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LAPLACIAN INTEGRAL SUBCUBIC SIGNED GRAPHS*

DIJIAN WANG† AND YAOPING HOU‡

Abstract. A (signed) graph is called Laplacian integral if all eigenvalues of its Laplacian matrix are integers. In this paper, we determine all connected Laplacian integral signed graphs of maximum degree 3; among these signed graphs, there are two classes of Laplacian integral signed graphs, one contains 4 infinite families of signed graphs and another contains 29 individual signed graphs.

Key words. Laplacian matrix, Signed graph, Integral spectral.

AMS subject classifications. 05C50, 05C22.

1. Introduction. All graphs considered here are simple and undirected. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. A *signed graph* $\Gamma = (G, \sigma)$ consists of an unsigned graph $G = (V, E)$ and $\sigma : E(G) \to \{+1, -1\}$. The G is its *underlying graph*, while σ is its sign function (or signature). An edge v_iv_j is *positive* (*negative*) if $\sigma(v_iv_j) = +1$ (resp. $\sigma(v_iv_j) = -1$), and denoted by $v_i \overset{+}{\sim} v_j$ (resp. $v_i \overset{-}{\sim} v_j$). If a signed graph Γ has the *all-positive* (resp. *all-negative*) signature, then it is denoted by $(G, +)$ (resp. $(G, -)$). The *negation* of Γ is a signed graph obtained from Γ by reversing the sign of every edge of Γ.

The *adjacency matrix* of a signed graph Γ is defined by $A(\Gamma) = (\sigma_{ij})$, where $\sigma_{ij} = \sigma(v_iv_j)$ if $v_i \sim v_j$, and $\sigma_{ij} = 0$ otherwise. The corresponding *Laplacian matrix* of Γ is $L_\sigma = L(\Gamma) = L(G, \sigma) = D(G) - A(\Gamma)$, where $D(G)$ is the diagonal matrix of vertex degrees. Note that $L(G, +) = L(G)$ and $L(G, -) = Q(G)$. Thus, $L(G, \sigma)$ may be viewed as a common generalization of the Laplacian matrix $L(G)$ and signless Laplacian matrix $Q(G) = D(G) + A(G)$ of the underlying graph G. The Laplacian eigenvalues of Γ are identified to be the eigenvalues of $L(\Gamma)$. If the distinct Laplacian eigenvalues of Γ are $\mu_1 > \mu_2 > \cdots > \mu_k$ and their respective multiplicities as $m_1, m_2, \ldots, m_k$, we write the spectrum of Γ as $spec_L(\Gamma) = \{\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_k^{m_k}\}$. Recently, some problems about the spectra of the signed graphs have attracted many studies, see [1, 5, 18].

A (signed) graph is called *Laplacian integral* (resp. *integral*) if all eigenvalues of its Laplacian matrix (resp. adjacency matrix) are integers. Which graphs have integral spectra? This problem was proposed by Harary and Schwenk [6]. Although this problem is easy to understand, it turns to be extremely hard, and from then on it attracts many mathematicians. In 1970s, Cvetković [2] and Schwenk [12] classified the connected integral graphs of maximum degree at most 3. Stevanović [13] determined the 4-regular integral graphs avoiding ± 3 in the spectrum and gave the possible spectrum of 4-regular bipartite graphs. In 2008, Kirkland [10] proved that there are 21 connected Laplacian integral graphs of maximum degree 3 on at least 6 vertices. For more results about (Laplacian) integral graphs, we refer the readers to [3, 4, 7, 13].

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†School of Science, Zhejiang University of Science and Technology, Hangzhou, Zhejiang 310023, China and College of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, China.

‡College of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, China (yphou@hunnu.edu.cn).

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Unsigned graphs can be considered as the particular case of signed graphs in which all edges are positive. Therefore, there is a natural question: Which signed graphs have integral spectra? Recently, Hou et al. [9] and Stanić [16] found out all integral signed graphs with vertex degree at most 4 and exactly 2 eigenvalues. Wang and Hou [14] gave all connected integral subcubic signed graphs. Stanić [15] investigated integral regular net-balanced signed graphs with vertex degree at most 4. Thus, it may be an interesting problem to investigate the Laplacian integral signed graphs with maximum degree 3.

Using Cayley–Hamilton theorem, Kirkland [10] showed that for a Laplacian integral graph G with spectral radius 5 on n vertices, value of n is restricted to be one of the divisors of 120. It is different from the unsigned graphs; the number of vertices of Laplacian integral signed graphs with spectral radius 5 may be arbitrary large.

The majority concepts defined for graphs can be directly extended to signed graphs. For example, the degree of a vertex v in G (denoted by d_v) is also its degree in Γ. As usual, we always write K_n, C_n (or n-cycle), and P_n for the complete graph, the cycle, and the path with order n, respectively. Let $\Delta(\Gamma)$ and $\delta(\Gamma)$ denote the maximum and minimum vertex degree of Γ, respectively. For $U \subseteq V(G)$, let $\Gamma[U]$ denote the induced subgraph by U, which is of course a signed graph.

A (signed) graph is called subcubic if its maximum degree is not more than 3. In this paper, we will determine all connected Laplacian integral subcubic signed graphs, which consist of 4 infinite families of signed graphs and 29 individual signed graphs.

This paper is organized as follows. In Section 2, we introduce the terminology and notation and give some preliminary results. In Section 3, we devote to the irregular Laplacian integral unbalanced signed graphs of maximum degree 3.

2. Preliminaries.

First we will bring some basic results about signed graphs. Let C_n be a cycle of Γ, the sign of C_n is $\sigma(C_n) = \prod_{e \in C_n} \sigma(e)$. A cycle whose sign is $+1$ (resp. -1) is called *positive* (resp. *negative*). A signed graph is called *balanced* if all its cycles are positive, otherwise it is called *unbalanced*. Throughout this paper, we denote a positive and a negative cycle of length n by C_n^+ and C_n^-, respectively.

For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be switching equivalent to Γ, write $\Gamma \sim \Gamma^U$. Furthermore, it is important to observe that switching equivalent signed graphs have similar Laplacian matrices. For more results on signed graphs, see [17].

The following lemma is used to check whether two signed graphs are switching equivalent.

Lemma 1 ([17, Lemma 3.1]). *Let G be a connected graph and T a spanning tree of G. The each switching equivalence class of signed graphs on the graph G has a unique representative which is $+1$ on T. Indeed, given any prescribed sign function $\sigma_T : T \longrightarrow \{+1, -1\}$, each switching class has a single representative which agrees with σ_T on T.*

If $\Gamma = (G, \sigma)$ is a connected signed graph of maximum degree at most 2, then G must be a path or cycle. It is then readily determined that Γ is Laplacian integral if and only if it is the signed graph (P_2, σ), (P_3, σ), C_3^+, C_3^-, C_4^+, or C_6^+ (see [1, Lemma 4.1]). Thus, in the following, we describe all connected Laplacian integral signed graphs of maximum degree 3.

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Let $\mu_1(\Gamma)$ and $\mu_n(\Gamma)$ be the largest and smallest Laplacian eigenvalues of the signed graph Γ with n vertices, respectively. Two basic results about $\mu_1(\Gamma)$ and $\mu_n(\Gamma)$ of the signed graphs are provided [8].

LEMMA 2 ([8, Theorem 2.5]). *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. Then $\mu_n(\Gamma) = 0$ if and only if Γ is balanced.*

LEMMA 3 ([8, Lemma 3.1]). *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. Then*

$$\mu_1(G, \sigma) \leq \mu_1(G, -),$$

with equality if and only if $(G, \sigma) \sim (G, -)$.

For a connected Laplacian integral signed graph $\Gamma = (G, \sigma)$ of maximum degree 3, it is straightforward to see that $\mu_1(G, \sigma) \leq \mu_1(G, -) \leq 6$, with equality holds if and only if $(G, \sigma) \sim (G, -)$ and G is 3-regular. Thus, the Laplacian spectrum of such a signed graph is

$$Spec_L(\Gamma) = \{0^{x_0}, 1^{x_1}, 2^{x_2}, 3^{x_3}, 4^{x_4}, 5^{x_5}, 6^{x_6}\}.$$

If $x_0 \neq 0$, by Lemma 2, then $\Gamma = (G, \sigma)$ is balanced. So Γ is one of the balanced Laplacian integral signed graphs of maximum degree 3 identified in [10] and [12]. Figs. 1 and 3 show such balanced signed graphs.

LEMMA 4 ([10, 12]). *Let $\Gamma = (G, \sigma)$ be a connected balanced signed graph of maximum degree 3, then Γ is Laplacian integral if and only if it is switching equivalent to one of the unsigned graphs in Figs. 1 and 3.*

If G is 3-regular then $\Gamma = (G, \sigma)$ is Laplacian integral if and only if it is integral, which are identified in [12] and [14].

LEMMA 5 ([12, 14]). *Let $\Gamma = (G, \sigma)$ be a connected 3-regular signed graph, then Γ is Laplacian integral if and only if it is switching equivalent to one of the signed graphs in Figs. 1 and 2, or their negations.*

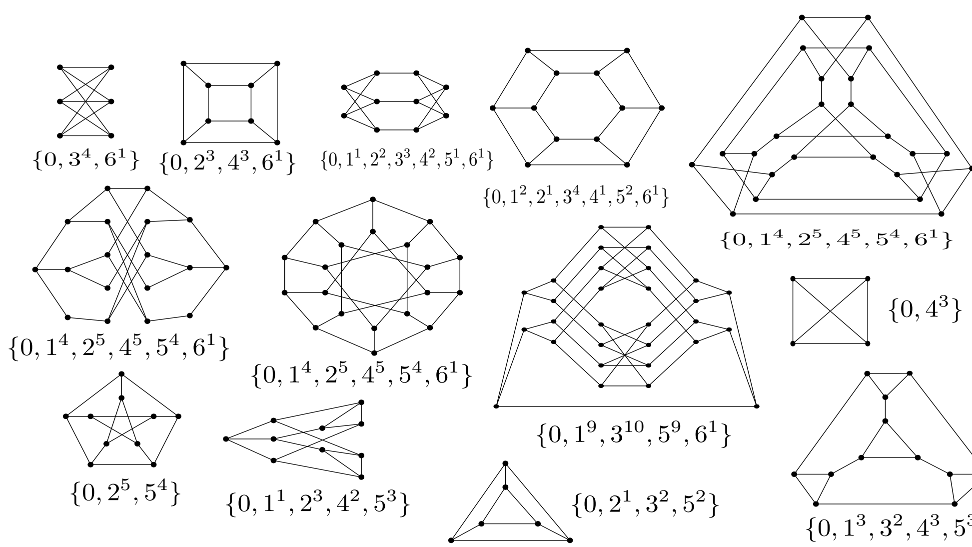

FIG. 1. *3-regular, Laplacian integral graphs.*

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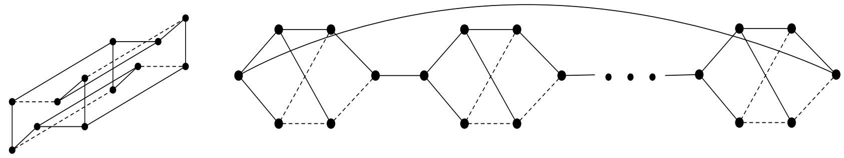

FIG. 2. *3-regular, Laplacian integral unbalanced signed graphs.*

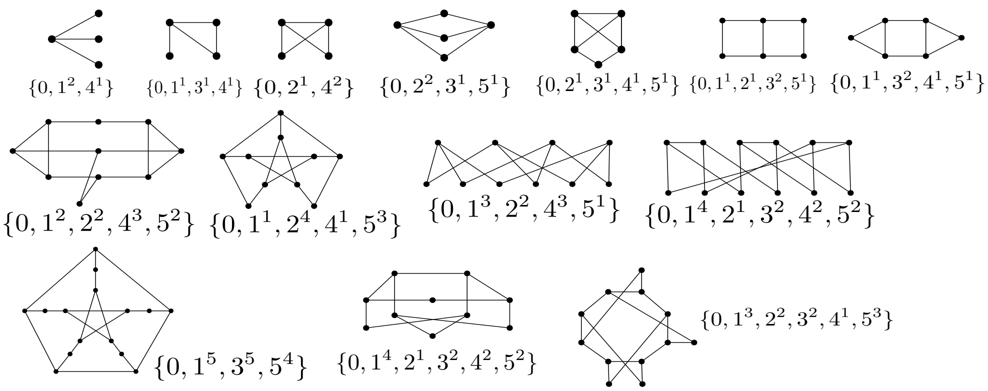

FIG. 3. *Irregular, Laplacian integral graphs of maximum degree 3.*

Thus, for our purposes, it is enough to focus on the case in which $\Gamma = (G, \sigma)$ is unbalanced and irregular. Let $\mathcal{G}$ denote the set of the connected irregular Laplacian integral unbalanced signed graphs of maximum degree 3. Obviously, Laplacian spectrum of a signed graph $\Gamma \in \mathcal{G}$ must be

$$Spec_L(\Gamma) = \{1^{x_1}, 2^{x_2}, 3^{x_3}, 4^{x_4}, 5^{x_5}\}.$$

Thus, we have the following observations, which will be used frequently in this paper.

PROPOSITION 2.1. *Let $\Gamma = (G, \sigma) \in \mathcal{G}$. Then*

(i) $\mu_n(\Gamma) \geq 1$ *and* $\mu_1(\Gamma) \leq 5$.
(ii) $L_\sigma - I$ *is positive semi-definite* (*if* $\mu_n(\Gamma) = 1$) *or positive definite* (*if* $\mu_n(\Gamma) > 1$).

LEMMA 6 ([8, Lemma 3.7]). *Let Γ be a signed graph with n vertices and let Γ' be a signed graph obtained from Γ by inserting a new* (*positive or negative*) *edge into Γ. Then the Laplacian eigenvalues of Γ and Γ' interlace, that is,*

$$\mu_1(\Gamma') \geq \mu_1(\Gamma) \geq \cdots \geq \mu_n(\Gamma') \geq \mu_n(\Gamma).$$

From Lemma 6, we know that $\mu_1(\mathcal{H}) \leq \mu_1(\Gamma) \leq 5$ for any subgraph $\mathcal{H}$ of signed graph $\Gamma \in \mathcal{G}$. Thus, in order to determine all signed graphs $\Gamma \in \mathcal{G}$, we need to present some forbidden subgraphs whose largest Laplacian eigenvalue is strictly greater than 5, as depicted in Fig. 4.

LEMMA 7. *Let $\Gamma \in \mathcal{G}$. Then Γ does not contain any signed graph in Fig. 4 as its subgraph.*

Proof. Since $\Gamma = (G, \sigma) \in \mathcal{G}$, then $\mu_1(\Gamma) \leq 5$ and Γ cannot contain the signed graph whose largest Laplacian eigenvalue is strictly greater than 5 as a subgraph (by Lemma 6). So Γ does not contain any signed graph in Fig. 4 as a subgraph. ∎

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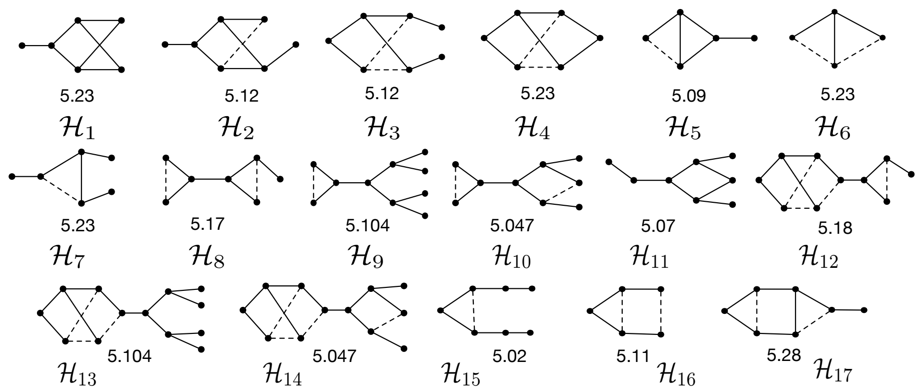

FIG. 4. *Forbidden subgraphs (the number denotes the largest Laplacian eigenvalue of the corresponding signed graph).*

Next we prove that if $\Gamma = (G, \sigma) \in \mathcal{G}$, then $\delta(\Gamma) \geq 2$.

LEMMA 8. *Let $\Gamma = (G, \sigma) \in \mathcal{G}$. Then Γ has no a pendant vertex.*

Proof. Suppose that u is a pendant vertex and v is the unique neighbor of u, then the 2×2 principal submatrix of $L_\sigma - I$ corresponding to u and v is

$$S = \begin{bmatrix} 0 & -1 \\ -1 & d_v - 1 \end{bmatrix}.$$

We have $\det(S) = -1$, which contradicts Proposition 2.1. This completes the proof. □

In the end of this section, we provide some results that will be useful to prove the main results of this paper.

LEMMA 9. *Let $\Gamma = (G, \sigma) \in \mathcal{G}$. If the underlying graph G contains the G_1 (see Fig. 5) as an induced subgraph, then there exists one new vertex x_7 that is adjacent to both x_5 and x_6 and $\Gamma[V(G_1) \cup \{x_7\}]$ is switching equivalent to G_2^σ (see Fig. 5).*

Proof. In view of Lemma 1, we may assume that $\sigma(x_1x_2) = \sigma(x_2x_3) = \sigma(x_2x_4) = \sigma(x_3x_5) = \sigma(x_3x_6) = +1$. First by forbidden subgraph $\mathcal{H}_1$, we have $\sigma(v_4v_5)\sigma(v_4v_6) = -1$ or $\sigma(v_4v_5) = \sigma(v_4v_6) = -1$.

Case 1. $\sigma(v_4v_5)\sigma(v_4v_6) = -1$. Without loss of generality, assume that $\sigma(v_4v_5) = -1$ and $\sigma(v_4v_6) = +1$. If $d_{v_5} = 3$, then Γ contains the subgraph $\mathcal{H}_2$, this is impossible. Thus, $d_{v_5} = 2$. The 4×4 principal submatrix of $L_\sigma - I$ corresponding to v_2, v_3, v_4, v_6 is

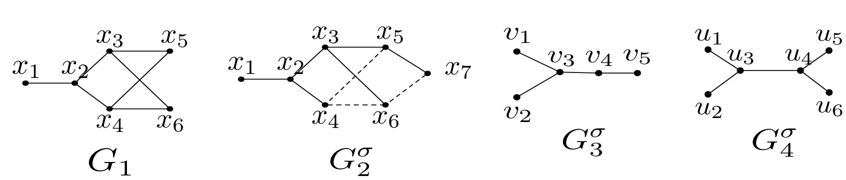

FIG. 5. *The graphs G_1 and the signed graphs G_2^σ, G_3^σ, G_4^σ.*

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$$S = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 1 \end{bmatrix}.$$

Then $\det(S) = -4$, which contradicts Proposition 2.1.

Case 2. $\sigma(x_4x_5) = \sigma(x_4x_6) = -1$. The 5×5 principal submatrix of $L_\sigma - I$ corresponding to vertices x_2, x_3, x_4, x_5, x_6 is

$$S = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 & 1 \\ 0 & -1 & 1 & d_{x_5} - 1 & 0 \\ 0 & -1 & 1 & 0 & d_{x_6} - 1 \end{bmatrix}, \quad \text{where } d_{x_i} \in \{2, 3\} \text{ for } i = 5, 6.$$

By direct computation, we have $\det(S) = 4((d_{x_5} - 2)(d_{x_6} - 2) - 1)$. Then $\det(S) \geq 0$ if and only if $d_{x_5} = d_{x_6} = 3$. Without loss of generality, assume that x_5 has a new neighbor x_7. By forbidden subgraphs $\mathcal{H}_3$ and $\mathcal{H}_4$, then x_6 is also adjacent to x_7 and $\sigma(x_5x_7)\sigma(x_6x_7) = -1$. This completes the proof. ∎

Lemma 10. *Let $\Gamma = (G, \sigma) \in \mathcal{G}$.*

(i) *If Γ contains G_3^σ (see Fig. 5) as an induced subgraph and $d_{v_1} = d_{v_2} = 3$, then $d_{v_4} = 3$.*
(ii) *If Γ contains G_4^σ (see Fig. 5) as an induced subgraph and $d_{u_1} = d_{u_2} = 3$, then $d_{u_5} = d_{u_6} = 3$.*

Proof. The 5×5 principal submatrix of $L_\sigma - I$ corresponding to vertices v_1, v_2, v_3, v_4, v_5 and the 6×6 principal submatrix of $L_\sigma - I$ corresponding to vertices $u_1, u_2, u_3, u_4, u_5, u_6$ are

$$S_1 = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & d_{v_4} - 1 & -1 \\ 0 & 0 & 0 & -1 & d_{v_5} - 1 \end{bmatrix}, S_2 = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & d_{u_5} - 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & d_{u_6} - 1 \end{bmatrix}.$$

Direct calculations show that $\det(S_1) = 4((d_{v_4} - 2)(d_{v_5} - 1) - 1)$ and $\det(S_2) = 4((d_{u_5} - 2)(d_{u_6} - 2) - 1)$. Then $\det(S_1) \geq 0$ if and only if $d_{v_4} = 3$ and $\det(S_2) \geq 0$ if and only if $d_{u_5} = d_{u_6} = 3$. Thus, the proofs of (i) and (ii) are completed. ∎

Lemma 11. *Let $\Gamma = (G, \sigma) \in \mathcal{G}$. If there is one pair of adjacent vertices u and v of degree 2, then there exists one C_3^- that contains two vertices u and v.*

Proof. Suppose that w is the another neighbor of v and $\sigma(uv) = \sigma(vw) = +1$, then the 3×3 principal submatrix of $L_\sigma - I$ corresponding to v, u, w is

$$S = \begin{bmatrix} 1 & -1 & -\sigma(wu) \\ -1 & 1 & -1 \\ -\sigma(wu) & -1 & d_w - 1 \end{bmatrix}, \quad \text{where } d_w \in \{2, 3\}.$$

By direct calculation, we have $\det(S) = -(\sigma(wu) + 1)^2$. Then $\det(S) \geq 0$ if and only if $\sigma(wu) = -1$. So $\{v, u, w\}$ is a negative 3-cycle. This completes the proof. ∎

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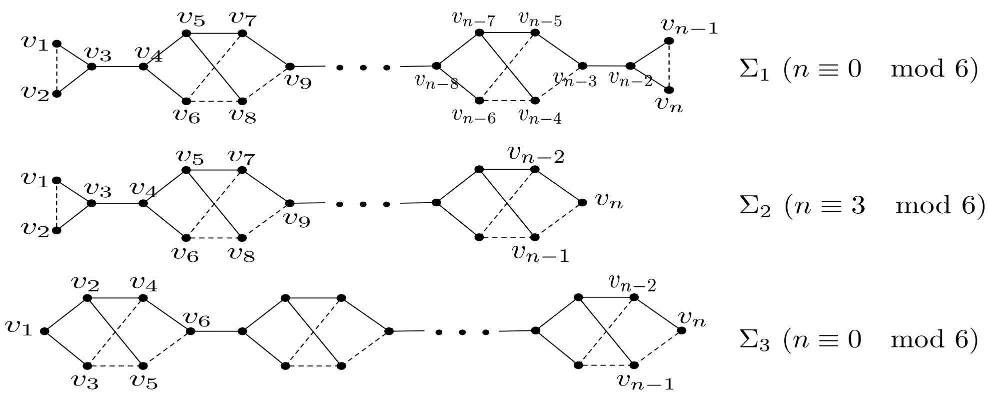

FIG. 6. *Irregular Laplacian integral unbalanced signed graphs of maximum degree* 3.

3. Signed graphs of $\mathcal{G}$. In this section, we determine the set $\mathcal{G}$ of all irregular Laplacian integral unbalanced signed graphs of maximum degree 3. We divide this section into two subsections: in Subsection 3.1, we consider that Γ contains the 3-cycle, and in Subsection 3.2, we consider that Γ contains no 3-cycle.

First we prove that three signed graphs $\Sigma_1, \Sigma_2, \Sigma_3$ of Fig. 6 are Laplacian integral. Let $\mathbf{0}_m$ be the all-zero vector of dimension m and take $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ to be an orthonormal basis of $\mathbb{R}^m$. We need to define some vectors:

$$\mathbf{t}_1 = (2,-2,0,2,0,-2),\ \mathbf{t}_2 = (2,-2,0,1,1,0), \mathbf{t}_3 = (2,-1,-1,0,0,0),$$
$$\mathbf{t}_4 = (1,1,0,1,0,1),\ \mathbf{t}_5 = (2,2,0,2,0,2), \mathbf{t}_6 = (2,2,0,1,1,0), \mathbf{t}_7 = (2,1,1,0,0,0).$$

LEMMA 12.

(i) *The spectrum of signed graph Σ_1 with order $n = 6k$ is*

$$spec_L(\Sigma_1) = \{5^{2k-1}, 4^k, 2^k, 1^{2k+1}\}.$$

(ii) *The spectrum of signed graph Σ_2 with order $n = 6k+3$ is*

$$spec_L(\Sigma_2) = \{5^{2k}, 4^{k+1}, 2^k, 1^{2k+2}\}.$$

(iii) *The spectrum of signed graph Σ_3 with order $n = 6k$ is*

$$spec_L(\Sigma_3) = \{5^{2k-1}, 4^{k+1}, 2^{k-1}, 1^{2k+1}\}.$$

Proof. (i) Let us define some n-dimensional vectors $\mathbf{a}_i$:

$$\mathbf{a}_1 = (1,1,-2,\mathbf{t}_1,\ldots,\mathbf{t}_1,2,-1,-1),$$
$$\mathbf{a}_i = (1,1,-2,\underbrace{\mathbf{t}_1,\ldots,\mathbf{t}_1}_{k-i},\mathbf{t}_2,\mathbf{0}_{6(i-2)+3}), \quad \text{for } i = 2,\ldots,k,$$
$$\mathbf{a}_i = (1,1,-2,\underbrace{\mathbf{t}_1,\ldots,\mathbf{t}_1}_{2k-1-i},\mathbf{t}_3,\mathbf{0}_{6(i-(k+1))+3}), \quad \text{for } i = k+1,k+2,\ldots,2k-1.$$

For example, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_{k+1}$ are displayed in Fig. 7, where the number in parentheses in Fig. 7 denotes the coordinate of the corresponding vector.

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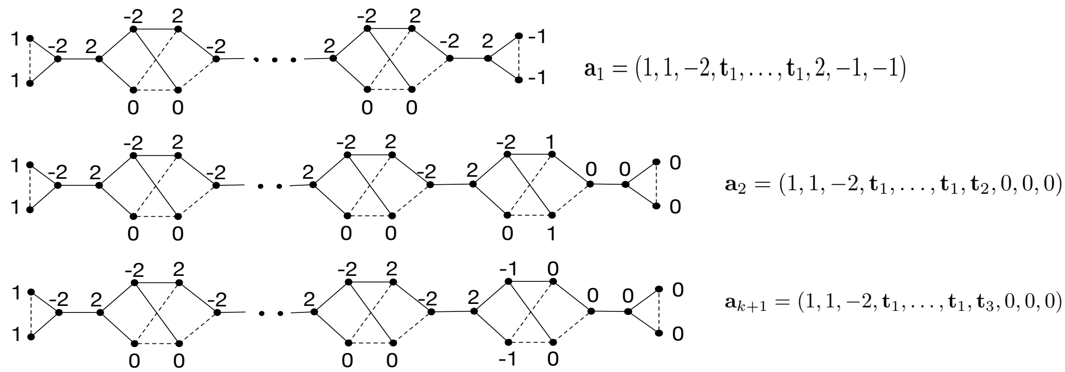

FIG. 7. *Three eigenvectors* $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_{k+1}$ *of* $L(\Sigma_1)$.

It is not difficult to check that $L(\Sigma_1)\mathbf{a}_i = 5\mathbf{a}_i$ for $i = 1, 2, \ldots, 2k-1$. That is, 5 is an eigenvalue of $L(\Sigma_1)$ and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{2k-1}$ are its eigenvectors.

We define the matrix $X = [\mathbf{a}_{2k-1}^T | \mathbf{a}_{2k-2}^T | \cdots | \mathbf{a}_1^T]$. Then we have $\text{row}_{6j}(X) = \mathbf{e}_{2j-1}$ for $j = 1, 2, \ldots, k$, $\text{row}_{6j+2}(X) = \mathbf{e}_{2j}$ for $j = 1, 2, \ldots, k-1$, where $\text{row}_i(X)$ denotes the i-row of the matrix X. So $\text{rank}(X)=2k-1$ and the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{2k-1}$ are linearly independent. Thus, 5 is an eigenvalue of $L(\Sigma_1)$ with multiplicity at least $2k-1$.

Let n-dimensional vectors $\mathbf{b}_i$ and $\mathbf{c}_i$ $(i = 1, 2, \ldots, k-1, k)$ be

$$\begin{aligned}
\mathbf{b}_i &= (\mathbf{0}_{6(k-i)}, 1, -1, -1, -1, 1, 1, \mathbf{0}_{6(i-1)}), \quad \text{for } i = 1, 2, \ldots, k-1, \\
\mathbf{b}_k &= (1, 1, -1, -1, 1, 1, \mathbf{0}_{6(k-1)}), \\
\mathbf{c}_i &= (\mathbf{0}_{6(k-i)}, -1, 1, -1, 1, 1, 1, \mathbf{0}_{6(i-1)}), \quad \text{for } i = 1, 2, \ldots, k-1, \\
\mathbf{c}_k &= (-1, -1, -1, 1, 1, 1, \mathbf{0}_{6(k-1)}).
\end{aligned}$$

For each of vectors $\mathbf{b}_i$, $\mathbf{c}_i$ $(i = 1, 2, \ldots, k)$, we can check that $L(\Sigma_1)\mathbf{b}_i^T = 4\mathbf{b}_i^T$ and $L(\Sigma_1)\mathbf{c}_i^T = 2\mathbf{c}_i^T$ for $i = 1, 2, \ldots, k$. It is easy to verify that the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ are linearly independent and the vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k$ are linearly independent. Thus, 4 and 2 are two eigenvalues of $L(\Sigma_1)$ with multiplicity at least k, respectively.

Let n-dimensional vectors $\mathbf{d}_i$ $(i = 1, 2, \ldots, 2k+1)$ be $\mathbf{d}_1 = (-1, 1, \mathbf{0}_{n-2})$,

$$\begin{aligned}
\mathbf{d}_i &= (2, 0, 2, \underbrace{\mathbf{t}_5, \ldots, \mathbf{t}_5}_{k-i}, \mathbf{t}_6, \mathbf{0}_{6(i-2)+3}), \quad \text{for } i = 2, \ldots, k, \\
\mathbf{d}_i &= (2, 0, 2, \underbrace{\mathbf{t}_5, \ldots, \mathbf{t}_5}_{2k-1-i}, \mathbf{t}_7, \mathbf{0}_{6(i-(k+1))+3}), \quad \text{for } i = k+1, k+2, \ldots, 2k-1, \\
\mathbf{d}_{2k} &= (1, 0, 1, \mathbf{t}_4, \ldots, \mathbf{t}_4, 1, 1, 0), \\
\mathbf{d}_{2k+1} &= (1, 0, 1, \mathbf{t}_4, \ldots, \mathbf{t}_4, 1, 0, 1).
\end{aligned}$$

In the similar way, we get that $L(\Sigma_1)\mathbf{d}_i^T = \mathbf{d}_i^T$ for $i = 1, 2, \ldots, 2k+1$, and the vectors $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_{2k+1}$ are linearly independent. So 1 is an eigenvalue of $L(\Sigma_1)$ with multiplicity at least $2k+1$. Thus, $spec_L(\Sigma_1) = \{5^{2k-1}, 4^k, 2^k, 1^{2k+1}\}$ and proves the (i).

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The proofs (ii) and (iii) are similar and omitted here. ◻

3.1. $\Gamma \in \mathcal{G}$ **contains a 3-cycle.** Before we give the main result of this subsection, we provide some useful lemmas.

Lemma 13. *Let $\Gamma = (G, \sigma) \in \mathcal{G}$. Then there is no two 3-cycles that share an edge.*

Proof. Suppose that there are two 3-cycles $v_1v_2v_3$ and $v_1v_2v_4$ that share an edge v_1v_2. If $v_3 \sim v_4$, then Γ is 3-regular. Next we assume that $v_3 \nsim v_4$. If $d_{v_3} = d_{v_4} = 2$, then $G = K_4 - v_3v_4$. It is easy to verify that $(K_4 - v_3v_4, \sigma)$ is Laplacian integral if and only if $\Gamma \sim (K_4 - v_3v_4, +)$, which have been done. So, without loss of generality, assume that $d_{v_4} = 3$ and v_5 is a new neighbor of v_4. By Lemma 1, we can assume that $v_1 \overset{+}{\sim} v_2$, $v_1 \overset{+}{\sim} v_3$, and $v_1 \overset{+}{\sim} v_4$. Then the 4×4 principal submatrix of $L_\sigma - I$ corresponding to v_1, v_2, v_3, v_4 is

$$S = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -\sigma(v_2v_3) & -\sigma(v_2v_4) \\ -1 & -\sigma(v_2v_3) & d_{v_3} - 1 & 0 \\ -1 & -\sigma(v_2v_4) & 0 & 2 \end{bmatrix}, \text{where } d_{v_3} \in \{2, 3\}.$$

By forbidden subgraphs $\mathcal{H}_5$ and $\mathcal{H}_6$, we have $\sigma(v_2v_3) = +1$.

If $d_{v_3} = 2$, then $\det(S) = -12$ (if $\sigma(v_2v_4) = +1$) and $\det(S) = -4$ (if $\sigma(v_2v_4) = -1$), which is a contradiction.

If $d_{v_3} = 3$, then $\sigma(v_2v_4) = +1$, otherwise Γ contains the subgraph $\mathcal{H}_5$, this is impossible. Now we have $\det(S) = -12$, which is also a contradiction.

So Γ is not Laplacian integral and $\Gamma \notin \mathcal{G}$. This completes the proof. ◻

The following result implies that for any $\Gamma \in \mathcal{G}$, it has no positive 3-cycle.

Lemma 14. *Let $\Gamma = (G, \sigma) \in \mathcal{G}$. Then there is no positive 3-cycle.*

Proof. For a contradiction, we assume that the 3-cycle $(V(C_3) = \{v_1, v_2, v_3\})$ is all-positive. Since Γ is connected and $\Delta(\Gamma) = 3$, we may assume that $d_{v_1} = 3$ and v_1 has a new neighbor v_1', $v_1 \overset{+}{\sim} v_1'$. By Lemma 13, we have $v_1' \nsim v_2, v_3$. The 4×4 principal submatrix of $L_\sigma - I$ corresponding to v_1, v_2, v_3, v_1' is

$$S = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & d_{v_2} - 1 & -1 & 0 \\ -1 & -1 & d_{v_3} - 1 & 0 \\ -1 & 0 & 0 & d_{v_1'} - 1 \end{bmatrix} = \begin{bmatrix} S' & \mathbf{x}^T \\ \mathbf{x} & d_{v_1'} - 1 \end{bmatrix}, \text{where } \mathbf{x} = (-1, 0, 0).$$

First we consider the 3×3 principal submatrix S' of $L_\sigma - I$ corresponding to v_1, v_2, v_3, direct calculations show that $\det(S') = (d_{v_2} - 2)(d_{v_3} - 2) + (d_{v_2} - 1)(d_{v_3} - 1) - 5 \geq 0$ if and only if $d_{v_2} = d_{v_3} = 3$. However, we have $\det(S) = -3$ for $d_{v_2} = d_{v_3} = 3$ and $d_{v_1'} \in \{2, 3\}$, which contradicts Proposition 2.1. So the 3-cycle must be negative. ◻

Remark 3.1. By Lemmas 13, 14 and forbidden subgraph $\mathcal{H}_7$, we can conclude that the negative 3-cycle has at least one vertex of degree 2.

By Remark 3.1, it suffices to consider that the negative 3-cycle has one or two vertices of degree 2 (see Lemmas 15 and 16).

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Lemma 15. *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. If Γ contains one negative 3-cycle which has exactly two vertices of degree 2, then $\Gamma \in \mathcal{G}$ if and only if Γ is switching equivalent to the signed graph Σ_1 or Σ_2 (see Fig. 6).*

Proof. Sufficiency. Lemma 12 shows that Σ_1 and Σ_2 are Laplacian integral.

Necessity. Suppose that a negative 3-cycle is on vertices v_1, v_2, v_3 such that $d_{v_1} = d_{v_2} = 2$ and $d_{v_3} = 3$. By Lemma 1, we can assume that $v_1 \overset{-}{\sim} v_2$, $v_1 \overset{+}{\sim} v_3$ and $v_2 \overset{+}{\sim} v_3$. Let v_4 be the new neighbor of v_3 and $v_3 \overset{+}{\sim} v_4$. Since $d_{v_4} \geq 2$, then v_4 has one new neighbor v_5, $v_4 \overset{+}{\sim} v_5$. The 4×4 principal submatrix of $L_\sigma - I$ corresponding to vertices v_1, v_3, v_4, v_5 is

$$S = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & d_{v_4} - 1 & -1 \\ 0 & 0 & -1 & d_{v_5} - 1 \end{bmatrix}, \quad \text{where } d_{v_i} \in \{2, 3\} \text{ for } i = 4, 5.$$

Direct calculations show that $\det(S) = (d_{v_4} - 2)(d_{v_5} - 1) - 1$. Then $\det(S) \geq 0$ if and only if $d_{v_4} = 3$. So there is one new vertex v_6 that is adjacent to v_4, $v_4 \overset{+}{\sim} v_6$. If $v_5 \sim v_6$, then $\{v_4, v_5, v_6\}$ is a negative 3-cycle and $v_5 \overset{-}{\sim} v_6$. By forbidden subgraph $\mathcal{H}_8$, we have $d_{v_5} = d_{v_6} = 2$. So $\Gamma \sim \Sigma_1$ with order 6. If $v_5 \not\sim v_6$, then the 5×5 principal submatrix of $L_\sigma - I$ corresponding to vertices v_1, v_3, v_4, v_5, v_6 is

$$S = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & d_{v_5} - 1 & 0 \\ 0 & 0 & -1 & 0 & d_{v_6} - 1 \end{bmatrix}, \quad \text{where } d_{v_i} \in \{2, 3\} \text{ for } i = 5, 6.$$

Direct calculations show that $\det(S) = (d_{x_5} - 2)(d_{x_6} - 2) - 1$. Then $\det(S) \geq 0$ if and only if $d_{v_5} = d_{v_6} = 3$. Without loss of generality, assume that v_7, v_8 are the new neighbors of v_5, $v_5 \overset{+}{\sim} v_7, v_8$. If v_6 is adjacent to none of v_7, v_8, or exactly one vertex of v_7, v_8, then $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ together with the neighbors of v_6 induces the forbidden subgraphs $\mathcal{H}_9, \mathcal{H}_{10}$, or $\mathcal{H}_{11}$, contradiction. So $v_6 \sim v_7, v_8$. By Lemma 13, we have $v_7 \not\sim v_8$. Note that $\{v_3, v_4, v_5, v_6, v_7, v_8\}$ induces the G_1 (see Fig. 5), by Lemma 9, then $v_6 \overset{-}{\sim} v_7, v_8$ and there is a new vertex v_9 that is adjacent to v_7 and v_8 with $\sigma(v_7v_9)\sigma(v_8v_9) = -1$.

Case 1. $d_{v_9} = 2$.

So $\Gamma \sim \Gamma_1$ (see Fig. 8), which is the signed graph Σ_2 with order 9.

Case 2. $d_{v_9} = 3$.

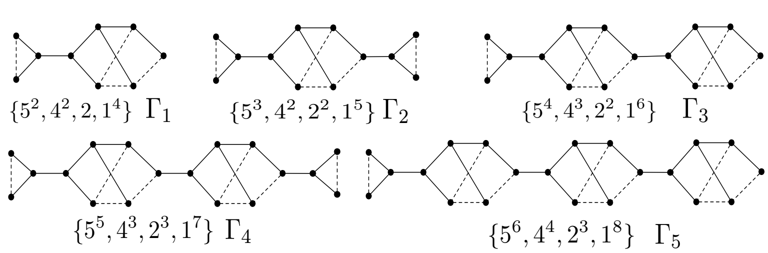

Fig. 8. *The signed graphs Γ_i $(i = 1, 2, 3, 4, 5)$ in the proof the Lemma 15.*

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Let v_{10} be the neighbor of v_9 and $v_9 \overset{+}{\sim} v_{10}$. As $d_{v_{10}} \geq 2$, let v_{11} be the new neighbor of v_{10}, $v_{10} \overset{+}{\sim} v_{11}$. Note that $\{v_7, v_8, v_9, v_{10}, v_{11}\}$ induces a subgraph G_3^σ (see Fig. 5), then $d_{v_{10}} = 3$ (by Lemma 10 (i)). So there is one new vertex v_{12} that is adjacent to v_{10}, $v_{10} \overset{+}{\sim} v_{12}$.

Subcase 2.1. $v_{11} \sim v_{12}$.

Then $\{v_{10}, v_{11}, v_{12}\}$ is a negative 3-cycle and $v_{11} \overset{-}{\sim} v_{12}$. By forbidden subgraph $\mathcal{H}_{12}$, we have $d_{v_{11}} = d_{v_{12}} = 2$. Hence $\Gamma \sim \Gamma_2$ (see Fig. 8), which is the signed graph Σ_1 with order 12.

Subcase 2.2. $v_{11} \not\sim v_{12}$.

As in the case of the vertices v_5, v_6, we have $d_{v_{11}} = d_{v_{12}} = 3$. Let v_{13}, v_{14} be two new neighbors of v_{11}, $v_{11} \overset{+}{\sim} v_{13}, v_{14}$. By forbidden subgraphs $\mathcal{H}_{11}$, $\mathcal{H}_{13}$, and $\mathcal{H}_{14}$, we have $v_{12} \sim v_{13}, v_{14}$. Similarly to the v_7, v_8, by Lemmas 13 and 9, we have $v_{13} \not\sim v_{14}$, $v_{12} \overset{-}{\sim} v_{13}, v_{14}$ and there is a new vertex v_{15} that is adjacent to v_{13} and v_{14} with $\sigma(v_{13}v_{15})\sigma(v_{14}v_{15}) = -1$.

If $d_{v_{15}} = 2$, then $\Gamma \sim \Gamma_3$ (see Fig. 8), which is the signed graph Σ_2 with order 15.

If $d_{v_{15}} = 3$, let v_{16} be the new neighbor of v_{15}, $v_{15} \overset{+}{\sim} v_{16}$. Similar to the v_{10}, we have $d_{v_{16}} = 3$, and let v_{17}, v_{18} be two new neighbors of v_{16}, $v_{16} \overset{+}{\sim} v_{17}, v_{18}$.

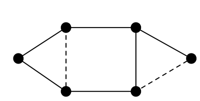
If $v_{17} \sim v_{18}$, similar to subcase 2.1, we have $v_{17} \overset{-}{\sim} v_{18}$ and $d_{v_{17}} = d_{v_{18}} = 2$. Hence $\Gamma \sim \Gamma_4$ (see Fig. 8), which is the signed graph Σ_1 with order 18. If $v_{17} \not\sim v_{18}$, similar to subcase 2.2, we have $d_{v_{17}} = d_{v_{18}} = 3$ and v_{17} has two new neighbors v_{19}, v_{20}, $v_{17} \overset{+}{\sim} v_{19}, v_{20}$, $v_{18} \overset{-}{\sim} v_{19}, v_{20}$, $v_{19} \not\sim v_{20}$ (by Lemma 13) and there is a new vertex v_{21} that is adjacent to v_{19} and v_{20} with $\sigma(v_{19}v_{21})\sigma(v_{20}v_{21}) = -1$ (by Lemma 9).

If $d_{v_{21}} = 2$, then $\Gamma \sim \Gamma_5$ (see Fig. 8) by similar discussions. So Γ is the signed graph Σ_2 with order 21. If $d_{v_{21}} = 3$, continuing the above process, then Γ is switching equivalent to the signed graph Σ_1 or Σ_2. ☐

Lemma 16. *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. If Γ contains one negative 3-cycle which has exactly one vertex of degree 2, then $\Gamma \in \mathcal{G}$ if and only if Γ is switching equivalent to the signed graph Σ_4 (see Fig. 9).*

Proof. Suppose that the negative 3-cycle is on vertices v_1, v_2, v_3 such that $d_{v_1} = 2$ and $d_{v_2} = d_{v_3} = 3$. By Lemma 1, we may assume that $v_1 \overset{+}{\sim} v_2$, $v_1 \overset{+}{\sim} v_3$ and $v_2 \overset{-}{\sim} v_3$. Let v_4 be the new neighbor of v_2 and $v_2 \overset{+}{\sim} v_4$. By Lemma 13, we have $v_3 \not\sim v_4$. So v_3 has a new neighbor v_5 and $v_3 \overset{+}{\sim} v_5$. If $v_4 \not\sim v_5$, since $d_{v_i} \geq 2$ for $i = 4, 5$, then each v_i has a new neighbor and Γ must contain the subgraph $\mathcal{H}_{15}$, which is a contradiction. So $v_4 \overset{+}{\sim} v_5$ (by forbidden subgraph $\mathcal{H}_{16}$). The 5×5 principal submatrix of $L_\sigma - I$ corresponding to vertices v_1, v_2, v_3, v_4, v_5 is

$$S = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 2 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & d_{v_5} - 1 \end{bmatrix}.$$



$\{1^2, 2^2, 5^2\}$

Fig. 9. *The signed graph Σ_4.*

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Then $\det(S) = d_{v_5} - 3 \geq 0$ if and only if $d_{v_5} = 3$. So $v_5 \sim v_6$, otherwise contains the forbidden subgraph $\mathcal{H}_{15}$. Further, we have $v_5 \overset{-}{\sim} v_6$ as $\{v_4, v_5, v_6\}$ is a negative 3-cycle. By forbidden subgraph $\mathcal{H}_{17}$, we have $d_{v_6} = 2$. So $\Gamma \sim \Sigma_4$ (see Fig. 9), which is Laplacian integral. This completes the proof. ∎

3.2. Γ contains no 3-cycle. We now attention to the signed graphs that contain no 3-cycle. Before giving the main result of this subsection, we present one lemma.

Lemma 17. *Let $\Gamma = (G, \sigma) \in \mathcal{G}$. If Γ contains no 3-cycle, then each vertex of degree 3 is adjacent to at most one vertex of degree 2.*

Proof. Let u be the vertex of degree 3 and its neighbor set be $\{w, x, y\}$. Since Γ contains no 3-cycle, then $x \not\sim y$, $x \not\sim w$, and $y \not\sim w$. By Lemma 1, assume that $u \overset{+}{\sim} w, x, y$. The 4×4 principal submatrix of $L_\sigma - I$ corresponding to u, w, x, y is

$$S = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & d_x - 1 & 0 & 0 \\ -1 & 0 & d_y - 1 & 0 \\ -1 & 0 & 0 & d_w - 1 \end{bmatrix}.$$

If at least two of d_x, d_y, d_w are 2, we have $\det(S) = -1$, which contradicts Proposition 2.1. So at most one of d_x, d_y, d_w is 2 and completes the proof. ∎

Now we are ready to give the main result of this subsection.

Lemma 18. *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. If Γ contains no 3-cycle, then $\Gamma \in \mathcal{G}$ if and only if Γ is switching equivalent to the signed graph Σ_3.*

Proof. Sufficiency. Lemma 12 shows that Σ_3 is Laplacian integral.

Necessity. Let v_1 be the vertex of degree 2, v_2 and v_3 be two neighbors of v_1 and $v_1 \overset{+}{\sim} v_2, v_3$. Since Γ contains no 3-cycle, then $v_2 \not\sim v_3$ and $d_{v_2} = d_{v_3} = 3$ (by Lemma 11). Without loss of generality, assume that v_2 has two new neighbors v_4, v_5 and $v_2 \overset{+}{\sim} v_4, v_5$. By Lemma 17, we have $d_{v_4} = d_{v_5} = 3$. The 5×5 principal submatrix of $L_\sigma - I$ corresponding to v_1, v_2, v_3, v_4, v_5 is

$$S = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & -x & -y \\ 0 & -1 & -x & 2 & 0 \\ 0 & -1 & -y & 0 & 2 \end{bmatrix}, \text{where } x = \sigma(v_3v_4) \text{ and } y = \sigma(v_3v_5).$$

Then $\det(S) = -(x+y+1)(x+y+3) - 1 \geq 0$ if and only if $x = y = -1$. So $v_3 \overset{-}{\sim} v_4, v_5$. Let v_6 be the new neighbor of v_4, $v_4 \overset{+}{\sim} v_6$, by forbidden subgraphs $\mathcal{H}_3$ and $\mathcal{H}_4$, we have $v_5 \overset{-}{\sim} v_6$. If $d_{v_6} = 2$, then $\Gamma \sim \Sigma_3$ with order 6. If $d_{v_6} = 3$, let v_7 be the new neighbor of v_6, $v_6 \overset{+}{\sim} v_7$. As $d_{v_7} \geq 2$, then v_7 has a new neighbor v_8, $v_7 \overset{+}{\sim} v_8$. Note that $\{v_4, v_5, v_6, v_7, v_8\}$ induces the G_3^σ, then $d_{v_7} = 3$ (by Lemma 10 (i)). So there is one new vertex v_9 that is adjacent to v_7, $v_7 \overset{+}{\sim} v_9$. Also note that $\{v_4, v_5, v_6, v_7, v_8, v_9\}$ induces the G_4^σ, then $d_{v_8} = d_{v_9} = 3$ (by Lemma 10 (ii)). Without loss of generality, assume that v_8 has two new neighbors v_{10}, v_{11}, $v_8 \overset{+}{\sim} v_{10}, v_{11}$. By forbidden subgraphs $\mathcal{H}_{11}$, $\mathcal{H}_{13}$, and $\mathcal{H}_{14}$, v_9 is adjacent to v_{10}, v_{11}. Note that $\{v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$ induces the G_1 (see Fig. 5), by Lemma 9, then $v_9 \overset{-}{\sim} v_{10}, v_{11}$ and there is one new vertex v_{12} that is adjacent to v_{10} and v_{11} with $\sigma(v_{10}v_{12})\sigma(v_{11}v_{12}) = -1$.

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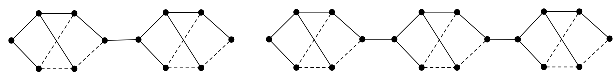

$\{5^3, 4^3, 2^1, 1^5\}$ Γ_6 $\{5^5, 4^4, 2^2, 1^7\}$ Γ_7

FIG. 10. *The signed graphs Γ_i $(i = 6, 7)$ in the proof of Lemma 18.*

If $d_{v_{12}} = 2$, then $\Gamma \sim \Gamma_6$ (see Fig. 10), which is the signed graph Σ_3 with order 12. If $d_{v_{12}} = 3$, then v_{12} has a new neighbor v_{13}, $v_{12} \overset{+}{\sim} v_{13}$. Similar to the v_7, we have $d_{v_{13}} = 3$, and let v_{14}, v_{15} be two new neighbors of v_{13} and $v_{13} \overset{+}{\sim} v_{14}, v_{15}$. Similar to the v_8, v_9, we have $d_{v_{14}} = d_{v_{15}} = 3$ and v_{14} has two new neighbors v_{16}, v_{17}, $v_{14} \overset{+}{\sim} v_{16}, v_{17}$. By forbidden subgraphs $\mathcal{H}_{11}$, $\mathcal{H}_{13}$, $\mathcal{H}_{14}$ and Lemma 9, we have $v_{15} \overset{-}{\sim} v_{16}, v_{17}$ and there is one new vertex v_{18} that is adjacent to v_{16} and v_{17} with $\sigma(v_{16}v_{18})\sigma(v_{17}v_{18}) = -1$.

If $d_{v_{18}} = 2$, then $\Gamma \sim \Gamma_7$ (see Fig. 10) by similar discussions. So Γ is the signed graph Σ_3 with order 18. If $d_{v_{18}} = 3$, continuing the above process, then Γ is switching equivalent to Σ_3. $\square$

Putting Lemmas 4, 5, 15, 16, and 18 together, we obtain the main result of this paper.

THEOREM 19. *Let $\Gamma = (G, \sigma)$ be a connected Laplacian integral signed graph of maximum degree 3, then Γ is switching equivalent to*

(i) *the signed graphs $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$,*
(ii) *the unsigned graphs of Fig. 3,*
(iii) *the 3-regular signed graphs in Figs. 1 and 2, or their negations.*

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