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BEYOND THE FAN–HOFFMAN INEQUALITY*

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Abstract. We establish a singular value inequality inspired by the Fan–Hoffman inequality and resolve the j-conjecture of Yang and Zhang.

Key words. Singular value, Eigenvalue, j-conjecture.

AMS subject classification. 15A42.

1. Introduction. Majorization inequalities for the eigenvalues and singular values of matrices are commonplace. Inequalities that compare the j^{th} eigenvalue or singular value of one matrix to that of another are more special. One such inequality is the Fan–Hoffman inequality [2] which asserts that

$$\lambda_j(\Re(A)) \le \sigma_j(|A|) \text{ for } j = 1, \ldots, n,$$

for an $n \times n$ complex matrix A. The notation A^* denotes the adjoint matrix of A, $\Re(A)$ denotes the hermitian part $\frac{1}{2}(A + A^*)$ of A, and $|A| = (A^*A)^{\frac{1}{2}}$ the absolute value of A. For a hermitian matrix H, $\lambda_j(H)$ denotes the j^{th} eigenvalue taken in decreasing order and $\sigma_j(A) = \lambda_j(|A|)$, the j^{th} singular value of A. For more details on these concepts, the reader may consult Bhatia [1].

In a related paper [3], R.C. Thompson establishes the following result.

Theorem 1. *For any pair A, B of $n \times n$ complex matrices, there exist unitary $n \times n$ matrices U and V such that $|A + B| \le U^*|A|U + V^*|B|V$ where $\le$ is understood in the positive semidefinite sense.*

An immediate consequence is the following.

Corollary 2. *For any $n \times n$ complex matrix A and any $n \times n$ contraction C, we have*

$$\sigma_j(C + A) \le \lambda_j(I + |A|) \text{ for } j = 1, \ldots, n.$$

Here, we have denoted by I the $n \times n$ identity matrix.

By a contraction C, we mean a complex matrix with its spectral norm $\|C\| \le 1$. It is this corollary that we intend to generalize by establishing the following.

Theorem 3. *For any $n \times n$ complex matrix A, any $n \times n$ contraction C and any positive nondecreasing function $f : [0, \|A\|] \to (0, \infty)$, the inequality*

$$\sigma_j\Big((C + A)f(|A|)\Big) \le \lambda_j\Big((I + |A|)f(|A|)\Big) \text{ for } j = 1, \ldots, n, \tag{1.1}$$

holds.

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For a hermitian matrix H with spectral decomposition

$$H = \sum_{j=1}^{n} \lambda_j(H) e_j e_j^*,$$

with $\{e_j, j = 1, \dots, n\}$ an orthonormal basis, we have denoted

$$f(H) = \sum_{j=1}^{n} f(\lambda_j(H)) e_j e_j^*,$$

the standard symbolic calculus for hermitian matrices. Note that $(I + |A|)f(|A|)$ is hermitian since $I + |A|$ and $f(|A|)$ are commuting hermitian matrices.

As a consequence, we will establish the j-conjecture of Yang and Zhang [4].

Theorem 4. *Let A be a strict contraction. Then*

$$\lambda_j\Big(\Re((I - A)^{-1})\Big) \geq \lambda_j\Big((I + |A|)^{-1}\Big), \text{ for } j = 1, 2, \dots, n.$$

2. Proofs of the results.

Proof. Proof of Theorem 3.

Since $|A|$ has only finitely many eigenvalues, we may always extend f to a continuous positive non-decreasing function defined on $(0, \infty)$ which we do without change of notation. The function g given by $g(t) = (1 + t)f(t)$ is continuous and strictly increasing tending to ∞ at ∞. The inverse function h of g is also a strictly increasing continuous function.

Equivalent to 1.1, we will show

$$\lambda_j(B(C^*C + C^*A + A^*C + A^*A)B) \leq \lambda_j((I + |A|)B)^2, \tag{2.1}$$

where $B = f(|A|)$. We will assume that 2.1 is false and find a contradiction. Then there exists a real number ν such that

$$\lambda_j(B(C^*C + C^*A + CA^* + A^*A)B) > \nu^2 \qquad \text{and} \qquad \nu > \lambda_j((I + |A|)B).$$

Also we have $B = f(|A|) \geq f(0)I$ so that $f(0) \leq \lambda_n(B) \leq \lambda_j(B) \leq \lambda_j((I+|A|)B)$. Hence by the intermediate value theorem, there exists μ such that $(1 + \mu)f(\mu) = g(\mu) = \nu$. Hence, there is a linear subspace E of dimension j such that

$$\xi^* B(C^*C + C^*A + CA^* + A^*A)B\xi > (1 + \mu)^2 f(\mu)^2 \|\xi\|^2,$$

for all nonzero $\xi \in E$.

Let

$$(I + |A|)B = \sum_{k=1}^{n} \lambda_k((I + |A|)B)\eta_k \eta_k^*,$$

be the spectral decomposition of $(I+|A|)B$ where the η_k are the mutually orthogonal unit eigenvectors. Let F be the linear subspace of dimension $n+1-j$ spanned by $\{\eta_k; j \leq k \leq n\}$. On F we have $(I+|A|)B \leq \nu I$.

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Since h is increasing and f is nondecreasing

$$|A| = h((I+|A|)B) \le h(\nu I) = \mu I \qquad \text{and} \qquad |B| = f(|A|) \le f(\mu)I,$$

on F. Note also that F is invariant under both $|A|$ and B since $|A| = h((I+|A|)B)$, $B = f(|A|)$ and using the symbolic calculus of hermitian matrices.

By dimensionality, the intersection $E \cap F$ is forced to be nonzero. We choose a unit vector $\xi \in E \cap F$. Then

$$\begin{aligned}
(1+\mu)^2 f(\mu)^2 &< \xi^* B(C^*C + C^*A + CA^* + A^*A)B\xi \\
&\le \|CB\xi\|^2 + 2\Re \xi^* BC^*AB\xi + \xi^* B|A|^2 B\xi \\
&\le \|B\xi\|^2 + 2|\xi^* BC^*AB\xi| + \mu^2 f(\mu)^2 \\
&\le \xi^* B^2 \xi + 2\|CB\xi\|\|AB\xi\| + \mu^2 f(\mu)^2 \\
&\le f(\mu)^2 + 2\|B\xi\|\|AB\xi\| + \mu^2 f(\mu)^2 \\
&\le (1+\mu^2) f(\mu)^2 + 2\sqrt{\xi^* B^2 \xi}\sqrt{\xi^* BA^*AB\xi} \\
&\le (1+\mu^2) f(\mu)^2 + 2\sqrt{f(\mu)^2}\sqrt{\xi^* B|A|^2 B\xi} \\
&\le (1+\mu^2) f(\mu)^2 + 2f(\mu)\sqrt{\mu^2 f(\mu)^2} = (1+\mu)^2 f(\mu)^2.
\end{aligned}$$

This contradiction establishes 2.1. ∎

Proof. Proof of Theorem 4

We replace A by $-A$ which is also a strict contraction, and we will therefore prove that

$$\lambda_j\Big(\Re((I+A)^{-1})\Big) \ge \lambda_j\Big((I+|A|)^{-1}\Big), \text{ for } j = 1, 2, \dots, n.$$

We have

$$\begin{aligned}
2\Re((I+A)^{-1}) &= (I+A^*)^{-1} + (I+A)^{-1} \\
&= (I+A^*)^{-1}(2I + A^* + A)(I+A)^{-1} \\
&= (I+A^*)^{-1}\Big((I + A^* + A + A^*A) + (I - A^*A)\Big)(I+A)^{-1} \\
&= I + (I+A^*)^{-1}(I - A^*A)(I+A)^{-1}.
\end{aligned}$$

On the other hand, we have

$$2(I+|A|)^{-1} = I + (I+|A|)^{-\frac{1}{2}}(I-|A|)(I+|A|)^{-\frac{1}{2}}.$$

Therefore, it suffices to show that

$$\lambda_j\Big((I+A^*)^{-1}(I-A^*A)(I+A)^{-1}\Big) \ge \lambda_j\Big((I+|A|)^{-\frac{1}{2}}(I-|A|)(I+|A|)^{-\frac{1}{2}}\Big),$$

or equivalently

$$\sigma_j\Big((I-|A|^2)^{\frac{1}{2}}(I+A)^{-1}\Big) \ge \sigma_j\Big((I-|A|)^{\frac{1}{2}}(I+|A|)^{-\frac{1}{2}}\Big),$$

or since $\sigma_j(X) = (\sigma_{n+1-j}(X^{-1}))^{-1}$ that

$$\sigma_{n+1-j}\Big((I+A)(I-|A|^2)^{-\frac{1}{2}}\Big) \le \sigma_{n+1-j}\Big((I+|A|)^{\frac{1}{2}}(I-|A|)^{-\frac{1}{2}}\Big),$$

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But

$$(I + |A|)^{\frac{1}{2}}(I - |A|)^{-\frac{1}{2}} = (I + |A|)(I - |A|^2)^{-\frac{1}{2}},$$

so finally, it remains to show that

$$\sigma_{n+1-j}\Big((I + A)f(|A|)\Big) \leq \sigma_{n+1-j}\Big((I + |A|)f(|A|)\Big),$$

where $f(t) = (1 - t^2)^{-\frac{1}{2}}$. But f is a positive increasing function on $[0, \|A\|]$ and the result follows from Theorem 3 taking $C = I$. ◻

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