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CENTROSYMMETRIC UNIVERSAL REALIZABILITY*

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Abstract. A list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers is said to be *realizable*, if it is the spectrum of an entrywise nonnegative matrix A. In this case, A is said to be a *realizing matrix.* Λ is said to be *universally realizable*, if it is realizable for each possible Jordan canonical form (JCF) allowed by Λ. The problem of the universal realizability of spectra is called the *universal realizability problem* (URP). Here, we study the centrosymmetric URP, that is, the problem of finding a nonnegative centrosymmetric matrix for each JCF allowed by a given list Λ. In particular, sufficient conditions for the centrosymmetric URP to have a solution are generated.

Key words. Universal realizability problem, nonnegative inverse eigenvalue problem, nonnegative matrix, centrosymmetric matrix, Jordan canonical form.

AMS subject classifications. 15A18, 15A20, 15A29.

1. Introduction. A list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers is said to be *realizable*, if Λ is the spectrum of an $n \times n$ entrywise nonnegative matrix A. In this case, A is said to be a realizing matrix for Λ. The problem of the realizability of spectra has been solved for $n = 3$ by Loewy and London [12] and, for $n = 4$, by Meehan [13] and independently by Torre-Mayo et al. [20]. For $n \geq 5$, the problem remains open. Λ is said to be *universally realizable* (UR), if there is a realizing matrix for each possible Jordan canonical form (JCF) allowed by Λ. The problem of the universal realizability of spectra is called the *universal realizability problem* (URP).

The first known results on the URP or the *nonnegative inverse elementary divisors problem*, as it was named formerly, are due to Minc [14]. In terms of the URP, Minc proved that if a list Λ of complex numbers is the spectrum of a diagonalizable positive matrix, then Λ is UR. The conditions that a realizing matrix be diagonalizable and positive are necessary for Minc's proof. Minc [14] says that "it is not known if the theorem holds for a general nonnegative matrix." Recently, two extensions of Minc's result were obtained in [3] and [7]. In particular, in [3] the authors proved that if a list Λ of complex numbers is the spectrum of a diagonalizable nonnegative matrix A with constant row sums λ_1 and a positive row or column, then Λ is UR. In [7], the authors proved that if a list Λ of complex numbers is diagonalizably ODP realizable, that is, Λ is the spectrum of a diagonalizable nonnegative matrix with only off-diagonal positive entries (zero entries are allowed on the diagonal), then Λ is also UR.

Throughout the paper, if $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is realizable by A, then $\lambda_1 = \rho(A) = \max\{|\lambda_i| : \lambda_i \in \Lambda\}$ is the Perron eigenvalue of A. A^{T}, $\mathbf{e}_k$, and J denote the transpose of A, the *k-th* canonical vector, and the counteridentity matrix (that is, $J = [\mathbf{e}_n \mid \mathbf{e}_{n-1} \mid \cdots \mid \mathbf{e}_1]$), respectively. We define the matrix

$$E = \sum_{i \in K} E_{i,i+1}, \ K \subset \{2, \ldots, n-1\}, \tag{1}$$

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where $E_{i,j}$ denotes the matrix with one in position (i,j) and zeros elsewhere. The set of all $n\times n$ real matrices with constant row sums equal to α is denoted by $\mathcal{CS}_\alpha$. It is clear that for any matrix in $\mathcal{CS}_\alpha$, $\mathbf{e}^{\mathrm{T}}=[1,\ldots,1]$ is an eigenvector corresponding to the eigenvalue α. The importance of matrices with constant row sums is due to the well-known fact that the problem of finding a nonnegative matrix A with spectrum $\Lambda=\{\lambda_1,\ldots,\lambda_n\}$ is equivalent to the problem of finding a nonnegative matrix $A\in\mathcal{CS}_{\lambda_1}$ with spectrum Λ [6].

In [10], it was proved that lists of nonnegative real numbers are always realizable by centrosymmetric matrices and, on the other hand, lists of complex numbers of Suleĭmanova type [19], that is, $\lambda_1>0$, $\mathrm{Re}(\lambda_i)\le 0$, $|\mathrm{Re}(\lambda_i)|\ge|\mathrm{Im}(\lambda_i)|$, $i=2,\ldots,n$, are centrosymmetrically realizable, except if they have only one real number (the Perron eigenvalue) and m pairs of conjugated complex numbers, with m being odd. For more general spectra, sufficient conditions were also obtained in [10].

In this paper, we study the centrosymmetric URP, that is, the problem of determining conditions for the existence and construction of a centrosymmetric realizing matrix, for each possible JCF allowed by a given list Λ of complex numbers. Centrosymmetric matrices have applications in many fields, such as physics, communication theory, differential equations, numerical analysis, engineering, and statistics. In particular, they are used in probability calculus and time series analysis, namely the transition matrices for the classification of states of periodic Markov chains and the smoothing matrices for signal extraction problems [4].

Outline of the paper: In Section 2, we introduce some background results. In Section 3, we start by showing a necessary condition for the 4×4 centrosymmetric realizability problem to have a solution. Based on it, we prove a necessary and sufficient condition for lists of four complex numbers of the form $\{\lambda_1,\lambda_2,a+ib,a-ib\}$ with $\lambda_1>0$, λ_2, $a\le 0$, and $b>0$, to be centrosymmetrically realizable. In particular, this result allows us to state that lists of complex numbers of Šmigoc type [16], that is, $\lambda_1>0$, $\mathrm{Re}(\lambda_i)\le 0$, $\sqrt{3}|\mathrm{Re}(\lambda_i)|\ge|\mathrm{Im}(\lambda_i)|$, $i=2,\ldots,n$, are not necessarily centrosymmetrically realizable. We also prove a centrosymmetric version of a result by Rado, published by Perfect in [15]. Finally, we give sufficient conditions for the centrosymmetric URP to have a solution.

2. Background results.

We start this section by giving the definition and certain properties of real centrosymmetric matrices.

Definition 1. *An $n\times n$ real matrix $C=[c_{ij}]$ is said to be centrosymmetric, if its entries satisfy the relation $c_{ij}=c_{n-i+1,n-j+1}$, or equivalently if $JCJ=C$.*

Definition 2. *An n-dimensional vector $\mathbf{x}$ is said to be symmetric if $J\mathbf{x}=\mathbf{x}$ and it is said to be skew-symmetric if $J\mathbf{x}=-\mathbf{x}$. For any n-dimensional vector $\mathbf{y}$, $\mathbf{y}^+=\frac{1}{2}(\mathbf{y}+J\mathbf{y})$ is symmetric and $\mathbf{y}^-=\frac{1}{2}(\mathbf{y}-J\mathbf{y})$ is skew-symmetric.*

The following results, some of which constitute important spectral properties of centrosymmetric matrices [2], will be applied to obtain our results.

Theorem 3 ([2]). *Let C be an $n\times n$ centrosymmetric matrix.*

(i) *If $n=2m$, then C can be written as $C=\begin{bmatrix}A & JBJ\\ B & JAJ\end{bmatrix}$, where A,B and J are $m\times m$ matrices. Moreover, C is orthogonally similar to the matrix $\begin{bmatrix}A+JB & \\ & A-JB\end{bmatrix}$ and the eigenvectors corresponding to the eigenvalues of $A+JB$ can be chosen to be symmetric, while the eigenvectors corresponding*

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to the eigenvalues of $A - JB$ can be chosen to be skew-symmetric. If C is nonnegative, with Perron eigenvalue λ_1, then λ_1 is the Perron eigenvalue of $A + JB$.

(ii) *If $n = 2m + 1$, then C can be written as $C = \begin{bmatrix} A & \mathbf{x} & JBJ \\ \mathbf{y}^T & c & \mathbf{y}^T J \\ B & J\mathbf{x} & JAJ \end{bmatrix}$, where A, B and J are $m \times m$ matrices, $\mathbf{x}$ and $\mathbf{y}$ are m-dimensional vectors, and c is a real number. Moreover, C is orthogonally similar to the matrix*

$$\begin{bmatrix} c & \sqrt{2}\mathbf{y}^T & \\ \sqrt{2}\mathbf{x} & A + JB & \\ & & A - JB \end{bmatrix},$$

and the eigenvectors corresponding to the eigenvalues of $\begin{bmatrix} c & \sqrt{2}\mathbf{y}^T \\ \sqrt{2}\mathbf{x} & A + JB \end{bmatrix}$ can be chosen to be symmetric, while the eigenvectors corresponding to the eigenvalues of $A - JB$ can be chosen to be skew-symmetric. If C is nonnegative with Perron eigenvalue λ_1, then λ_1 is the Perron eigenvalue of $\begin{bmatrix} c & \sqrt{2}\mathbf{y}^T \\ \sqrt{2}\mathbf{x} & A + JB \end{bmatrix}$.

Theorem 4 ([1]). *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\mathbf{v}^T = [v_1, \ldots, v_n]$ be an eigenvector of A corresponding to the eigenvalue λ_k and let $\mathbf{q}$ be any n-dimensional vector. Then the matrix $A + \mathbf{v}\mathbf{q}^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, \ldots, \lambda_n$.*

Lemma 5 ([17]). *Let $\mathbf{q}^T = [q_1, \ldots, q_n]$ be an arbitrary n-dimensional vector and let E_{11} be an $n \times n$ matrix with 1 in position $(1, 1)$ and zeros elsewhere. Let $A \in \mathcal{CS}_{\lambda_1}$ with JCF $J(A) = S^{-1}AS = diag\{J_1(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k)\}$. If $\lambda_1 + \sum_{i=1}^n q_i \neq \lambda_i, \; i = 2, \ldots, n$, then the matrix $A + \mathbf{e}\mathbf{q}^T$ has JCF $J(A) + (\sum_{i=1}^n q_i)E_{11}$. In particular, if $\sum_{i=1}^n q_i = 0$, then A and $A + \mathbf{e}\mathbf{q}^T$ are similar.*

3. Centrosymmetric universal realizability. We start this section with a necessary condition for a list $\Lambda = \{\lambda_1, \lambda_2, a + ib, a - ib\}$ be centrosymmetrically realizable.

Lemma 6. *Let $\Lambda = \{\lambda_1, \lambda_2, a + ib, a - ib\}$ be a list of complex numbers, such that $\lambda_1 > 0$, $b > 0$. If Λ is centrosymmetrically realizable, then $\left(\frac{\lambda_1+\lambda_2}{2}\right)^2 - b^2 - \lambda_1\lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 - 2a \geq 0$.*

Proof. Let $C = \begin{bmatrix} A & JBJ \\ B & JAJ \end{bmatrix}$ be a nonnegative centrosymmetric matrix with spectrum Λ, such that

$$A + JB = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad and \quad A - JB = \begin{bmatrix} p_1 & q_1 \\ r_1 & s_1 \end{bmatrix}. \tag{2}$$

From (2)

$$p + s = \lambda_1 + \lambda_2 \tag{3}$$

$$ps - rq = \lambda_1\lambda_2 \tag{4}$$

$$p_1 + s_1 = 2a \tag{5}$$

$$p_1 s_1 - r_1 q_1 = a^2 + b^2. \tag{6}$$

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From (5) and (6), it is clear that $q_1 \neq 0$ and

$$r_1 = \frac{(p_1 - a)^2 + b^2}{-q_1}. \tag{7}$$

Since C is nonnegative,

$$p \geq |p_1|, \quad q \geq |q_1|, \quad r \geq |r_1| \text{ and } s \geq |s_1|.$$

So, the first inequality it follows from (3), (4), $r \geq |r_1|$, $q \geq |q_1|$:

$$\begin{aligned}\left(\frac{\lambda_1 + \lambda_2}{2}\right)^2 - b^2 - \lambda_1\lambda_2 &= \frac{(p+s)^2}{4} - b^2 - (ps - rq) \\ &= \frac{(p-s)^2}{4} - b^2 + rq \\ &\geq \frac{(p-s)^2}{4} - b^2 + |r_1||q_1|,\end{aligned}$$

and using (7) completes the proof.

For the second inequality, we have $\lambda_1 + \lambda_2 + 2a \geq 0$, that is, the sum of the elements in Λ. If $a \leq 0$, then $\lambda_1 + \lambda_2 - 2a \geq \lambda_1 + \lambda_2 + 2a \geq 0$. If $a \geq 0$, then (3), (5), $p \geq |p_1|$, $s \geq |s_1|$ imply $\lambda_1 + \lambda_2 = p + s \geq |p_1| + |s_1| \geq |p_1 + s_1| = 2a$. ∎

For lists of four complex numbers with $\lambda_1 > 0$ and $\mathrm{Re}(\lambda_i) \leq 0$, $i = 2, 3, 4$, we have the following necessary and sufficient condition:

Proposition 7. *Let $\Lambda = \{\lambda_1, \lambda_2, a + ib, a - ib\}$ with $\lambda_1 > 0,\ \lambda_2, a \leq 0, b > 0$ and $\lambda_1 + \lambda_2 + 2a = 0$. Then Λ is centrosymmetrically realizable if and only if $a^2 - b^2 - \lambda_1\lambda_2 \geq 0$.*

Proof. From Lemma 6, the condition is necessary. Suppose that $a^2 - b^2 - \lambda_1\lambda_2 \geq 0$, then the matrices

$$A + JB = \begin{bmatrix} -a & 1 \\ a^2 - \lambda_1\lambda_2 & -a \end{bmatrix} \quad \textit{and} \quad A - JB = \begin{bmatrix} a & -1 \\ b^2 & a \end{bmatrix},$$

have eigenvalues λ_1, λ_2, and $a \pm ib$, respectively. Thus,

$$A = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ a^2 + b^2 - \lambda_1\lambda_2 & 0 \end{bmatrix} \quad \textit{and} \quad B = \frac{1}{2}J\begin{bmatrix} -2a & 2 \\ a^2 - b^2 - \lambda_1\lambda_2 & -2a \end{bmatrix},$$

are nonnegative matrices. Therefore, $C = \begin{bmatrix} A & JBJ \\ B & JAJ \end{bmatrix}$ is nonnegative centrosymmetric with spectrum Λ. ∎

Proposition 7 allows us to show that there are lists which are realizable, but not centrosymmetrically realizable.

Example 1. The list $\Lambda = \{7, -1, -3 + 5i, -3 - 5i\}$ is Šmigoc realizable [16] by

$$\begin{bmatrix} 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \\ \frac{34}{49} & \frac{246}{49} & \frac{9}{7} & 0 \end{bmatrix}.$$

However, Λ does not satisfy the condition of Proposition 7. Hence, Λ is not centrosymmetrically realizable.

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A perturbation result by Rado, published by Perfect in [15], shows how to change r eigenvalues of an $n \times n$ matrix without changing any of the remaining eigenvalues. Different versions of Rado's result have been obtained (see [8, 11, 18]). Here we give a centrosymmetric version:

THEOREM 8. *Let A be an $n \times n$ centrosymmetric matrix with complex eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $X = [\mathbf{x}_1 \mid \cdots \mid \mathbf{x}_r]$ be such that $rank(X) = r$ and $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$, where the eigenvectors $\mathbf{x}_i$ are all symmetric or all skew-symmetric, $i = 1, \ldots, r$. Let $J(A) = S^{-1}AS$ be a JCF of A with*

$$S = [X \mid Y], \quad S^{-1} = \begin{bmatrix} \widetilde{X} \\ \widetilde{Y} \end{bmatrix}.$$

Let $\mathcal{C}$ be any $r \times r$ matrix and $\Omega = diag\{\lambda_1, \ldots, \lambda_r\}$. Then, $A + X\mathcal{C}\widetilde{X}$ is centrosymmetric with eigenvalues $\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$, where $\mu_1, \ldots, \mu_r$ are eigenvalues of the matrix $\Omega + \mathcal{C}$.

Proof. Since $S^{-1}S = I$, then

$$\widetilde{X}X = I_r, \quad \widetilde{X}Y = 0, \quad \widetilde{Y}X = 0, \quad \widetilde{Y}Y = I_{n-r}.$$

Moreover, since $AX = X\Omega$ we have

$$S^{-1}AS = \begin{bmatrix} \widetilde{X} \\ \widetilde{Y} \end{bmatrix} A\,[X \mid Y] = \begin{bmatrix} \Omega & \widetilde{X}AY \\ 0 & \widetilde{Y}AY \end{bmatrix},$$

$$S^{-1}X\mathcal{C}\widetilde{X}S = \begin{bmatrix} \widetilde{X} \\ \widetilde{Y} \end{bmatrix} X\mathcal{C}\widetilde{X}\,[X \mid Y] = \begin{bmatrix} \mathcal{C} & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$S^{-1}\left(A + X\mathcal{C}\widetilde{X}\right)S = \begin{bmatrix} \Omega + \mathcal{C} & \widetilde{X}AY \\ 0 & \widetilde{Y}AY \end{bmatrix}.$$

Since $\mathbf{x}_i$ are all symmetric (or all skew-symmetric) the rows of $\widetilde{X}$ are also all symmetric (or all skew-symmetric), that is ,$JX = X$ (or $JX = -X$), $\widetilde{X}J = \widetilde{X}$ $\left(\text{or } \widetilde{X}J = -\widetilde{X}\right)$. So $JX\mathcal{C}\widetilde{X}J = X\mathcal{C}\widetilde{X}$, that is, $X\mathcal{C}\widetilde{X}$ is centrosymmetric and therefore $A + X\mathcal{C}\widetilde{X}$ is also centrosymmetric with eigenvalues $\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$, where $\mu_1, \ldots, \mu_r$ are eigenvalues of $\Omega + \mathcal{C}$. ∎

EXAMPLE 2. The centrosymmetric matrix

$$C = \frac{1}{2}\begin{bmatrix} A & JBJ \\ B & JAJ \end{bmatrix} \quad \text{with } A = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 2 & 6 \\ 6 & -3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 3 & -1 \\ 2 & 0 & 0 \\ 6 & 0 & 3 \end{bmatrix},$$

has eigenvalues 6, 1, -3, -3, $1+3i$, $1-3i$. Suppose we want to change the eigenvalues 6 and 1 by 10 and -3, respectively.

Since $\mathbf{x}_1^{\mathrm{T}} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$, $\mathbf{x}_2^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$ are eigenvectors associated with 6 and 1, respectively, then

$$X = \begin{bmatrix} \mathbf{x}_1 \mid \mathbf{x}_2 \end{bmatrix}, \ \widetilde{X} = \begin{bmatrix} -\frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} 0 & 4 \\ 9 & 0 \end{bmatrix},$$

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and

$$C + X\mathcal{C}\widetilde{X} = \frac{1}{2}\begin{bmatrix} 4 & -4 & 3 & 3 & -4 & 10 \\ 0 & -2 & 3 & -3 & -4 & 0 \\ 10 & -7 & 1 & -1 & -1 & 10 \\ 10 & -1 & -1 & 1 & -7 & 10 \\ 0 & -4 & -3 & 3 & -2 & 0 \\ 10 & -4 & 3 & 3 & -4 & 4 \end{bmatrix},$$

is centrosymmetric with eigenvalues $10, -3, -3, -3, 1+3i, 1-3i$.

Next, we adapt a well-known lemma by Fiedler [5], to the centrosymmetric case to obtain the following corollary, which gives a procedure to construct centrosymmetric realizations of complex numbers.

Corollary 9. *Let $\Lambda = \{\lambda_1, \lambda_2, \alpha_2, \alpha_3 \ldots, \alpha_{\frac{n}{2}}, \alpha_2, \alpha_3 \ldots, \alpha_{\frac{n}{2}}\}$ be a realizable list of complex numbers with even n and λ_1, λ_2 being real numbers. If $\Lambda_1 = \{\mu, \alpha_2, \alpha_3, \ldots, \alpha_{\frac{n}{2}}\}$ is realizable, where $2\mu = \lambda_1 + \lambda_2$, then Λ is centrosymmetrically realizable.*

Proof. Let A_1 be a realizing matrix for Λ_1. Without loss of generality, we assume that $A_1 \in \mathcal{CS}_\mu$, that is, $A_1\mathbf{e} = \mu\mathbf{e}$. Then, the matrix

$$A = \begin{bmatrix} A_1 & \frac{2}{n}\rho\mathbf{e}\mathbf{e}^{\mathrm{T}} \\ \frac{2}{n}\rho\mathbf{e}\mathbf{e}^{\mathrm{T}} & JA_1J \end{bmatrix},$$

is nonnegative centrosymmetric with spectrum Λ, where $\rho \geq 0$ and $\begin{bmatrix} \mu & \rho \\ \rho & \mu \end{bmatrix}$ has eigenvalues λ_1, λ_2. ◻

Example 3. Consider the Šmigoc type list

$$\Lambda = \{10, -2, -2+3i, -2-3i, -2+3i, -2-3i\},$$

with $\Lambda_1 = \{4, -2+3i, -2-3i\}$ being the spectrum of

$$A_1 = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ \frac{13}{4} & \frac{3}{4} & 0 \end{bmatrix}.$$

Then, $\rho = 6$ and

$$A = \begin{bmatrix} A_1 & 2\mathbf{e}\mathbf{e}^{\mathrm{T}} \\ 2\mathbf{e}\mathbf{e}^{\mathrm{T}} & JA_1J \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 2 & 2 & 2 \\ 0 & 0 & 4 & 2 & 2 & 2 \\ \frac{13}{4} & \frac{3}{4} & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & \frac{3}{4} & \frac{13}{4} \\ 2 & 2 & 2 & 4 & 0 & 0 \\ 2 & 2 & 2 & 0 & 4 & 0 \end{bmatrix},$$

is nonnegative centrosymmetric with spectrum Λ.

The odd case $n = 2m+1$ comes trivially from Corollary 9, by taking an appropriate direct sum with a 1×1 matrix.

The following two results about centrosymmetric URP use ODP matrices, that is, nonnegative matrices having all their off-diagonal entries being positive (see [7]). ODP matrices are important, not only because they allow that spectra of nonnegative matrices become UR, but also because they allow proving that a

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certain list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of complex numbers with $\sum_{i=1}^{n} \lambda_i = 0$ can be shown to be UR, which is not possible using Minc's result [14]. These conditions are constructive in the sense that if they are satisfied, then a centrosymmetric realizing matrix with spectrum Λ can be constructed for each JCF associated with Λ.

THEOREM 10. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a realizable list of complex numbers with λ_1 simple, $n = 2m$. Suppose $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$, where Λ_1 is diagonalizably realizable by an $m \times m$ matrix W_1 and Λ_2 is the spectrum of an $m \times m$ real diagonalizable matrix W_2 (not necessarily nonnegative). If $W_1 + W_2$ is ODP and $W_1 - W_2$ is positive, then Λ is centrosymmetrically UR.*

Proof. Since the $n \times n$ centrosymmetric matrix

$$C = \frac{1}{2} \begin{bmatrix} W_1 + W_2 & (W_1 - W_2)J \\ J(W_1 - W_2) & J(W_1 + W_2)J \end{bmatrix},$$

is orthogonally similar to

$$\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix},$$

C has spectrum $\Lambda_1 \cup \Lambda_2 = \Lambda$. Moreover, as W_1 and W_2 are diagonalizable with $W_1 + W_2$ ODP and $W_1 - W_2$ positive, C is a centrosymmetric ODP matrix with diagonal JCF $J(C) = S^{-1}CS$. As C is ODP, we may assume without loss of generality that $C \in \mathcal{CS}_{\lambda_1}$ and S can be chosen with its first column being the vector $\mathbf{e}$. Let $J(C) + E = diag\{\lambda_1, J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k)\}$ be any desired JCF allowed by Λ, where E is the matrix in (1), that is, $E = \sum_{i \in K} E_{i,i+1},\ K \subset \{2, \ldots, n-1\}$ with $1's$ in certain desired positions $(i, i+1)$ and zeros elsewhere. Then,

$$J(C) + E = S^{-1}CS + E = S^{-1}(C + SES^{-1})S.$$

Observe that S has complex entries. Then, for an adequate ordering of the columns of S (eigenvectors of C) it is always possible to obtain SES^{-1} as a real matrix. This was proved by Minc [14].

It is clear that $C + SES^{-1}$ has spectrum Λ and the desired JCF, although $C + SES^{-1}$ is not necessarily nonnegative, and $tr(SES^{-1}) = 0$. Now we have two situations: if SES^{-1} has all its diagonal entries equal to zero then, for $\epsilon > 0$ small enough the matrix $C + \epsilon SES^{-1}$ is nonnegative and has the same eigenvalues as C, but the desired JCF. If SES^{-1} has negative entries on its main diagonal, we define $\mathbf{q}^{\mathrm{T}} = [q_1, \ldots, q_n] = -\mathbf{d}$, where $\mathbf{d}$ is the vector of diagonal entries of SES^{-1} and therefore $\mathbf{e}^{\mathrm{T}}\mathbf{q} = 0$. Since $S\mathbf{e_1} = \mathbf{e}$, $SES^{-1} \in CS_0$. Then, from Theorem 4, $SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}}$ has all its diagonal entries equal to zero and therefore $C + \epsilon(SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}})$, for $\epsilon > 0$ small enough, is nonnegative. Note that $C + \epsilon(SES^{-1}) \in \mathcal{CS}_{\lambda_1}$ with JCF $J(C) + E$, then from Lemma 5 $C + \epsilon(SES^{-1})$ and $C + \epsilon(SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}})$ are similar, that is, $C + \epsilon(SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}})$ has the desired JCF $J(C) + E$.

It remains showing that $C + \epsilon SES^{-1}$ and $C + \epsilon(SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}})$ are both centrosymmetric. We only need to prove that SES^{-1} and $SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}}$ are both centrosymmetric. From Theorem 3, S has m symmetric columns and m skew-symmetric columns. It is clear that if S has its first m columns being symmetric, then S^{-1} has its first m rows being also symmetric. Observe that if $E = E_{i,i+1}$ for some $i \in K$ fixed, $K \subset \{2, \ldots, n-1\}$, then $SES^{-1} = \mathbf{u}\mathbf{v}^{\mathrm{T}}$ where $\mathbf{u}$ is the i-th column of S and $\mathbf{v}^{\mathrm{T}}$ is the $(i+1)$-th row of

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S^{-1}. Thus, since $\Lambda_1 \cap \Lambda_2 = \emptyset$, $\mathbf{u}$ and $\mathbf{v}^{\mathrm{T}}$ are both symmetric or skew-symmetric. Therefore, $J(SES^{-1})J = J(\mathbf{u}\mathbf{v}^{\mathrm{T}})J = (J\mathbf{u})(\mathbf{v}^{\mathrm{T}}J) = \mathbf{u}\mathbf{v}^{\mathrm{T}} = SES^{-1}$, that is, SES^{-1} is centrosymmetric and so $C + \epsilon SES^{-1}$ is also centrosymmetric. If $E = \sum_{i\in K} E_{i,i+1}, K \subset \{2,\dots,n-1\}$, SES^{-1} will also be centrosymmetric because it is the sum of centrosymmetric matrices. Finally, since SES^{-1} is centrosymmetric then $\mathbf{d}$ (the vector of diagonal entries of SES^{-1}) is a symmetric vector and $\mathbf{e}\mathbf{q}^{\mathrm{T}}$ is a centrosymmetric matrix. Thus, $C + \epsilon(SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}})$ is also centrosymmetric. Then Λ is centrosymmetrically UR. ∎

Theorem 11. *Let $\Lambda = \{\lambda_1,\dots,\lambda_n\}$ be a realizable list of complex numbers with λ_1 simple, $n = 2m+1$. Suppose $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$, where Λ_1 is diagonalizably realizable by the $(m+1)\times(m+1)$ ODP matrix*

$$\begin{bmatrix} W_1 & \mathbf{a} \\ \mathbf{b}^T & c \end{bmatrix},$$

and Λ_2 is the spectrum of an $m \times m$ real diagonalizable matrix W_2. If $W_1 + W_2$ is ODP and $W_1 - W_2$ is positive, then Λ is centrosymmetrically UR.

Proof. Since the $n \times n$ centrosymmetric matrix

$$C = \frac{1}{2}\begin{bmatrix} W_1 + W_2 & \sqrt{2}\mathbf{a} & (W_1 - W_2)J \\ \sqrt{2}\mathbf{b}^{\mathrm{T}} & 2c & \sqrt{2}\mathbf{b}^{\mathrm{T}}J \\ J(W_1 - W_2) & \sqrt{2}J\mathbf{a} & J(W_1 + W_2)J \end{bmatrix},$$

is orthogonally similar to

$$\begin{bmatrix} W_1 & \mathbf{a} & \\ \mathbf{b}^{\mathrm{T}} & c & \\ & & W_2 \end{bmatrix}.$$

C has the spectrum $\Lambda_1 \cup \Lambda_2 = \Lambda$. As $\mathbf{a}, \mathbf{b}^{\mathrm{T}} > 0$, $W_1 + W_2$ is ODP and $W_1 - W_2$ is positive with W_1 and W_2 diagonalizable matrices, then C is a centrosymmetric ODP matrix with diagonal JCF $J(C) = S^{-1}CS$. To obtain a nonnegative centrosymmetric matrix with spectrum Λ, for each one of the possible nondiagonal JCF allowed by Λ, we proceed as in the proof of Theorem 10. Then for an appropriate matrix $E = \sum_{i\in K} E_{i,i+1}, K \subset \{2,\dots,n-1\}$, $C + \epsilon SES^{-1}$ and $C + \epsilon(SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}})$ are both centrosymmetric with JCF $J(C) + E$ and therefore Λ is centrosymmetrically UR. ∎

Example 4. Consider the list

$$\Lambda = \{10, -2, -2, -2, -1+2i, -1-2i, -1+2i, -1-2i\}.$$

We apply Theorem 10 to show that Λ is centrosymmetrically UR. We take

$$\Lambda_1 = \{10, -2, -2, -2\},\ \Lambda_2 = \{-1+2i, -1-2i, -1+2i, -1-2i\},$$

which are the spectrum of

$$W_1 = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 3 & 1 & 3 & 3 \\ 3 & 3 & 1 & 3 \\ 3 & 3 & 3 & 1 \end{bmatrix} \text{ and } W_2 = \begin{bmatrix} -1 & -2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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respectively. Next we compute the centrosymmetric ODP matrix

$$C_1 = \frac{1}{2}\begin{bmatrix} W_1+W_2 & (W_1-W_2)J \\ J(W_1-W_2) & J(W_1+W_2)J \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 3 & 3 & 3 & 3 & 5 & 2 \\ 5 & 0 & 3 & 3 & 3 & 3 & 2 & 1 \\ 3 & 3 & 0 & 1 & 5 & 2 & 3 & 3 \\ 3 & 3 & 5 & 0 & 2 & 1 & 3 & 3 \\ 3 & 3 & 1 & 2 & 0 & 5 & 3 & 3 \\ 3 & 3 & 2 & 5 & 1 & 0 & 3 & 3 \\ 1 & 2 & 3 & 3 & 3 & 3 & 0 & 5 \\ 2 & 5 & 3 & 3 & 3 & 3 & 1 & 0 \end{bmatrix},$$

with diagonal JCF. Now, we compute a centrosymmetric ODP matrix C_2 with JCF

$$J(C_2) = diag\{J_1(10), J_2(-2), J_1(-2), J_2(-1+2i), J_2(-1-2i)\}.$$

To do this, we first compute the matrix of eigenvectors of C_1 :

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & -i & 0 & i \\ 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & -1 & -1 & -1 & -i & 0 & i & 0 \\ 1 & -1 & -1 & -1 & i & 0 & -i & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & i & 0 & -i \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then for $E = E_{2,3} + E_{5,6} + E_{7,8}$, we have

$$SES^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{8} & \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & -\frac{5}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & -\frac{7}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{5}{8} & \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and taking $\mathbf{q} = -\mathbf{d}$, where $\mathbf{d}$ is the vector of the diagonal entries of SES^{-1} we obtain that $SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}}$ has all its diagonal entries being zero. Thus,

$$C_2 = C_1 + (SES^{-1} + \mathbf{e}\mathbf{q}^{\mathrm{T}}) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{13}{8} & \frac{11}{8} & \frac{11}{8} & \frac{13}{8} & \frac{5}{2} & 1 \\ \frac{5}{2} & 0 & \frac{13}{8} & \frac{11}{8} & \frac{11}{8} & \frac{13}{8} & 1 & \frac{1}{2} \\ \frac{15}{8} & \frac{15}{8} & 0 & \frac{1}{4} & \frac{9}{4} & 1 & \frac{15}{8} & \frac{7}{8} \\ \frac{13}{8} & \frac{13}{8} & \frac{11}{4} & 0 & 1 & \frac{3}{4} & \frac{5}{8} & \frac{13}{8} \\ \frac{13}{8} & \frac{5}{8} & \frac{3}{4} & 1 & 0 & \frac{11}{4} & \frac{13}{8} & \frac{13}{8} \\ \frac{7}{8} & \frac{15}{8} & 1 & \frac{9}{4} & \frac{1}{4} & 0 & \frac{15}{8} & \frac{15}{8} \\ \frac{1}{2} & 1 & \frac{13}{8} & \frac{11}{8} & \frac{11}{8} & \frac{13}{8} & 0 & \frac{5}{2} \\ 1 & \frac{5}{2} & \frac{13}{8} & \frac{11}{8} & \frac{11}{8} & \frac{13}{8} & \frac{1}{2} & 0 \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 37, pp. 680-691, October 2021.

is nonnegative centrosymmetric with spectrum Λ and with the desired JCF $J(C_2)$. Applying the same procedure, changing the matrix E, say by, $E_1 = E_{2,3}$, $E_2 = E_{2,3} + E_{3,4}$, $E_3 = E_{2,3} + E_{3,4} + E_{5,6} + E_{7,8}$, $E_4 = E_{5,6} + E_{7,8}$, we may construct a nonnegative centrosymmetric matrix with spectrum Λ, for each one of the other four JCF allowed by Λ.

Remark 1. In Example 1, we show that there are spectra that are realizable but not centrosymmetrically realizable. Moreover, a centrosymmetrically realizable spectrum is not necessarily centrosymmetrically UR. In fact, the spectrum $\Lambda = \{1, 1, -1, -1\}$ is centrosymmetrically realizable by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

However, it has been shown in [9] that Λ is not UR.

Theorems 10 and 11 allow us to show that certain real spectra of nonnegative numbers and of the Suleĭmanova type are centrosymmetrically UR (Corollaries 12 and 13 below).

In [15], Perfect introduces the $n \times n$ matrix

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & -1 \\ 1 & 1 & \cdots & -1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & -1 & \cdots & 0 & 0 \end{bmatrix}, \tag{8}$$

and she proves that if $D = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, then PDP^{-1} is a positive matrix in $\mathcal{CS}_{\lambda_1}$.

Corollary 12. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a list of nonnegative real numbers with $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. If $\lambda_m > \lambda_{m+1}$ when $n = 2m$ and $\lambda_{m+1} > \lambda_{m+2}$ when $n = 2m + 1$, then Λ is centrosymmetrically UR.*

Proof. For $n = 2m$, we define the $m \times m$ diagonalizable positive matrix with spectrum $\Lambda_1 = \{\lambda_1, \ldots, \lambda_m\}$ as $W_1 = PDP^{-1}$, where $D = diag\{\lambda_1, \ldots, \lambda_m\}$ and P is the matrix in (8). Let $W_2 = diag\{\lambda_{m+1}, \ldots, \lambda_n\}$ with spectrum Λ_2. Note that $\Lambda = \Lambda_1 \cup \Lambda_2$ and since $\lambda_m > \lambda_{m+1}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$. Moreover, it is clear that $W_1 + W_2$ is ODP. On the other hand, in [10, Theorem 3.2], it was proved that if d_{jj}, $j = 1, 2, \ldots, m$, are the diagonal entries of PDP^{-1}, then $d_{jj} > \lambda_{m+j}$, for all $j = 1, 2, \ldots, m$. Thus, $W_1 - W_2$ is positive. Then, from Theorem 10, Λ is centrosymmetrically UR.

For $n = 2m + 1$, we define the $(m + 1) \times (m + 1)$ diagonalizable positive matrix with spectrum $\Lambda_1 = \{\lambda_1, \ldots, \lambda_{m+1}\}$ as $PDP^{-1} = \begin{bmatrix} W_1 & \mathbf{a} \\ \mathbf{b}^T & c \end{bmatrix}$, where $D = diag\{\lambda_1, \ldots, \lambda_{m+1}\}$ and P is the $(m + 1) \times (m + 1)$ matrix in (8). Let $W_2 = diag\{\lambda_{m+2}, \ldots, \lambda_n\}$ with spectrum Λ_2. Again $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, $W_1 + W_2$ is ODP and $W_1 - W_2$ is positive. Then, from Theorem 11, Λ is centrosymmetrically UR. $\square$

Corollary 13. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a realizable list of real numbers with $\lambda_1 > 0 > \lambda_2 \geq \cdots \geq \lambda_n$. If $\lambda_m > \lambda_{m+1}$ when $n = 2m$ and $\lambda_{m+1} > \lambda_{m+2}$ when $n = 2m + 1$, then Λ is centrosymmetrically UR.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Proof. We assume without loss of generality that $\sum_{i=1}^{n} \lambda_i = 0$. For $n = 2m$, we define

$$W_1 = \begin{bmatrix} \lambda_1 & & & \\ \lambda_1 - \lambda_2 & \lambda_2 & & \\ \vdots & \vdots & \ddots & \\ \lambda_1 - \lambda_m & 0 & \cdots & \lambda_m \end{bmatrix} + \mathbf{e}\mathbf{q}^{\mathrm{T}}$$

$$= \begin{bmatrix} -\lambda_{m+1} & -\lambda_{m+2} - \lambda_2 & \cdots & -\lambda_n - \lambda_m \\ -\lambda_{m+1} - \lambda_2 & -\lambda_{m+2} & \cdots & -\lambda_n - \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_{m+1} - \lambda_m & -\lambda_{m+2} - \lambda_2 & \cdots & -\lambda_n \end{bmatrix},$$

where $\mathbf{q}^{\mathrm{T}} = [-\lambda_{m+1} - \lambda_1, -\lambda_{m+2} - \lambda_2, \cdots, -\lambda_n - \lambda_m]$. Note that W_1 is an $m \times m$ diagonalizable positive matrix with spectrum $\Lambda_1 = \{\lambda_1, \ldots, \lambda_m\}$. Let $W_2 = diag\{\lambda_{m+1}, \ldots, \lambda_n\}$ with spectrum Λ_2. It is clear that $\Lambda = \Lambda_1 \cup \Lambda_2$ and since $\lambda_m > \lambda_{m+1}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$. Moreover,

$$W_1 + W_2 = \begin{bmatrix} 0 & -\lambda_{m+2} - \lambda_2 & \cdots & -\lambda_n - \lambda_m \\ -\lambda_{m+1} - \lambda_2 & 0 & \cdots & -\lambda_n - \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_{m+1} - \lambda_m & -\lambda_{m+2} - \lambda_2 & \cdots & 0 \end{bmatrix},$$

is ODP and

$$W_1 - W_2 = \begin{bmatrix} -2\lambda_{m+1} & -\lambda_{m+2} - \lambda_2 & \cdots & -\lambda_n - \lambda_m \\ -\lambda_{m+1} - \lambda_2 & -2\lambda_{m+2} & \cdots & -\lambda_n - \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_{m+1} - \lambda_m & -\lambda_{m+2} - \lambda_2 & \cdots & -2\lambda_n \end{bmatrix},$$

is positive. Then, from Theorem 10, Λ is centrosymmetrically UR.

For $n = 2m + 1$, we define the $(m + 1) \times (m + 1)$ diagonalizable nonnegative matrix with spectrum $\Lambda_1 = \{\lambda_1, \ldots, \lambda_{m+1}\}$ as

$$\begin{bmatrix} W_1 & \mathbf{a} \\ \mathbf{b}^{\mathrm{T}} & c \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & & \\ \lambda_1 - \lambda_2 & \lambda_2 & & & \\ \vdots & \vdots & \ddots & & \\ \lambda_1 - \lambda_m & 0 & \cdots & \lambda_m & \\ \lambda_1 - \lambda_{m+1} & 0 & \cdots & 0 & \lambda_{m+1} \end{bmatrix} + \mathbf{e}\mathbf{q}^{\mathrm{T}}$$

$$= \begin{bmatrix} -\lambda_{m+2} & -\lambda_{m+3} - \lambda_2 & \cdots & -\lambda_n - \lambda_m & -\lambda_{m+1} \\ -\lambda_{m+2} - \lambda_2 & -\lambda_{m+3} & \cdots & -\lambda_n - \lambda_m & -\lambda_{m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda_{m+2} - \lambda_m & -\lambda_{m+3} - \lambda_2 & \cdots & -\lambda_n & -\lambda_{m+1} \\ -\lambda_{m+2} - \lambda_{m+1} & -\lambda_{m+3} - \lambda_2 & \cdots & -\lambda_n - \lambda_m & 0 \end{bmatrix},$$

where $\mathbf{q}^{\mathrm{T}} = [-\lambda_{m+2} - \lambda_1, -\lambda_{m+3} - \lambda_2, \cdots, -\lambda_n - \lambda_m, -\lambda_{m+1}]$.

Let $W_2 = diag\{\lambda_{m+2}, \ldots, \lambda_n\}$ with spectrum Λ_2. Then from Theorem 11, the result follows. $\square$

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REFERENCES

[1] A. Brauer. Limits for the characteristic roots of a matrix, IV: Applications to stochastic matrices. *Duke Math. J.*, 19:75–91, 1952.

[2] A. Cantoni and P. Butler. Eigenvalues and eigenvectors of symmetric centrosymmetric matrices. *Linear Algebra Appl.,* 13:275–288, 1976.

[3] M. Collao, M. Salas, and R.L. Soto. Spectra universally realizable by doubly stochastic matrices. *Spec. Mat.,* 6:301–309, 2018.

[4] E.B. Dagum, L. Guidotti, and A. Luati. Some statistical applications of centrosymmetric matrices. In: H.H. Bock et al. (editors), *New Developments in Classification and Data Analysis*. Springer, Berlin, Heidelberg, 97–104, 2005.

[5] M. Fiedler. Eigenvalues of nonnegative symmetric matrices. *Linear Algebra Appl.,* 9:119–142, 1974.

[6] C.R. Johnson. Row stochastic matrices similar to doubly stochastic matrices. *Linear Multilinear Algebra,* 10:113–130, 1981.

[7] C.R. Johnson, A.I. Julio, and R.L. Soto. Nonnegative realizability with Jordan structure. *Linear Algebra Appl.* 587:302–313, 2020.

[8] A.I. Julio, C.B. Manzaneda, and R.L. Soto. Normal nonnegative realization of spectra. *Linear Multilinear Algebra,* 63:1204–1215, 2015.

[9] A.I. Julio, C. Marijuán, M. Pisonero, and R.L. Soto. On universal realizability of spectra. *Linear Algebra Appl.,* 563:353–372, 2019.

[10] A.I. Julio, O. Rojo, and R.L. Soto. Centrosymmetric nonnegative realization of spectra. *Linear Algebra Appl.,* 581:260–284, 2019.

[11] A.I. Julio and R.L. Soto. Persymmetric and bisymmetric nonnegative inverse eigenvalue problem. *Linear Algebra Appl.*, 469:130–152, 2015.

[12] R. Loewy and D. London. A note on the inverse problem for nonnegative matrices. *Linear Multilinear Algebra.*, 6:83–90, 1978.

[13] M.E. Meehan. Some results on matrix spectra, Ph.D. thesis, National University of Ireland, Dublin, 1998.

[14] H. Minc. Inverse elementary divisor problem for nonnegative matrices. *Proc. Amer. Math. Soc.*, 83(4):665–669, 1981.

[15] H. Perfect. Methods of constructing certain stochastic matrices II. *Duke Math. J.,* 22:305–311, 1955.

[16] H. Šmigoc. The inverse eigenvalue problem for nonnegative matrices. *Linear Algebra Appl.*, 393:365–374, 2004.

[17] R.L. Soto and J. Ccapa. Nonnegative matrices with prescribed elementary divisors. *Electron. J. Linear Algebra.* 17:287–303, 2008.

[18] R.L. Soto, O. Rojo, J. Moro, and A. Borobia. Symmetric nonnegative realization of spectra. *Electronic J. Linear Algebra*, 16:1–18, 2007.

[19] H.R. Suleĭmanova. Stochastic matrices with real characteristic values. *Dokl. Akad. Nauk SSSR*, 66:343–345, 1949.

[20] J. Torre-Mayo, M.R. Abril-Raymundo, E. Alarcía-Estévez, C. Marijuán, and M. Pisonero. The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs. *Linear Algebra Appl.* 426:729–773, 2007.