greatest in length among those s-paths consisting precisely of p 1-arcs which end in a vertex belonging to $V \setminus \{n_1\}$ and this vertex is n.

Analogously, $\mathcal{P}_0 \sqcup (p-1)\mathcal{C}_1$ is the longest s-path among those of composition consisting precisely of p 1-arcs which terminate at $V \setminus \{n_1\}$, namely at vertex 1, when $n = n_1$.

LEMMA 3.4. *Let* p *be any natural number, then there is an* s*-path of composition* $(p, \beta), \beta \in \mathbb{Z}_+$*, ending in a vertex belonging to* $V \setminus \{n_1\}$ *if and only if* $0 \leq \beta < (n-n_1+1)p$*. Besides that, there is a unique* s*-path of composition* $(p, (n-n_1+1)p-1)$ *terminating at a vertex belonging to* $V \setminus \{n_1\}$.

Proof. *Case* $n > n_1 \geq 3$: The proof is by induction on p. Initially, Lemma 3.2

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 1-12, January 2009

enables us to assert that we can find s-paths of composition $(1, \beta)$ with $0 \leq \beta < n - n_1 - 1$ ending in a vertex of $V \setminus \{n_1\}$ because the path $\mathcal{P}_0$ terminates at vertex 1 and has composition $(1, 0)$. Conversely, according to the 2-D influence digraph given in Fig. 3.2, the s-paths of composition $(1, \beta)$ are only $\mathcal{P}_0$ (with $\beta = 0$) and $\mathcal{P}_0 \sqcup \mathcal{P}_2^{n_1+\beta}$ where $1 \leq \beta \leq n - n_1$. We observe that each of them ends in a vertex belonging to $V \setminus \{n_1\}$ and in addition, for $\beta = n - n_1$, the corresponding s-path is exactly $\mathcal{P}_0 \sqcup \mathcal{P}_2$ whose final vertex is n. Hence, the assertion of this lemma for $p = 1$ is proven.

Let us assume that the same statement holds for $p = \alpha$, that is,

- There exist s-paths of composition (α, β) ending in a vertex of $V \setminus \{n_1\}$ if and only if

$$0 \leq \beta < (n - n_1 + 1)\alpha. \tag{3.2}$$

- There is a unique s-path of composition $(\alpha, (n - n_1 + 1)\alpha - 1)$ ending in a vertex belonging to $V \setminus \{n_1\}$. In fact, this s-path is precisely $\mathcal{P}_0 \sqcup (\alpha - 1)\mathcal{C}_4 \sqcup \mathcal{P}_2$ which terminates at vertex n.

Let us take now the case in which $p = \alpha + 1$. We know that for every $t \in \mathbb{N}$ there exist s-paths $\mathcal{P}_0 \sqcup (t-1)\mathcal{C}_4$ of composition $(t, (t-1)(n - n_1 + 1))$ terminating at vertex 1. In particular for each $t \in \mathbb{N}$ satisfying $1 \leq t \leq \alpha + 1$, we can apply successively Lemma 3.2 to each one of these paths, taking $p = t$, $q = (t-1)(n - n_1 + 1)$ and setting $\lambda = \alpha + 1 - t$, that is, such that $p + \lambda = \alpha + 1$. Hence, we can find at least an s-path of composition $(\alpha + 1, \beta)$ ending in a vertex of $V \setminus \{n_1\}$ for each $\beta \in \mathbb{Z}_+$ such that $q \leq \beta \leq q + n - n_1 + \lambda$, so:

$$(t-1)(n - n_1 + 1) \leq \beta \leq t(n - n_1 + 1) + \alpha - t\ . \tag{3.3}$$

Plugging the different values of t into equation (3.3), we can assure that there is at least an s-path of composition $(\alpha + 1, \beta)$ for each $0 \leq \beta < (n - n_1 + 1)(\alpha + 1)$ terminating at a vertex belonging to $V \setminus \{n_1\}$.

Let us suppose that there exists an s-path of composition $(\alpha + 1, \beta)$ ending in a vertex of $V \setminus \{n_1\}$ with $\beta \geq (n - n_1 + 1)(\alpha + 1)$ and that such a vertex is v. If $v \in V_1$ then we could construct an s-path of composition (α, β) for this same β simply removing the last 1-arc of the initial path. This contradicts our inductive assumption. If $v \in V_2$ then $v = n_1 + k$ where $k \in \{1, \ldots, n - n_1\}$. Hence, the path constructed by eliminating the last k 2-arcs of the starting path terminates at vertex 1. Moreover, it has composition $(\alpha + 1, \beta - k)$ with $\beta - k \geq (n - n_1 + 1)(\alpha + 1) - k > \alpha(n - n_1 + 1)$ but this is impossible since $\mathcal{P}' := \mathcal{P}_0 \sqcup \alpha\mathcal{C}_4$ of composition $(\alpha + 1, \alpha(n - n_1 + 1))$ is the longest among those s-paths consisting of $\alpha + 1$ 1-arcs and ending in vertex 1 (see remark 3.3).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 1-12, January 2009

We also conclude from remark 3.3 that $\mathcal{P}' \sqcup \mathcal{P}_2 = \mathcal{P}_0 \sqcup \alpha\mathcal{C}_4 \sqcup \mathcal{P}_2$ is the greatest in length among those s-paths consisting exactly of $\alpha + 1$ 1-arcs which terminate at vertex n. Thus, the proof is complete.

Case $n = n_1 \geq 3$: The same conclusion can be drawn for this situation, with the s-paths $\mathcal{P}_0 \sqcup (p-1)\mathcal{C}_4$ replaced by the s-paths $\mathcal{P}_0 \sqcup (p-1)\mathcal{C}_1$. It deserves to be mentioned that the unique s-path of composition $(\alpha, (n - n_1 + 1)\alpha - 1) = (\alpha, \alpha - 1)$ ends in vertex 1 instead of vertex n. □

The following theorem provides us an expression of the reachability index for the specific class of systems we are considering in this paper.

THEOREM 3.5. *Let (A_1, A_2, B_1, B_2) be a positive 2-D system given as in* (3.1). *Then, it is (positively) locally reachable and its local reachability index I_{LR} is:*

$$I_{LR} = n_1(n - n_1 + 2). \tag{3.4}$$

Proof. Case $n > n_1 \geq 3$: Let us examine the deterministically reachable vertices v and their corresponding determination indices $I_D(v)$. In particular, let us check that every vertex is deterministically reachable and that $I_D(n_1)$ provides the maximum determination index, that is, $I_D(n_1) = I_{LR}$.

We start by noting that we can find a unique path of composition $(\ell, 0)$ ending in ℓ, for all $\ell \in V_1 = \{1, \ldots, n_1 - 1\}$. Then each ℓ, $\ell \in V_1$, is deterministically reachable with composition $(\ell, 0)$. In fact,

$$(A_1{}^{\ell-1} \sqcup\!\sqcup^0 A_2)B_1 = (A_1{}^{\ell-1} \sqcup\!\sqcup^0 A_2)\alpha e_1 = \beta e_\ell.$$

where $\beta > 0$.

Likewise, every vertex ℓ with $\ell \in V_2 = \{n_1 + 1, \ldots, n\}$ is deterministically reachable with composition $(1, \ell - n_1)$ since

$$(A_1{}^0 \sqcup\!\sqcup^{\ell-n_1} A_2)B_1 = (A_1{}^0 \sqcup\!\sqcup^{\ell-n_1} A_2)\alpha e_1 = \gamma e_\ell.$$

where $\gamma > 0$. That is, there exists a unique path of composition $(1, \ell - n_1)$ terminating at ℓ, for all $\ell \in V_2$. Hence, obviously

$$\begin{gathered} I_D(1) = 1\ , I_D(2) = 2\ , \ldots, I_D(n_1 - 1) = n_1 - 1, \\ I_D(n_1 + 1) = 2\ , I_D(n_1 + 2) = 3, \ldots, I_D(n) = n - n_1 + 1. \end{gathered} \tag{3.5}$$

Therefore, to complete the local reachability analysis of this system, it is necessary to analyze whether the vertex n_1 is deterministically reachable or not as well as to evaluate its determination index $I_D(n_1)$ if possible. For this immediate purpose

Electronic Journal of Linear Algebra ISSN 1081-3810
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we must calculate the shortest length among those s-paths of 2-D influence digraph deterministically reaching n_1.

In order to obtain an s-path ending in vertex n_1, we emphasize that an s-path may end in vertex n_1 only if it contains at least n_1 1-arcs. Notice that the s-path $\mathcal{P}_0 \sqcup \mathcal{P}_1 \sqcup \mathcal{P}_4$ of composition $(n_1, 0)$ terminates at vertex n_1. Furthermore, in accordance with Lemma 3.4, there is an s-path of composition $(n_1 + \lambda, \beta)$ ending in $V \setminus \{n_1\}$ if and only if $0 \leq \beta < (n_1 + \lambda)(n - n_1 + 1)$. Hence, if there exists an s-path of composition $(n_1 + \lambda, \beta)$ with $\beta \geq (n_1 + \lambda)(n - n_1 + 1)$, then this one can terminate uniquely at vertex n_1. Note that it is straightforward that if there is an s-path of composition $(n_1, n_1(n - n_1 + 1))$ then it is the shortest among those s-paths of the 2-D influence digraph deterministically reaching n_1. Such an s-path of composition $(n_1, n_1(n-n_1+1))$ deterministically reaching n_1 is $\mathcal{P}_0 \sqcup \mathcal{P}_1 \sqcup \mathcal{P}_4 \sqcup (n_1(n-n_1+1))\mathcal{C}_3$. Thus, $I_D(n_1) = n_1 + (n - n_1 + 1)n_1 = n_1(n - n_1 + 2)$. Consequently, all vertices are deterministically reachable and the system is locally reachable. It is obvious that the determination index $I_D(n_1)$ is greater than the determination indices found in (3.5). Hence,

$$I_{LR} = I_D(n_1) = n_1 + n_1(n - n_1 + 1) = n_1(n - n_1 + 2),$$

which completes the proof.

Case $n = n_1 \geq 3$: The details are left to the reader since similar considerations as formerly indicated show that

$$I_D(1) = 1\ , I_D(2) = 2\ , \ldots , I_D(n-1) = n-1, I_D(n) = I_D(n_1) = 2n.$$

and hence the theorem follows. □

From this result we conclude:

Corollary 3.6. *The maximum local reachability index I_{LR} for a positive 2-D system given as in* (3.1) *for a natural number n_1 is attained at*

$$\begin{cases} n_1 = \dfrac{n}{2} + 1 & \text{if } n \text{ is an even natural number and} \\ n_1 = \dfrac{n+1}{2} \text{ or } \dfrac{n+3}{2} & \text{if } n \text{ is an odd natural number.} \end{cases} \tag{3.6}$$

Hence, the maximum I_{LR} is achieved for those systems satisfying $n_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and can be written uniformly like:

$$I_{LR} = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \left(n + 1 - \left\lfloor \frac{n}{2} \right\rfloor \right). \tag{3.7}$$

Moreover, I_{LR} is upper bounded by $\frac{(n+2)^2}{4}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 1-12, January 2009

Proof. For a fixed natural number n, we obtain that $n_1 = \frac{(n+2)}{2}$ is the point of maximum of the function $I_{LR} = I_D(n_1) = n_1(n - n_1 + 2)$, since its derivative with respect to n_1, $\frac{dI_D(n_1)}{dn_1}$, is positive for $n_1 < (n+2)/2$ and negative for $n_1 > (n+2)/2$. Thus, if n is an even natural number then, $n_1 = \frac{(n+2)}{2} = \frac{n}{2} + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$, but if n is an odd natural number then n_1 cannot take the value $\frac{(n+2)}{2}$, since it is not a natural number and in this case, we deduce that the vertex n_1 has to be $\frac{n+1}{2} = \left\lfloor \frac{n}{2} \right\rfloor + 1$ or $\frac{n+3}{2} = \left\lfloor \frac{n}{2} \right\rfloor + 2$, studying the closest natural values in this parabola to the aforementioned one corresponding to n_1 which is the desired initial assertion. In fact,

$$I_D(\frac{n+1}{2}) = \frac{n+1}{2}(n - \frac{n+1}{2} + 2) = \frac{(n+1)}{2}\frac{(n+3)}{2} = \frac{n+3}{2}(n - \frac{n+3}{2} + 2) = I_D(\frac{n+3}{2}).$$

In addition, it is clear that if n is an even natural number, an upper bound on I_{LR} given in expression (3.7) is $\frac{(n+2)^2}{4}$. In the same way, if n is an odd natural number, an upper bound on I_{LR} given in expression (3.7) is $\frac{(n+1)(n+3)}{4}$. Hence $I_{LR} \leq \frac{(n+2)^2}{4}$ since $(n+2)^2 = (n+1)(n+3) + 1$ which proves this corollary. $\square$

Dimension	n	4	5	6	7	10	11	15	20
I_{LR}	$\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(n + 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)$	9	12	16	20	36	42	72	121

TABLE 3.1
Minimum values corresponding to the upper bounds on I_{LR} for any positive 2-D system.

REMARK 3.7. As a consequence of the preceding corollary, we can state that an upper bound on I_{LR} for any (positively) locally reachable 2-D system must be greater than or equal to $\frac{(n+2)^2}{4}$.

The study of an upper bound on the whole is still an open problem.

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