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CERTAIN MATRICES RELATED TO THE FIBONACCI SEQUENCE HAVING RECURSIVE ENTRIES*

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Abstract. Let $\phi = (\phi_i)_{i\geq 1}$ and $\psi = (\psi_i)_{i\geq 1}$ be two arbitrary sequences with $\phi_1 = \psi_1$. Let $A_{\phi,\psi}(n)$ denote the matrix of order n with entries $a_{i,j}$, $1 \leq i, j \leq n$, where $a_{1,j} = \phi_j$ and $a_{i,1} = \psi_i$ for $1 \leq i \leq n$, and where $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ for $2 \leq i, j \leq n$. It is of interest to evaluate the determinant of $A_{\phi,\psi}(n)$, where one of the sequences ϕ or ψ is the Fibonacci sequence (i.e., $1, 1, 2, 3, 5, 8, \ldots$) and the other is one of the following sequences:

$$\alpha^{(k)} = (\overbrace{1, 1, \ldots, 1}^{k-\text{times}}, 0, 0, 0, \ldots)\ ,$$
$$\chi^{(k)} = (1^k, 2^k, 3^k, \ldots, i^k, \ldots),$$
$$\xi^{(k)} = (1, k, k^2, \ldots, k^{i-1}, \ldots) \quad \text{(a geometric sequence)},$$
$$\gamma^{(k)} = (1, 1+k, 1+2k, \ldots, 1+(i-1)k, \ldots) \quad \text{(an arithmetic sequence)}.$$

For some sequences of the above type the inverse of $A_{\phi,\psi}(n)$ is found. In the final part of this paper, the determinant of a generalized Pascal triangle associated to the Fibonacci sequence is found.

Key words. Inverse matrix, Determinant, LU-factorization, Fibonacci sequence, Generalized Pascal triangle, Recursive relation.

AMS subject classifications. 15A09, 11B39.

1. Introduction. Generally, finding the inverse of a matrix plays an important part in many areas of science. For example, decrypting an encoded message or solving a square system of linear equations deals with finding the inverse of a matrix. In some instances, authors have discovered special matrix inversions; see, e.g., [2], [3], [4] and [5]). Also, Milne [9], focuses on inverses of lower triangular matrices.

Furthermore, when one is faced with matrices whose entries obey a recursive relation, inversion can be complicated. Such matrices have been studied e.g., in [1], [6], [8], [10], [11], [12] and [14]. The research done in these papers is mostly related to the determinants; rarely are the inverses discussed. One of the main purposes of

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this paper is to consider matrices whose entries are obtained by recursive formulas and are related to two sequences, one of which is the Fibonacci sequence, and then try to invert them. Since some of the matrices considered are lower triangular, it may seem that their inverse matrices can be found by Milne's results [9], but our approach to the problem is new. Another aim of ours is to study the determinants of these matrices.

2. Notations and Definitions. The generalized Pascal triangles were first introduced by Bacher in [1], as follows. Let $\phi = (\phi_i)_{i\geq 1}$ and $\psi = (\psi_i)_{i\geq 1}$ be two sequences starting with a common first term $\phi_1 = \psi_1$. We define a matrix $P_{\phi,\psi}(n)$ of order n with entries $p_{i,j}$ by setting

$$p_{i,j} = \begin{cases} \psi_j & \text{if} \quad i = 1, 1 \leq j \leq n, \\ \phi_i & \text{if} \quad j = 1, 1 \leq i \leq n, \\ p_{i-1,j} + p_{i,j-1} & \text{if} \quad 2 \leq i, j \leq n. \end{cases}$$

The infinite matrix $P_{\phi,\psi}(\infty)$ is called the *generalized Pascal triangle* associated to the sequences ϕ and ψ.

Following the ideas of Bacher in [1], we define in a similar way some matrices whose entries are determined recursively and are associated with some sequences.

Definitions. Let $\phi = (\phi_i)_{i\geq 1}$, $\psi = (\psi_i)_{i\geq 1}$, $\lambda = (\lambda_i)_{i\geq 1}$ and $\delta = (\delta_i)_{i\geq 1}$ be sequences such that $\phi_1 = \psi_1$. We denote by $A^c_{\phi,\psi,\lambda,\delta}(m,n)$ the matrix of size m by n with entries $a_{i,j}$, $1 \leq i \leq m, 1 \leq j \leq n$ by setting $a_{1,j} = \phi_j$ for $1 \leq j \leq n$, $a_{i,1} = \psi_i$ for $1 \leq i \leq m$ and

$$a_{i,j} = \delta_j a_{i-1,j-1} + \lambda_j a_{i-1,j}, \tag{2.1}$$

for $2 \leq i \leq m$ and $2 \leq j \leq n$.

Similarly, we denote by $A^r_{\phi,\psi,\lambda,\delta}(m,n)$ the matrix of size m by n with entries $b_{i,j}$, $1 \leq i \leq m, 1 \leq j \leq n$ by setting $b_{1,j} = \phi_j$ for $1 \leq j \leq n$, $b_{i,1} = \psi_i$ for $1 \leq i \leq m$ and

$$b_{i,j} = \delta_i (b_{i-1,j-1} + \lambda_{i-1} b_{i-1,j}), \tag{2.2}$$

for $2 \leq i \leq m$ and $2 \leq j \leq n$.

When $\lambda = \delta = 1$ and $m = n$, for convenience, we put $A^c_{\phi,\psi,\lambda,\delta}(m,n) = A_{\phi,\psi}(n)$.

We recall that a sequence $\sigma = (\sigma_i)_{i\geq 1}$ satisfies a *linear recursion of order k* if there exist constants $c_1, c_2, \ldots, c_k$ (with $c_k \neq 0$) such that

$$\sigma_n = \sum_{i=1}^{k} c_i \sigma_{n-i},$$

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for all $n > k$. *Initial conditions* for the sequence σ are explicitly given values for a finite number of the terms of the sequence.

We will use the following notations: The *Fibonacci numbers* $F(n)$ satisfy

$$\begin{cases} F(0) = 0,\, F(1) = 1 \\ F(n+2) = F(n+1) + F(n) \quad (n \geq 0). \end{cases}$$

Throughout this article we assume that

$$\begin{aligned}
&\mathcal{F} = (\mathcal{F}_i)_{i\geq 1} = (1,1,2,3,5,8,\ldots,\mathcal{F}_i = F(i),\ldots) && \text{(Fibonacci numbers} \neq 0),\\
&\alpha^{(k)} = (\alpha_i^{(k)})_{i\geq 1} = (\overbrace{1,1,\ldots,1}^{k-\text{times}},0,0,0,\ldots) && \text{(when } k=1, \text{ we put } \alpha^{(1)} = \alpha),\\
&\chi^{(k)} = (\chi_i^{(k)})_{i\geq 1} = (1^k,2^k,3^k,\ldots,\chi_i^{(k)} = i^k,\ldots) && \text{(when } k=1, \text{ we put } \chi^{(1)} = \chi),\\
&\xi^{(k)} = (\xi_i^{(k)})_{i\geq 1} = (1,k,k^2,\ldots,\xi_i^{(k)} = k^{i-1},\ldots) && \text{(a geometric sequence)},\\
&\gamma^{(k)} = (\gamma_i^{(k)})_{i\geq 1} = (1,1+k,\ldots,\gamma_i^{(k)} = 1+(i-1)k,\ldots) && \text{(an arithmetic sequence)}.
\end{aligned}$$

Moreover, we assume that 1 denotes the all ones vector. Given a matrix A we denote by

$$A_{[i_1,i_2,\ldots,i_l]}^{[j_1,j_2,\ldots,j_k]}$$

the submatrix of A obtained by erasing rows $i_1, i_2, \ldots, i_l$ and columns $j_1, j_2, \ldots, j_k$. We use the notation A^T for the transpose of A. Also, we denote by $\mathrm{R}_i(A)$ and $\mathrm{C}_j(A)$ the row i and the column j of A, respectively. A matrix $T = (t_{i,j})_{1\leq i,j\leq n}$ is said to be Töeplitz if $t_{i,j} = t_{k,l}$ whenever $i - j = k - l$. For a Töeplitz matrix $T = (t_{i,j})_{1\leq i,j\leq n}$, if $\eta = (t_{1,j})_{1\leq j\leq n}$ and $\theta = (t_{i,1})_{1\leq i\leq n}$, then we write $T = T_{\eta,\theta}(n)$.

3. A Preliminary Result. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a finite sequence with $x_1 \neq 0$. We define the matrix $A_{\mathbf{x}}(n)$ of order n with entries $a_{i,j}$, $1 \leq i, j \leq n$, by setting

$$a_{i,j} = \begin{cases} x_{l+1} & j = i + l,\ 0 \leq l \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now that the sequence $\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots)$ satisfies the recurrence

$$\sigma_i = -x_1^{-1} \sum_{l=2}^{n} x_l \sigma_{i-l+1},$$

for all $i > n-1$, and the initial conditions $\sigma_i = 0$ $(1 \leq i \leq n-2)$, $\sigma_{n-1} = x_1^{-1}$. We put $\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$, where $\dot{x}_i = \sigma_{n+i-2}$ $(1 \leq i \leq n)$. In the following lemma we show that $A_{\mathbf{x}}(n)$ and $A_{\dot{\mathbf{x}}}(n)$ are matrix inverses.

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LEMMA 3.1. *Let* $\mathbf{x}$ *and* $\dot{\mathbf{x}}$ *defined as above. Then* $A_{\mathbf{x}}(n)$ *and* $A_{\dot{\mathbf{x}}}(n)$ *are matrix inverses.*

Proof. Let $A_{\mathbf{x}}(n) = (a_{i,j})_{1\leq i,j\leq n}$ and $A_{\dot{\mathbf{x}}}(n) = (b_{i,j})_{1\leq i,j\leq n}$. Indeed, we must establish

$$A_{\mathbf{x}}(n)\cdot A_{\dot{\mathbf{x}}}(n) = I,$$

where I is the identity matrix of order n. Since both matrices $A_{\mathbf{x}}(n)$ and $A_{\dot{\mathbf{x}}}(n)$ are upper triangular matrices with x_1 and x_1^{-1} on their diagonals, respectively, $A_{\mathbf{x}}(n)\cdot A_{\dot{\mathbf{x}}}(n)$ is an upper triangular matrix with 1's on its diagonal. Hence, we have to show that the product of row i of $A_{\mathbf{x}}(n)$ with column j of $A_{\dot{\mathbf{x}}}(n)$ when $i < j$ is always 0. To see this, we observe that

$$\mathrm{R}_i(A_{\mathbf{x}}(n))\cdot \mathrm{C}_j(A_{\dot{\mathbf{x}}}(n)) = \sum_{k=i}^{j} a_{i,k}b_{k,j} = \sum_{l=1}^{j-i+1} x_l\dot{x}_{j-i-l+2} = \sum_{l=1}^{j-i+1} x_l\sigma_{n+j-i-l}$$

$$= x_1\sigma_{n+j-i-1} + \sum_{l=2}^{j-i+1} x_l\sigma_{n+j-i-l}$$

$$= x_1(-x_1^{-1}\sum_{l=2}^{j-i+1} x_l\sigma_{n+j-i-l}) + \sum_{l=2}^{j-i-1} x_l\sigma_{n+j-i-l} = 0,$$

as desired. ☐

Example. Take the sequence $\mathbf{x} = (1,-1,-1,0,0,0,\ldots,0)$. Through direct calculations we obtain

$$\dot{\mathbf{x}} = (F(1), F(2), F(3), \ldots, F(n)).$$

Moreover, the matrices $A_{\mathbf{x}}(n)$ and $A_{\dot{\mathbf{x}}}(n)$ are given by

$$A_{\mathbf{x}}(n) = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

and

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$$A_{\grave{x}}(n)=\begin{bmatrix} F(1) & F(2) & F(3) & \dots & F(n-1) & F(n) \\ 0 & F(1) & F(2) & \dots & F(n-2) & F(n-1) \\ 0 & 0 & F(1) & \dots & F(n-3) & F(n-2) \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & F(1) & F(2) \\ 0 & 0 & 0 & \dots & 0 & F(1) \end{bmatrix}.$$

4. Main Results. This section consists of six parts where in each subdivision the matrices of type $A^c_{\phi,\psi,\lambda,\delta}(n,n)$, for certain sequences ϕ, ψ, λ and δ, will be treated separately. In this treatment we have evaluated the determinants of such matrices and in some instances we have been able to determine the inverse matrix.

4.1. On the matrices $A^c_{\alpha,\mathcal{F},\lambda,\delta}(n,n)$. Given two arbitrary sequences $\lambda=(\lambda_i)_{i\ge 1}$ with $\lambda_1=1$, $\delta=(\delta_i)_{i\ge 1}$ with $\delta_i\neq 0$ for every i, and the sequence $\gamma=(\gamma_i)_{i\ge 1}=(-1,0,1,0,0,0,\ldots)$, we consider the matrices

$$L=A^c_{\alpha,\mathcal{F},\lambda,\delta}(n,n) \qquad \text{and} \qquad \tilde{L}=A^r_{\gamma,-\alpha,-\lambda,\delta^{-1}}(n,n+2)^{[1,2]},$$

where $-\alpha=(-\alpha_i)_{i\ge 1}$, $-\lambda=(-\lambda_i)_{i\ge 1}$ and $\delta^{-1}=(\delta_i^{-1})_{i\ge 1}$. The matrices L and $\tilde{L}$ are lower triangular matrices of order n with diagonal entries

$$1,\ \delta_2,\ \delta_2\delta_3,\ \delta_2\delta_3\delta_4,\ldots,\ \delta_2\delta_3\ldots\delta_n,$$

and

$$1,\ \delta_2^{-1},\ \delta_2^{-1}\delta_3^{-1},\ \delta_2^{-1}\delta_3^{-1}\delta_4^{-1},\ldots,\ \delta_2^{-1}\delta_3^{-1}\ldots\delta_n^{-1},$$

respectively. Hence, $\det L\cdot\det\tilde{L}=1$ from which it follows that L and $\tilde{L}$ are invertible.

Let

$$\Psi(j,k)=\sum\lambda_2^{n_2}\lambda_3^{n_3}\ldots\lambda_j^{n_j} \qquad (j\ge 2,k\ge 0), \tag{4.1}$$

where the summation is over all possible choices of $n_2,n_3,\ldots,n_j$ satisfying $n_2+n_3+\cdots+n_j=k$. Suppose $L=(\hat{a}_{i,j})_{1\le i,j\le n}$. It is easy to see that the entry $\hat{a}_{i,j}$ is also given by the formula

$$\hat{a}_{i,j}=\begin{cases} 0 & \text{if} \quad i<j, \\ \delta_2\delta_3\ldots\delta_j & \text{if} \quad i=j>1, \\ F(i) & \text{if} \quad j=1, \\ \delta_2\delta_3\ldots\delta_j\sum_{k=0}^{i-j}F(i-j+1-k)\Psi(j,k) & \text{if} \quad i>j>1. \end{cases} \tag{4.2}$$

In particular, we have

$$\hat{a}_{i,i-1}=\begin{cases} 1 & i=2 \\ \delta_2\delta_3\ldots\delta_{i-1}(\lambda_1+\lambda_2+\cdots+\lambda_{i-1}) & 3\le i\le n. \end{cases}$$

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Similarly, we assume that $\tilde{L} = (\hat{b}_{i,j})_{1\leq i,j\leq n}$. Here, we can also easily see that

$$\hat{b}_{i,i-1} = \delta_2^{-1}\delta_3^{-1}\dots\delta_i^{-1}(-\lambda_1 - \lambda_2 - \cdots - \lambda_{i-1}), \qquad (2 \leq i \leq n).$$

Theorem 4.1. *Let L and $\tilde{L}$ defined as above. Then L and $\tilde{L}$ are matrix inverses. Moreover, we have*

$$(L^{[n,n-1,\dots,m]}_{[n,n-1,\dots,m]})^{-1} = \tilde{L}^{[n,n-1,\dots,m]}_{[n,n-1,\dots,m]}, \quad 1 < m \leq n$$

and

$$(L^{[1,2,\dots,m]}_{[1,2,\dots,m]})^{-1} = \tilde{L}^{[1,2,\dots,m]}_{[1,2,\dots,m]}, \quad 1 \leq m < n.$$

Proof. It is easy to see that the matrix L and $\tilde{L}$ can be written as MD and $D^{-1}\tilde{M}$ respectively, where

$$M = A^c_{\alpha,\mathcal{F},\lambda,1}(n,n) = (a_{i,j})_{1\leq i,j\leq n} \quad , \quad \tilde{M} = A^r_{\gamma,-\alpha,-\lambda,1}(n,n+2)^{[1,2]} = (b_{i,j})_{1\leq i,j\leq n},$$

and $D = \mathrm{diag}(1, \delta_2, \delta_2\delta_3, \dots, \delta_2\delta_3\dots\delta_n)$. Since $L^{-1} = D^{-1}M^{-1}$, it is enough to prove the theorem for the simpler matrix M. In fact, we will prove that $M^{-1} = \tilde{M}$. Therefore, we must establish $\tilde{M} \cdot M = I$, where I is the identity matrix of order n. First of all, since both matrices $\tilde{M}$ and M are lower triangular matrices with 1's on their diagonals, thus $\tilde{M} \cdot M$ is also a lower triangular matrix with 1's on its diagonal. Hence, we must show that the product of row i of $\tilde{M}$ with column j of M when $i > j$ is always 0. Therefore we assume that $i > j$ and proceed by induction on j. This will be done below by going through a sequence of separately stated lemmas.

Since the matrix $\tilde{M}$ is determined after the omission of the first two columns of matrix

$$A^r_{\gamma,-\alpha,-\lambda,1}(n, n+2),$$

for convenience, we put

$$A^r_{\gamma,-\alpha,-\lambda,1}(n, n+2) = \begin{bmatrix} b_{1,-1} & b_{1,0} & b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,-1} & b_{2,0} & b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ b_{3,-1} & b_{3,0} & b_{3,1} & b_{3,2} & \cdots & b_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n,-1} & b_{n,0} & b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix}.$$

Moreover, we notice that

$$a_{i,i-1} = -b_{i,i-1} = \sum_{k=1}^{i-1} \lambda_k \ \text{ for every } \ i \geq 2.$$

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LEMMA 4.2. *Let* $1 < i \leq n$. *Then we have* $\mathrm{R}_i(\tilde{M}) \cdot \mathrm{C}_1(M) = 0$.

Proof. The proof is by induction on i. We first prove the result for $i = 2, 3$. Since, for $i = 2$ we have

$$\mathrm{R}_2(\tilde{M}) = (-\lambda_1, 1, 0, 0, \ldots, 0) \quad \text{and} \quad \mathrm{C}_1(M) = (F(1), F(2), \ldots, F(n))^T,$$

it follows that

$$\mathrm{R}_2(\tilde{M}) \cdot \mathrm{C}_1(M) = -F(1) + F(2) = 0.$$

Similarly, when $i = 3$ we have

$$\mathrm{R}_3(\tilde{M}) = (-1 + \lambda_1\lambda_2, -\lambda_1 - \lambda_2, 1, 0, 0, \ldots, 0),$$

and hence

$$\mathrm{R}_3(\tilde{M}) \cdot \mathrm{C}_1(M) = (-1 + \lambda_1\lambda_2)F(1) - (\lambda_1 + \lambda_2)F(2) + F(3) = 0.$$

We now assume the result inductively for $i = 2, 3, \ldots, k-1$ and prove it for $i = k$. Therefore we see that it is sufficient to establish that

$$\mathrm{R}_k(\tilde{M}) \cdot \mathrm{C}_1(M) = (b_{k,1}, b_{k,2}, \ldots, b_{k,k}, 0, 0, \ldots, 0) \cdot (F(1), F(2), \ldots, F(n))^T = 0,$$

or, equivalently,

$$\sum_{l=1}^{k} b_{k,l}F(l) = 0. \tag{4.3}$$

Before starting the proof of (4.3), we provide some remarks. We know by induction hypothesis $\mathrm{R}_{k-2}(\tilde{M}) \cdot \mathrm{C}_1(M) = 0$, hence

$$\sum_{l=1}^{k-2} b_{k-2,l}F(l) = 0. \tag{4.4}$$

Similarly, since $\mathrm{R}_{k-1}(\tilde{M}) \cdot \mathrm{C}_1(M) = 0$, we obtain

$$\sum_{l=1}^{k-1} b_{k-1,l}F(l) = 0. \tag{4.5}$$

On the other hand, by the definition of $b_{i,j}$ as given by (2.2), we have

$$b_{k-1,l} = b_{k-2,l-1} - \lambda_{k-2}b_{k-2,l}, \quad l = 1, 2, \ldots, k-1.$$

After having substituted these values in the left hand side of (4.5) and by using (4.4), we deduce that

$$\sum_{l=1}^{k-1} b_{k-2,l-1}F(l) = 0. \tag{4.6}$$

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We are now ready to prove (4.3). We observe again as above that

$$b_{k,l} = b_{k-2,l-2} - \lambda_{k-2} b_{k-2,l-1} - \lambda_{k-1} b_{k-1,l}, \quad l = 1, 2, \ldots, k.$$

If these values are substituted in (4.3) and the sums are put together, then we obtain that

$$\sum_{l=1}^{k} b_{k,l} F(l) = \sum_{l=1}^{k} b_{k-2,l-2} F(l) - \lambda_{k-2} \sum_{l=1}^{k} b_{k-2,l-1} F(l) - \lambda_{k-1} \sum_{l=1}^{k} b_{k-1,l} F(l)$$

$$= \sum_{l=2}^{k} b_{k-2,l-2} F(l) - \lambda_{k-2} \sum_{l=1}^{k-1} b_{k-2,l-1} F(l)$$

(by (4.5) and since $b_{k-2,-1} = 0$)

$$= \sum_{l=2}^{k} b_{k-2,l-2} \{F(l-2) + F(l-1)\} - \lambda_{k-2} \sum_{l=1}^{k-1} b_{k-2,l-1} F(l)$$

$$= \sum_{l=0}^{k-2} b_{k-2,l} F(l) + \sum_{l=1}^{k-1} b_{k-2,l-1} F(l) - \lambda_{k-2} \sum_{l=1}^{k-1} b_{k-2,l-1} F(l)$$

$$= \sum_{l=1}^{k-2} b_{k-2,l} F(l) + (1 - \lambda_{k-2}) \sum_{l=1}^{k-1} b_{k-2,l-1} F(l) = 0 \ \text{(by (4.4) and (4.6))}.$$

The proof is now completed. □

LEMMA 4.3. *Let j be fixed with $1 \leq j < n$. Furthermore, we assume that*

$$\text{“for all } r \text{ with } j < r \leq n \text{ we have } \mathrm{R}_r(\tilde{M}) \cdot \mathrm{C}_j(M) = 0\text{”}. \qquad (\star)$$

Then for all l with $j+1 < l \leq n$ we have $\mathrm{R}_l(\tilde{M}) \cdot \mathrm{C}_{j+1}(M) = 0$.

Proof. The proof is by induction on l. We first assume that $l = j+2$. In this case we show that

$$\mathrm{R}_{j+2}(\tilde{M}) \cdot \mathrm{C}_{j+1}(M) = 0.$$

Indeed, we have

$$\mathrm{R}_{j+2}(\tilde{M}) \cdot \mathrm{C}_{j+1}(M) = \sum_{k=1}^{n} b_{j+2,k} a_{k,j+1} = b_{j+2,j+1} + a_{j+2,j+1} = \sum_{r=1}^{j+1} (-\lambda_r) + \sum_{r=1}^{j+1} \lambda_r = 0,$$

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as desired.

Let us assume that $j+2<l\le n$. Now we must show that

$$\mathrm{R}_l(\tilde{M})\cdot \mathrm{C}_{j+1}(M)=\sum_{k=j+1}^{l} b_{l,k}a_{k,j+1}, \tag{4.7}$$

By (2.2) we have

$$b_{l,k}=b_{l-1,k-1}-\lambda_{_{l-1}}b_{l-1,k},\qquad k=j+1,\dots,l.$$

If these are substituted in (4.7) and the sums are put together, then we obtain

$$\mathrm{R}_l(\tilde{M})\cdot \mathrm{C}_{j+1}(M)=\sum_{k=j}^{l-1} b_{l-1,k}a_{k+1,j+1}-\lambda_{_{l-1}}\sum_{k=j+1}^{l-1} b_{l-1,k}a_{k,j+1}.$$

Since by induction hypothesis we have

$$\sum_{k=j+1}^{l-1} b_{l-1,k}a_{k,j+1}=0,$$

hence we obtain

$$\mathrm{R}_l(\tilde{M})\cdot \mathrm{C}_{j+1}(M)=\sum_{k=j}^{l-1} b_{l-1,k}a_{k+1,j+1}. \tag{4.8}$$

We now show that the sum on the right-hand side of (4.8) is equal to 0. Since $l>j+2$, it follows that $l-1>j+1>j$ and by $(\star)$ we have $\mathrm{R}_{l-1}(\tilde{M})\cdot \mathrm{C}_j(M)=0$ or, equivalently,

$$\sum_{k=j}^{l-1} b_{l-1,k}a_{k,j}=0. \tag{4.9}$$

We also recall that by the induction hypothesis $\mathrm{R}_{l-1}(\tilde{M})\cdot \mathrm{C}_{j+1}(M)=0$ from which we deduce that

$$\sum_{k=j+1}^{l-1} b_{l-1,k}a_{k,j+1}=0. \tag{4.10}$$

On the other hand by (2.1) we have

$$a_{k,j}=a_{k+1,j+1}-\lambda_{j+1}a_{k,j+1},\qquad k=j,j+1,\dots,l-1.$$

Substituting this in (4.9) and using (4.10) we obtain

$$0=\sum_{k=j}^{l-1} b_{l-1,k}a_{k,j}=\sum_{k=j}^{l-1} b_{l-1,k}(a_{k+1,j+1}-\lambda_{j+1}a_{k,j+1})$$

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$$= \sum_{k=j}^{l-1} b_{l-1,k} a_{k+1,j+1} - \lambda_{j+1} \sum_{k=j}^{l-1} b_{l-1,k} a_{k,j+1}$$

$$= \sum_{k=j}^{l-1} b_{l-1,k} a_{k+1,j+1} - \lambda_{j+1} \sum_{k=j+1}^{l-1} b_{l-1,k} a_{k,j+1}$$

$$= \sum_{k=j}^{l-1} b_{l-1,k} a_{k+1,j+1}.$$

Now, it can be deduced that $\sum_{k=j}^{l-1} b_{l-1,k} a_{k+1,j+1} = 0$, as desired. □

Therefore, from Lemmas 4.2 and 4.3 we conclude the first part of the theorem. The second part of the theorem is easily verified and we leave it to the reader. □

Remark 1. Another *proof* for the claim $\tilde{M} \cdot M = I$ in Theorem 4.1 is as follows. First, we easily see that

$$\mathrm{R}_1(\tilde{M} \cdot M) = \mathrm{R}_1(I) = (1, 0, 0, \ldots, 0).$$

Next, by Lemma 4.2, we observe

$$\mathrm{C}_1(\tilde{M} \cdot M) = \mathrm{C}_1(I) = (1, 0, 0, \ldots, 0)^T.$$

Finally, we show

$$(\tilde{M} \cdot M)_{i,j} = (\tilde{M} \cdot M)_{i-1,j-1} + (\lambda_j - \lambda_{i-1})(\tilde{M} \cdot M)_{i-1,j} \quad (2 \leq i, j \leq n).$$

In fact, by direct calculations we obtain

$$\begin{array}{rcl}
(\tilde{M} \cdot M)_{i,j} & = & \sum_{k=j}^{n} b_{i,k} a_{k,j} \\
& = & \sum_{k=j}^{n} (b_{i-1,k-1} - \lambda_{i-1} b_{i-1,k}) a_{k,j} \\
& = & \sum_{k=j}^{n} b_{i-1,k-1} a_{k,j} - \lambda_{i-1} \sum_{k=j}^{n} b_{i-1,k} a_{k,j} \\
& = & \sum_{k=j}^{n} b_{i-1,k-1} (a_{k-1,j-1} + \lambda_j a_{k-1,j}) - \lambda_{i-1} \sum_{k=j}^{n} b_{i-1,k} a_{k,j} \\
& = & \sum_{k=j-1}^{n} b_{i-1,k} a_{k,j-1} + \lambda_j \sum_{k=j}^{n} b_{i-1,k-1} a_{k-1,j} - \lambda_{i-1} \sum_{k=j}^{n} b_{i-1,k} a_{k,j} \\
& = & (\tilde{M} \cdot M)_{i-1,j-1} + \lambda_j \sum_{k=j-1}^{n} b_{i-1,k} a_{k,j} - \lambda_{i-1} \sum_{k=j}^{n} b_{i-1,k} a_{k,j} \\
& = & (\tilde{M} \cdot M)_{i-1,j-1} + (\lambda_j - \lambda_{i-1}) \sum_{k=j}^{n} b_{i-1,k} a_{k,j} \\
& = & (\tilde{M} \cdot M)_{i-1,j-1} + (\lambda_j - \lambda_{i-1})(\tilde{M} \cdot M)_{i-1,j},
\end{array}$$

as desired.

Remark 2. It is worth mentioning that Theorem 4.1 is quite close to a result due to Neuwirth (Theorem 13 in [13]). In fact, Theorem 4.1 can be deduced from Neuwirth's Theorem with just a little effort. We will discuss it now briefly.

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Let L be the matrix in Theorem 4.1. The first column of L is not a geometric sequence, but it is a sum of two geometric sequences, of quotients q and $\bar{q}$. The numbers q and $\bar{q}$ are the roots of the equation

$$x^2 - x - 1 = 0.$$

Hence, to resolve the situation, we do two things:

(i) We modify $\lambda_1 = 0$. Notice that this modification does not affect the matrix L.

(ii) We extend the matrix L by adding 3 rows and columns, indexed by $-2, -1, 0$, as follows. First, we define $(\lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1) = (0, q, \bar{q}, 0)$ and $\delta_{-2} = \delta_{-1} = \delta_0 = \delta_1 = 1$. Next, we complete the rows and columns in the unique way so we obtain an extended matrix L^e which is lower triangular and satisfies the same recursion. Notice that the first column of L^e is $(1, 0, 0, \ldots, 0)^T$ and hence geometric. It is thus a *"pure Galton array"* in Neuwirth's language, and Theorem 13 in [13] may be applied. Notice that L is a triangular block inside L^e, so that the inverse of L can be deduced easily from that of L^e. Constructing $(L^e)^{-1}$ by the recipe given in Theorem 13 in [13], we obtain the result, with a little difference. The sequence γ is modified to $\gamma' = (-1, -1, 1, 0, \ldots, 0)$, and as mentioned above $\lambda_1 = 0$. But by changing back from γ' to γ and setting $\lambda_1 = 1$, we arrive at exactly the same second row, hence Theorem 4.1 is confirmed.

4.2. On the matrices $A_{\alpha^{(k)},\mathcal{F}}(n)$. In what follows we consider the matrices $A_k(n) := A_{\alpha^{(k)},\mathcal{F}}(n)$. As a matter of fact, we are interested in finding the sequence of determinants

$$(\det A_k(1),\ \det A_k(2),\ \det A_k(3),\ \ldots,\ \det A_k(n),\ \ldots),$$

and the inverse matrix $A_k^{-1}(n)$. Here, for every pair of integers k and n, with $k \leq n$, we will determine $\det A_k(n)$ and $A_k^{-1}(n)$.

When $k = 1$, $\det A_1(n) = 1$ and by Theorem 4.1 we obtain that $A_1(n)^{-1} = A^r_{\gamma,-\alpha,-1,1}(n, n+2)^{[1,2]}$. Hence, from now on we assume that $k \geq 2$. Using *LU-factorization method* (see [7], Sec. 4 or [6], Sec. 2.6), we first find the unique factorization of $A_k(n)$ as $A_k(n) = L \cdot U$, where $L = (L_{i,j})_{1 \leq i,j \leq n}$ is a lower triangular matrix and $U = (U_{i,j})_{1 \leq i,j \leq n}$ is an upper triangular one. Then, clearly

$$\det A = \det L \cdot \det U = \prod_{i=1}^{n} L_{i,i} \cdot \prod_{i=1}^{n} U_{i,i}.$$

THEOREM 4.4. *Let $A_2(n) = A_{\alpha^{(2)},\mathcal{F}}(n)$. Then we have*

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(a) $A_2(n) = L \cdot U$, *where*

$$L = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \hat{\mathcal{F}} & A_{\alpha,\mathcal{F}}(n-1) \end{array}\right],$$

with $\hat{\mathcal{F}} = (F(2), F(3), \ldots, F(n))^T$ *and where* $U = (U_{i,j})_{1\le i,j\le n}$ *with*

$$U_{i,j} = \begin{cases} 1 & i = j \le 2 \\ 2 & i = j \ge 3 \\ 1 & j = i+1 \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, it follows that

$$\det A_2(n) = \begin{cases} 1 & \textit{if} \quad n = 1 \\ 2^{n-2} & \textit{if} \quad n > 1. \end{cases}$$

(b) $A_2(n)^{-1} = U^{-1} \cdot L^{-1}$, *where*

$$U^{-1} = \left[\begin{array}{cc|cccccc} 1 & \omega_0 & \omega_1 & \omega_2 & \omega_3 & \ldots & \omega_{n-3} & \omega_{n-2} \\ 0 & \tilde{\omega}_0 & \tilde{\omega}_1 & \tilde{\omega}_2 & \tilde{\omega}_3 & \ldots & \tilde{\omega}_{n-3} & \tilde{\omega}_{n-2} \\ \hline 0 & 0 & \omega_1 & \omega_2 & \omega_3 & \ldots & \omega_{n-3} & \omega_{n-2} \\ 0 & 0 & 0 & \omega_1 & \omega_2 & \ldots & \omega_{n-4} & \omega_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & \omega_1 & \omega_2 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \omega_1 \end{array}\right],$$

with $\omega_i = (-1)^{i+1}(\frac{1}{2})^i$ *and* $\tilde{\omega}_i = -\omega_i$ $(0 \le i \le n-2)$, *and where*

$$L^{-1} = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline x & A^r_{\gamma,-\alpha,-1,1}(n-1, n+1)^{[1,2]} \end{array}\right],$$

with $x = (-1, \underbrace{-1, 1, -1, 1, \ldots, (-1)^n}_{2-\text{periodic}})^T$.

Proof. (a) The matrix L is a lower triangular matrix with 1's on the diagonal, whereas U is an upper triangular matrix with diagonal entries $1, 1, 2, 2, 2, \ldots, 2$.

By definition, $\mathrm{R}_1(L) = (1, 0, 0, 0, \ldots, 0)$, $\mathrm{C}_1(L) = (F(1), F(2), \ldots, F(n))^T$, $\mathrm{C}_2(L) = (0, F(1), F(2), \ldots, F(n-1))^T$ and

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \tag{4.11}$$

for $2 \le i \le n$ and $3 \le j \le n$. Also we have

$$(U_{1,j}, \ldots, U_{n,j})^T = \begin{cases} (1, 0, 0, \ldots, 0)^T & j = 1, \\ (1, 1, 0, \ldots, 0)^T & j = 2, \\ (0, 0, \ldots, 0, U_{j-1,j} = 1, U_{j,j} = 2, 0, \ldots, 0)^T & j > 2. \end{cases} \tag{4.12}$$

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For the proof of the claimed factorization we compute the $(i,j)-$entry of $L\cdot U$, that is

$$(L\cdot U)_{i,j}=\sum_{k=1}^{n}L_{i,k}U_{k,j}.$$

In fact, so as to prove the theorem, we should establish $\mathrm{R}_1(L\cdot U)=(1,1,0,0,\ldots,0)$, $\mathrm{C}_1(L\cdot U)=(F(1),F(2),\ldots,F(n))^T$, $\mathrm{C}_2(L\cdot U)=(F(2),F(3),\ldots,F(n+1))^T$ and

$$(L\cdot U)_{i,j}=(L\cdot U)_{i-1,j-1}+(L\cdot U)_{i-1,j},$$

for $2\le i\le n$ and $3\le j\le n$.

Let us do the required calculations. First, suppose that $i=1$. Then

$$(L\cdot U)_{1,j}=\sum_{k=1}^{n}L_{1,k}U_{k,j}=L_{1,1}U_{1,j}=U_{1,j},$$

and so $\mathrm{R}_1(L\cdot U)=(1,1,0,0,\ldots,0)$.

Next, suppose that $j=1$ or 2. In this case we obtain

$$(L\cdot U)_{i,1}=\sum_{k=1}^{n}L_{i,k}U_{k,1}=L_{i,1}U_{1,1}=F(i),$$

from which we deduce that $\mathrm{C}_1(L\cdot U)=(F(1),F(2),\ldots,F(n))^T$. Similarly, we have

$$(L\cdot U)_{i,2}=\sum_{k=1}^{n}L_{i,k}U_{k,2}=L_{i,1}U_{1,2}+L_{i,2}U_{2,2}=F(i)+F(i-1)=F(i+1),$$

and we get $\mathrm{C}_2(L\cdot U)=(F(2),F(3),\ldots,F(n+1))^T$ again.

Finally, we assume that $2\le i\le n$ and $3\le j\le n$. Now, we claim that

$$(L\cdot U)_{i,j}=(L\cdot U)_{i-1,j-1}+(L\cdot U)_{i-1,j}. \tag{4.13}$$

If $j=3$, then we have

$$\begin{aligned}(L\cdot U)_{i-1,2}+(L\cdot U)_{i-1,3}&=(L_{i-1,1}+L_{i-1,2})+(L_{i-1,2}+2L_{i-1,3})\\&=L_{i-1,1}+2(L_{i-1,2}+L_{i-1,3})\\&=L_{i,2}+2L_{i,3}\\&=(L\cdot U)_{i,3}.\end{aligned}$$

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Suppose $j > 3$. In this case we have

$$(L \cdot U)_{i-1,j-1} = \sum_{k=1}^{n} L_{i-1,k} U_{k,j-1} = L_{i-1,j-2} + 2L_{i-1,j-1},$$

and

$$(L \cdot U)_{i-1,j} = \sum_{k=1}^{n} L_{i-1,k} U_{k,j} = L_{i-1,j-1} + 2L_{i-1,j}.$$

Consequently, the sum on the right-hand side of (4.13) is equal to

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = (L_{i-1,j-2} + L_{i-1,j-1}) + 2(L_{i-1,j-1} + L_{i-1,j}),$$

and by using (4.11), we obtain

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = L_{i,j-1} + 2L_{i,j}.$$

On the other hand, we have

$$(L \cdot U)_{i,j} = \sum_{k=1}^{n} L_{i,k} U_{k,j} = L_{i,j-1} + 2L_{i,j}.$$

Now, if we compare with the last equation, we obtain the claimed (4.13).

It is obvious that the claimed factorization of $A_2(n)$ immediately implies that

$$\det A_2(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 2^{n-2} & \text{if} \quad n > 1. \end{cases}$$

(b) One can readily check $U \cdot U^{-1} = I$. Now, we show $L \cdot L^{-1} = I$. First of all, we notice that by Theorem 4.1, the inverse of $A_{\alpha,\mathcal{F}}(n-1)$ is $A^r_{\gamma,-\alpha,-1,1}(n-1, n+1)^{[1,2]}$. Therefore, we obtain

$$L \cdot L^{-1} = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \hat{\mathcal{F}} + A_{\alpha,\mathcal{F}}(n-1) \times x & I \end{array}\right].$$

Now, it is enough to show

$$\hat{\mathcal{F}} + A_{\alpha,\mathcal{F}}(n-1) \times x = \mathbf{0}. \tag{4.14}$$

Let $A_{\alpha,\mathcal{F}}(n-1) = (a_{i,j})_{1 \le i,j \le n-1}$, $\hat{\mathcal{F}} = (\hat{\mathcal{F}}_{1,1}, \hat{\mathcal{F}}_{2,1}, \ldots, \hat{\mathcal{F}}_{n-1,1})^T$ where $\hat{\mathcal{F}}_{i,1} = F(i+1)$, and

$$x = (x_{1,1}, x_{2,1}, \ldots, x_{n-1,1})^T = (-1, \underbrace{-1, 1, -1, 1, \ldots, (-1)^n}_{2-\text{periodic}})^T.$$

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From (4.1) and (4.2), with $\delta_i = \lambda_i = 1$, it follows that $\Psi(j,k) = \binom{k+j-2}{j-2}$ and

$$a_{i,j} = \begin{cases} 0 & \text{if} \quad i < j, \\ F(i) & \text{if} \quad j = 1, \\ \sum_{k=0}^{i-j} F(i-j+1-k)\binom{k+j-2}{j-2} & \text{if} \quad i \geq j > 1. \end{cases}$$

The proof of the claim (4.14) requires some calculations. The $(i,1)$-entry on the left-hand side of (4.14) is $\hat{\mathcal{F}}_{i,1} + \mathrm{R}_i(A_{\alpha,\mathcal{F}}(n-1)) \times x$. First, we assume that $i = 1$. Then the $(1,1)$-entry is equal to $\hat{\mathcal{F}}_{1,1} + a_{1,1} \times (-1) = F(2) - F(1) = 0$, and in this case the result is true. Next, we assume that $i > 1$. Then the $(i,1)$-entry is equal to

$$\hat{\mathcal{F}}_{i,1} + a_{i,1}x_{1,1} + a_{i,2}x_{2,1} + \cdots + a_{i,j}x_{j,1} + \cdots + a_{i,i}x_{i,1} =$$

$$F(i+1) - a_{i,1} - a_{i,2} + a_{i,3} - a_{i,4} + \cdots + (-1)^{j+1}a_{i,j} + \cdots + (-1)^{i+1}a_{i,i} =$$

$$F(i+1) - F(i) + \sum_{j=2}^{i}\sum_{k=0}^{i-j}(-1)^{j+1}F(i-j+1-k)\binom{k+j-2}{j-2} =$$

$$F(i+1) - F(i) - F(i-1) + \sum_{j=3}^{i}\left(\sum_{k=0}^{j-2}(-1)^{k+1}\binom{j-2}{k}\right)F(i-j+1),$$

and since $\sum_{k=0}^{j-2}(-1)^{k+1}\binom{j-2}{k} = 0$, this entry is 0. $\square$

Theorem 4.5. *Let $A_k(n) = A_{\alpha^{(k)},\mathcal{F}}(n)$, where $2 < k \leq n$ is a natural number. Then we have*

(a) $A_k(n) = L \cdot U$, where

$$L = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \hat{\mathcal{F}} & A_{\alpha,\mathcal{F}}(n-1) \end{array}\right],$$

with $\hat{\mathcal{F}} = (F(2), F(3), \ldots, F(n))^T$ and where $U = (U_{i,j})_{1\leq i,j\leq n}$ with

$$(U_{1,j},\ldots,U_{n,j})^T = \begin{cases} (\overbrace{1,1,\ldots,1}^{j-\text{times}},0,0,0,\ldots,0)^T & j \leq k, \\ (\underbrace{0,0,\ldots,0}_{(j-k)-\text{times}},\underbrace{1,2,1,1,\ldots,1}_{k-\text{times}},\underbrace{0,0,\ldots,0}_{(n-j)-\text{times}})^T & j > k. \end{cases}$$

In particular, it follows that $\det A_k(n) = 1$.

(b) If we partition U in the following manner

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$$U = \left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right],$$

where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, *then* $C = A_{\mathbf{x}}(n-2)$ *with* $\mathbf{x} = (\underbrace{1, 1, \ldots, 1, 2, 1}_{k-\text{times}}, 0, \ldots, 0)$. *Moreover, we have* $A_k(n)^{-1} = U^{-1} \cdot L^{-1}$, *where*

$$U^{-1} = \left[\begin{array}{c|c} A^{-1} & -A^{-1}BA_{\dot{\mathbf{x}}}(n-2) \\ \hline \mathbf{0} & A_{\dot{\mathbf{x}}}(n-2) \end{array}\right],$$

and where

$$L^{-1} = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline x & A^r_{\gamma,-\alpha,-1,1}(n-1, n+1)^{[1,2]} \end{array}\right],$$

with $x = (-1, \underbrace{-1, 1, -1, 1, \ldots, (-1)^n}_{2-\text{periodic}})^T$.

Proof. (a) The matrix L is a lower triangular matrix and U is an upper triangular one with 1's on their diagonals. It is obvious that the claim immediately implies that $\det A_k(n) = 1$.

By definition, $\mathrm{R}_1(L) = (1, 0, 0, 0, \ldots, 0)$, $\mathrm{C}_1(L) = (F(1), F(2), \ldots, F(n))^T$, $\mathrm{C}_2(L) = (0, F(1), F(2), \ldots, F(n-1))^T$ and

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \tag{4.15}$$

for $2 \leq i \leq n$ and $3 \leq j \leq n$.

Similar to the proof of Theorem 4.4, here we also compute the $(i, j)-$entry of $L \cdot U$, that is

$$(L \cdot U)_{i,j} = \sum_{k=1}^{n} L_{i,k} U_{k,j}.$$

In fact, we must prove that $\mathrm{R}_1(L \cdot U) = (\overbrace{1, 1, \ldots, 1}^{k-\text{times}}, 0, 0, \ldots, 0)$, $\mathrm{C}_1(L \cdot U) = (F(1), F(2), \ldots, F(n))^T$, $\mathrm{C}_2(L \cdot U) = (F(2), F(3), \ldots, F(n+1))^T$ and

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j},$$

for $2 \leq i \leq n$ and $3 \leq j \leq n$.

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First, suppose that $i = 1$. Then

$$(L \cdot U)_{1,j} = \sum_{m=1}^{n} L_{1,m} U_{m,j} = L_{1,1} U_{1,j} = U_{1,j},$$

and so $\mathrm{R}_1(L \cdot U) = (\overbrace{1, 1, \ldots, 1}^{k-\text{times}}, 0, 0, \ldots, 0)$.

Next, suppose that $j = 1$ or 2. If $j = 1$, then we have

$$(L \cdot U)_{i,1} = \sum_{m=1}^{n} L_{i,m} U_{m,1} = L_{i,1} U_{1,1} = F(i),$$

from which we deduce that $\mathrm{C}_1(L \cdot U) = (F(1), F(2), \ldots, F(n))^T$. Similarly, if $j = 2$ then we have

$$(L \cdot U)_{i,2} = \sum_{k=1}^{n} L_{i,k} U_{k,2} = L_{i,1} U_{1,2} + L_{i,2} U_{2,2} = F(i) + F(i-1) = F(i+1),$$

and so we get $\mathrm{C}_2(L \cdot U) = (F(2), F(3), \ldots, F(n+1))^T$.

Finally, we assume that $2 \le i \le n$ and $3 \le j \le n$. Now, we claim that

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}. \tag{4.16}$$

We split the proof into two cases, according to the following possibilities for j.

Case 1. $3 \le j \le k$. In this case we have

$$(L \cdot U)_{i-1,j-1} = \sum_{m=1}^{j-1} L_{i-1,m} U_{m,j-1} = \sum_{m=1}^{j-1} L_{i-1,m} = \sum_{m=2}^{j} L_{i-1,m-1},$$

and

$$(L \cdot U)_{i-1,j} = \sum_{m=1}^{j} L_{i-1,m} U_{m,j} = \sum_{m=1}^{j} L_{i-1,m}.$$

Consequently, the sum on the right-hand side of (4.16) is equal to

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = \sum_{m=2}^{j} L_{i-1,m-1} + \sum_{m=1}^{j} L_{i-1,m},$$

and by using (4.15), we obtain

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = 2F(i-1) + F(i-2) + \sum_{m=3}^{j} L_{i,m} = F(i) + F(i-1) + \sum_{m=3}^{j} L_{i,m}.$$

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On the other hand, we have

$$(L \cdot U)_{i,j} = \sum_{m=1}^{n} L_{i,m} U_{m,j} = \sum_{m=1}^{j} L_{i,m} = F(i) + F(i-1) + \sum_{m=3}^{j} L_{i,m}.$$

Now, if we compare with the last equation, we obtain the claimed (4.16).

Case 2. $j = k+1$. In this case we have

$$\begin{aligned}
(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} &= \textstyle\sum_{m=1}^{k} L_{i-1,m} + \sum_{m=2}^{k+1} L_{i-1,m} U_{m,j} \\
&= \textstyle\sum_{m=2}^{k+1} L_{i-1,m-1} + L_{i-1,2} + 2L_{i-1,3} + \sum_{m=4}^{k+1} L_{i-1,m} \\
&= L_{i-1,1} + 2(L_{i-1,2} + L_{i-1,3}) + \textstyle\sum_{m=4}^{k+1} (L_{i-1,m-1} + L_{i-1,m}) \\
&= L_{i,2} + 2L_{i,3} + \textstyle\sum_{m=4}^{k+1} L_{i,m} \\
&= (L \cdot U)_{i,j}.
\end{aligned}$$

as required.

Case 3. $k+1 < j \leq n$. In this case we obtain

$$(L \cdot U)_{i-1,j-1} = \sum_{m=j-k}^{j-1} L_{i-1,m} U_{m,j-1} = L_{i-1,j-k} + 2L_{i-1,j-k+1} + \sum_{m=j-k+3}^{j} L_{i-1,m-1},$$

and

$$(L \cdot U)_{i-1,j} = \sum_{m=j-k+1}^{j} L_{i-1,m} U_{m,j} = L_{i-1,j-k+1} + 2L_{i-1,j-k+2} + \sum_{m=j-k+3}^{j} L_{i-1,m}.$$

Consequently, using (4.15) we get

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = L_{i,j-k+1} + 2L_{i,j-k+2} + \sum_{m=j-k+3}^{j} L_{i,m} = (L \cdot U)_{i,j}.$$

This completes the proof of part (a).

(b) Using Lemma 3.1 and easy matrix-calculations, we deduce that $U \cdot U^{-1} = I$. On the other hand, by Theorem 4.4, it follows that $L \cdot L^{-1} = I$. Therefore, we have $A_k(n)^{-1} = U^{-1} \cdot L^{-1}$. □

4.3. On the matrices $A_{\chi^{(k)},\mathcal{F}}(n)$. In this subsection, we concentrate on the evaluation of determinant of matrices of the kind $A_{\chi^{(k)},\mathcal{F}}(n)$. Moreover, we restrict our evaluations to the cases $k = 1, 2$. Our observations show that these evaluations for $k \geq 3$ are complicated and we leave them for future investigation.

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THEOREM 4.6. *Let* $A = A_{\chi,\mathcal{F}}(n)$ *with* $n \geq 2$. *Then* $\det A = 2^{n-2}$.

Proof. We apply LU-factorization. Here, we claim that

$$A = L \cdot U,$$

where

$$L = \left[\begin{array}{cc|c} 1 & 0 & \mathbf{0} \\ 1 & 1 & \mathbf{0} \\ \hline \hat{\mathcal{F}} & \tilde{\mathcal{F}} & A_{\alpha,\breve{\mathcal{F}}}(n-2) \end{array}\right],$$

with

$$\hat{\mathcal{F}} = (\hat{\mathcal{F}}_i)_{3\leq i\leq n} = (F(3), F(4), \ldots, F(i), \ldots, F(n))^T,$$

$$\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_i)_{3\leq i\leq n} = (0, 0, -1, -2, \ldots, 1 - F(i-2), \ldots, 1 - F(n-2))^T,$$

$$\breve{\mathcal{F}} = (\breve{\mathcal{F}}_i)_{3\leq i\leq n} = (1, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{F(i-2)+i-2}{2}, \ldots, \frac{F(n-2)+n-2}{2}),$$

and where

$$U = \left[\begin{array}{cc|cccc} 1 & 2 & 3 & 4 & \ldots & n \\ 0 & 1 & 2 & 3 & \ldots & n-1 \\ \hline & & & & & \\ \mathbf{0} & & & \overline{U} & & \end{array}\right],$$

with

$$\overline{U}_{i,j} = \begin{cases} 0 & i > j, \\ 2(j-i+1) & i \leq j. \end{cases}$$

The matrix L is a lower triangular matrix with 1's on the diagonal, whereas U is an upper triangular matrix with diagonal entries

$$1, 1, 2, 2, 2, \ldots, 2.$$

It is obvious that the claim immediately implies the theorem.

For the proof of the claimed factorization we compute the $(i,j)-$entry of $L \cdot U$, that is

$$(L \cdot U)_{i,j} = \sum_{k=1}^{n} L_{i,k} U_{k,j},$$

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and then we show that $(L \cdot U)_{i,j} = A_{i,j}$. It is not difficult to see that

$$\mathrm{R}_1(L \cdot U) = \mathrm{R}_1(A) = (1, 2, 3, \ldots, n),$$

$$\mathrm{R}_2(L \cdot U) = \mathrm{R}_2(A) = (1, 3, 5, \ldots, 2n-1),$$

$$\mathrm{C}_1(L \cdot U) = \mathrm{C}_1(A) = (1, 1, 2, \ldots, F(n))^T.$$

Moreover, for $i \geq 3$ we obtain

$$(L \cdot U)_{i,2} = \sum_{k=1}^{n} L_{i,k} U_{k,2} = 2L_{i,1} + L_{i,2} = 2F(i) + 1 - F(i-2) = F(i+1) + 1,$$

which implies that $\mathrm{C}_2(L \cdot U) = \mathrm{C}_2(A) = (2, 3, F(4)+1, \ldots, F(i+1)+1, \ldots, F(n+1)+1)^T$.

Similarly, for $i, j \geq 3$ we get

$$(L \cdot U)_{3,j} = \sum_{k=1}^{n} L_{3,k} U_{k,j} = 2j + 2(j-3+1) = 4(j-1),$$

and

$$(L \cdot U)_{i,3} = \sum_{k=1}^{n} L_{i,k} U_{k,3} = 3L_{i,1} + 2L_{i,2} + 2L_{i,3}$$

$$= 3F(i) + 2(1 - F(i-2)) + 2 \times \frac{F(i-2) + i - 2}{2} = F(i+2) + i.$$

Hence, we have $\mathrm{R}_3(L \cdot U) = \mathrm{R}_3(A)$ and $\mathrm{C}_3(L \cdot U) = \mathrm{C}_3(A)$.

Now, it suffices to establish the equation

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}, \tag{4.17}$$

for $4 \leq i, j \leq n$.

Here, we recall that

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \tag{4.18}$$

for $4 \leq i, j \leq n$.

Let us do the required calculations. First, we obtain that

$$(L \cdot U)_{i-1,j-1} = \sum_{k=1}^{i-1} L_{i-1,k} U_{k,j-1}$$

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$$= (j-1)F(i-1) + (j-2)(1-F(i-3)) + \sum_{k=3}^{i-1} 2(j-k)L_{i-1,k}$$

$$= (j-1)F(i-1) + (j-2)(1-F(i-3)) + \sum_{k=4}^{i} 2(j-k+1)L_{i-1,k-1}.$$

Similarly, we obtain that

$$\begin{aligned}(L\cdot U)_{i-1,j} &= \sum_{k=1}^{i-1} L_{i-1,k}U_{k,j}\\ &= jF(i-1) + (j-1)(1-F(i-3)) + (j-2)(F(i-3)+i-3)\\ &\quad + \sum_{k=4}^{i-1} 2(j-k+1)L_{i-1,k}.\end{aligned}$$

From (4.18), the sum on the right-hand side of (4.17) is equal to

$$(2j-1)F(i-1) + (2j-3)(1-F(i-3)) + (j-2)(F(i-3)+i-3) + \sum_{k=4}^{i} 2(j-k+1)L_{i,k}. \tag{4.19}$$

On the other hand, we have

$$(L\cdot U)_{i,j} = jF(i) + (j-1)(1-F(i-2)) + (j-2)(F(i-2)+i-2) + \sum_{k=4}^{i} 2(j-k+1)L_{i,k}. \tag{4.20}$$

But since $F(i) = F(i-1)+F(i-2)$ and $F(i-2) = F(i-1)-F(i-3)$, easy calculations show that

$$\begin{aligned}&jF(i) + (j-1)(1-F(i-2)) + (j-2)(F(i-2)+i-2)\\ =\,& j[F(i-1)+F(i-2)] + (j-1)(1-F(i-2)) + (j-2)[F(i-1)-F(i-3)+i-2]\\ =\,&(2j-2)F(i-1) + j-1 + F(i-2) - (j-2)F(i-3) + (j-2)(i-2)\\ =\,&(2j-2)F(i-1) + j-1 + F(i-1) - F(i-3) - (j-2)F(i-3) + (j-2)(i-2)\\ =\,&(2j-1)F(i-1) + j-1 - (j-1)F(i-3) + (j-2)(i-2)\\ =\,&(2j-1)F(i-1) + (2j-3-j+2) - (2j-3-j+2)F(i-3) + (j-2)(i-2)\\ =\,&(2j-1)F(i-1) + (2j-3)(1-F(i-3)) + (j-2)(F(i-3)+i-3).\end{aligned}$$

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By comparing (4.19), (4.20) and the above relation we obtain (4.17). The proof of the theorem is now complete. □

THEOREM 4.7. *Let* $A = A_{\chi^{(2)},\mathcal{F}}(n)$. *Then*

$$\det A = \begin{cases} 1 & \text{if} \quad n \leq 2 \\ 2^3.5^{n-3} & \text{if} \quad n > 2. \end{cases}$$

Proof. We apply LU-factorization. Here, we claim that

$$A = L \cdot U,$$

where

$$L = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 1 & 1 & 0 & \mathbf{0}_{3\times(n-3)} \\ 2 & \text{-}2 & 1 & \\ \hline \hat{\mathcal{F}} & \bar{\mathcal{F}} & \tilde{\mathcal{F}} & A_{\alpha,\breve{\mathcal{F}}}(n-3) \end{array}\right],$$

with

$$\hat{\mathcal{F}} = (\hat{\mathcal{F}}_i)_{4\leq i\leq n} = (F(4), F(5), \ldots, \mathcal{F}_i = F(i), \ldots, F(n))^T,$$

$$\bar{\mathcal{F}} = (\bar{\mathcal{F}}_i)_{4\leq i\leq n} = (-4, -9, \ldots, \bar{\mathcal{F}}_i = F(i+1) - 4F(i) + 3, \ldots, \bar{\mathcal{F}}_n)^T,$$

$$\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_i)_{4\leq i\leq n} = (\frac{13}{8}, \frac{23}{8}, \ldots, \tilde{\mathcal{F}}_i = \frac{1}{8}\big[F(i+2) - 4F(i+1) + 7F(i) + 3i - 8\big], \ldots, \tilde{\mathcal{F}}_n)^T,$$

$$\breve{\mathcal{F}} = (\breve{\mathcal{F}}_i)_{4\leq i\leq n} = (1, \frac{12}{5}, \ldots, \breve{\mathcal{F}}_i = \frac{1}{5}\big[F(i+3) - 4F(i) + \frac{3}{2}i(i-1) - 5i + 6\big], \ldots, \breve{\mathcal{F}}_n)^T,$$

and where

$$U = \left[\begin{array}{ccc|ccccccc} 1 & 4 & 9 & 16 & 25 & \ldots & j^2 & \ldots & n^2 \\ 0 & 1 & 4 & 16 & 9 & \ldots & (j-1)^2 & \ldots & (n-1)^2 \\ 0 & 0 & 8 & 24 & 48 & \ldots & 4(j-1)(j-2) & \ldots & 4(n-1)(n-2) \\ \hline & \mathbf{0} & & & & & \overline{U} & & \end{array}\right],$$

with

$$\overline{U}_{i,j} = \begin{cases} 0 & i > j, \\ 5\binom{j-i+2}{2} & i \leq j. \end{cases}$$

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The matrix L is a lower triangular matrix with 1's on the diagonal, whereas U is an upper triangular matrix with diagonal entries

$$1, 1, 8, 5, 5, 5, \ldots, 5.$$

It is obvious that the claim immediately implies the theorem. Moreover, the proof of the claim is similar to the proof of Theorem 4.6 and we leave it to the reader. ☐

Remark 3. Computing LU-factorizations of finite matrices is well known and plenty of software can do this. However, let us explain how we compute the LU-factorization of $A_{\chi^{(2)},\mathcal{F}}(n)$ in Theorem 4.7. We put $A(n) = A_{\chi^{(2)},\mathcal{F}}(n) = (A_{i,j})_{1\le i,j\le n}$. Let $A(n) = L(n) \cdot U(n)$ be the unique LU-factorization of $A(n)$ where $L(n) = (L_{i,j})_{1\le i,j\le n}$ is a lower triangular matrix with all entries along the diagonal equal to 1 and $U = (U_{i,j})_{1\le i,j\le n}$ is an upper triangular matrix. Obviously, by the structure of $A(n)$, we have

$$A(n-1) = A(n)^{[n]}_{[n]},$$

and also

$$L(n-1) = L(n)^{[n]}_{[n]} \quad \text{and} \quad U(n-1) = U(n)^{[n]}_{[n]}.$$

First of all, we go to the computer and crank out the matrices $L(n)$ and $U(n)$ for small values of n. Then, we try to guess what the entries of the matrices $L(n)$ and $U(n)$ are. For the purpose of guessing, it suffices that we just display the matrices $L(8)$ and $U(8)$. They are

$$L(8) = \left[\begin{array}{rrr|rrrrr}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
3 & -4 & 13/8 & 1 & 0 & 0 & 0 & 0 \\
5 & -9 & 23/8 & 12/5 & 1 & 0 & 0 & 0 \\
8 & -16 & 35/8 & 23/5 & 17/5 & 1 & 0 & 0 \\
13 & -28 & 27/4 & 37/5 & 8 & 22/5 & 1 & 0 \\
21 & -47 & 41/4 & 11 & 77/5 & 62/5 & 27/5 & 1
\end{array}\right]$$

and

$$U(8) = \left[\begin{array}{rrr|rrrrr}
1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 \\
0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 \\
0 & 0 & 8 & 24 & 48 & 80 & 120 & 168 \\
\hline
0 & 0 & 0 & 5 & 15 & 30 & 50 & 75 \\
0 & 0 & 0 & 0 & 5 & 15 & 30 & 50 \\
0 & 0 & 0 & 0 & 0 & 5 & 15 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 15 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{array}\right].$$

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Having seen that, it will not take you for long to guess that, apparently, $L(n)$ has the following form:

$$L(n) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 1 & 1 & 0 & \mathbf{0}_{3\times(n-3)} \\ 2 & -2 & 1 & \\ \hline \hat{\mathcal{F}} & \bar{\mathcal{F}} & \tilde{\mathcal{F}} & A_{\alpha,\breve{\mathcal{F}}}(n-3) \end{array}\right],$$

where $\hat{\mathcal{F}}$, $\bar{\mathcal{F}}$, $\tilde{\mathcal{F}}$ and $\breve{\mathcal{F}}$ are certain finite sequences, and that $U(n)$ has the following form:

$$U(n) = \left[\begin{array}{ccc|c} 1 & 4 & 9 & \hat{\mathcal{S}} \\ 0 & 1 & 4 & \bar{\mathcal{S}} \\ 0 & 0 & 8 & \tilde{\mathcal{S}} \\ \hline \multicolumn{3}{c|}{\mathbf{0}_{(n-3)\times 3}} & T_{\breve{\mathcal{S}},\beta}(n-3) \end{array}\right],$$

where $\hat{\mathcal{S}}$, $\bar{\mathcal{S}}$, $\tilde{\mathcal{S}}$ and $\breve{\mathcal{S}}$ are again certain finite sequences. Therefore, it is enough to show that these sequences are as in the proof of Theorem 4.7. To do this, we determine the entries of columns 1, 2, 3, 4 of $L(n)$ and rows 1, 2, 3, 4 of $U(n)$.

It is not difficult to see that

$$\begin{aligned}
&\mathrm{C}_1(A(n)) = (1, 1, 2, \ldots, A_{i,1} = F(i), \ldots, A_{n,1})^T, \\
&\mathrm{C}_2(A(n)) = (4, 5, 6, \ldots, A_{i,2} = F(i+1) + 3, \ldots, A_{n,2})^T, \\
&\mathrm{C}_3(A(n)) = (9, 13, 18, \ldots, A_{i,3} = F(i+2) + 3i + 4, \ldots, A_{n,3})^T, \\
&\mathrm{C}_4(A(n)) = (16, 25, 38, \ldots, A_{i,4} = F(i+3) + \tfrac{3}{2}i(i-1) + 4i + 9, \ldots, A_{n,4})^T, \\
&\mathrm{R}_1(A(n)) = (1, 4, 9, \ldots, A_{1,j} = j^2, \ldots, A_{1,n}), \\
&\mathrm{R}_2(A(n)) = (1, 5, 13, \ldots, A_{2,j} = 2j^2 - 2j + 1, \ldots, A_{2,n}), \\
&\mathrm{R}_3(A(n)) = (2, 6, 18, \ldots, A_{3,j} = 4j^2 - 8j + 6, \ldots, A_{3,n}), \\
&\mathrm{R}_4(A(n)) = (3, 8, 24, \ldots, A_{4,j} = 8j^2 - 24j + 24, \ldots, A_{4,n}).
\end{aligned}$$

Notice that, to obtain $\mathrm{C}_i(A(n))$, $i = 2, 3, 4$, we use the fact that

$$\sum_{i=1}^{k} F(i) = F(k+2) - 1.$$

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Moreover, it is easy to see that

$$\begin{aligned}
&\mathrm{R}_1(L(n)) = (1,0,\ldots,0),\\
&\mathrm{R}_2(L(n)) = (1,1,0,\ldots,0),\\
&\mathrm{R}_3(L(n)) = (2,-2,1,0,\ldots,0),\\
&\mathrm{R}_4(L(n)) = (3,-4,13/8,1,0,\ldots,0),\\
&\mathrm{C}_1(U(n)) = (1,0,\ldots,0)^T,\\
&\mathrm{C}_2(U(n)) = (4,1,0,\ldots,0)^T,\\
&\mathrm{C}_3(U(n)) = (9,4,8,0,\ldots,0)^T,\\
&\mathrm{C}_4(U(n)) = (16,9,24,5,0,\ldots,0)^T.
\end{aligned}$$

In the sequel, first we try to compute the columns $\mathrm{C}_j(L(n))$, for $j = 1,2,3,4$.

Let us do the required calculations. First, we observe that

$$F(i) = A_{i,1} = \sum_{l=1}^{1} L_{i,l}\cdot U_{l,1} = L_{i,1}, \tag{4.21}$$

and hence $\mathrm{C}_1(L(n)) = (1,1,2,\ldots,L_{i,1} = F(i),\ldots,F(n))^T$. Similarly, we obtain

$$F(i+1)+3 = A_{i,2} = \sum_{l=1}^{2} L_{i,l}\cdot U_{l,2} = 4L_{i,1} + L_{i,2} = 4F(i) + L_{i,2}, \quad \text{(by (4.21))}$$

which implies that

$$L_{i,2} = F(i+1) - 4F(i) + 3. \tag{4.22}$$

Therefore, we get

$$\mathrm{C}_2(L(n)) = (0,1,-2,\ldots,L_{i,2} = F(i+1)-4F(i)+3,\ldots,F(n+1)-4F(n)+3)^T.$$

Also, we have

$$\begin{aligned}
F(i+2)+3i+4 &= A_{i,3}\\
&= \textstyle\sum_{l=1}^{3} L_{i,l}\cdot U_{l,3}\\
&= 9L_{i,1} + 4L_{i,2} + 8L_{i,3}\\
&= 9F(i) + 4\big[F(i+1) - 4F(i) + 3\big] + 8L_{i,3}\\
&\qquad \text{(by (4.21) and (4.22))},
\end{aligned}$$

which implies that

$$L_{i,3} = \frac{1}{8}\big[F(i+2) - 4F(i+1) + 7F(i) + 3i - 8\big], \tag{4.23}$$

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and so, we obtain

$$\mathrm{C}_3(L(n)) = (0, 0, 1, \ldots, L_{i,3} = \frac{1}{8}\big[F(i+2) - 4F(i+1) + 7F(i) + 3i - 8\big], \ldots, L_{n,3})^T.$$

Finally, from

$$\begin{array}{rcl} F(i+3) + \frac{3}{2}i(i-1) + 4i + 9 & = & A_{i,4} \\ & = & \sum_{l=1}^{4} L_{i,l} \cdot U_{l,4} \\ & = & 16L_{i,1} + 9L_{i,2} + 24L_{i,3} + 5L_{i,4} \\ & = & 16F(i) + 9\big[F(i+1) - 4F(i) + 3\big] \\ & & +24 \cdot \frac{1}{8}\big[F(i+2) - 4F(i+1) + 7F(i) + 3i - 8\big] \\ & & +5L_{i,4} \quad (\text{ by } (4.21), (4.22) \text{ and } (4.23)) \\ & = & F(i) - 3F(i+1) + 3F(i+2) + 9i + 3 + 5L_{i,4} \\ & = & 4F(i) + 9i + 3 + 5L_{i,4}, \end{array}$$

we deduce that

$$L_{i,4} = \frac{1}{5}\big[F(i+3) - 4F(i) + \frac{3}{2}i(i-1) - 5i + 6\big],$$

and hence

$$\mathrm{C}_4(L(n)) = (0, 0, 0, 1, \ldots, L_{i,4} = \frac{1}{5}\big[F(i+3) - 4F(i) + \frac{3}{2}i(i-1) - 5i + 6\big], \ldots, L_{n,4})^T.$$

Now, in a similar way we determine the coefficients of rows 1, 2, 3, 4 of $U(n)$. Again, we observe that

$$j^2 = A_{1,j} = \sum_{l=1}^{1} L_{1,l} U_{l,j} = U_{1,j}, \tag{4.24}$$

and as a result $\mathrm{R}_1(U(n)) = (1, 4, 9, \ldots, U_{1,j} = j^2, \ldots, n^2)$. Similarly, we have

$$2j^2 - 2j + 1 = A_{2,j} = \sum_{l=1}^{2} L_{2,l} \cdot U_{l,j} = U_{1,j} + U_{2,j} = j^2 + U_{2,j}, \quad (\text{by } (4.24))$$

from which we deduce that

$$U_{2,j} = (j-1)^2, \tag{4.25}$$

and so $\mathrm{R}_2(U(n)) = (0, 1, 4, \ldots, U_{2,j} = (j-1)^2, \ldots, (n-1)^2)$. Also, from

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$$\begin{aligned} 4j^2-8j+6 &= A_{3,j} \\ &= \textstyle\sum_{l=1}^{3} L_{3,l}\cdot U_{l,j} \\ &= 2U_{1,j}-2U_{2,j}+U_{3,j} \\ &= 2j^2-2(j-1)^2+U_{3,j} \quad \text{(by (4.24) and (4.25))} \end{aligned}$$

we conclude that

$$U_{3,j} = 4(j-1)(j-2), \tag{4.26}$$

and hence $\mathrm{R}_3(U(n)) = (0,0,8,24,\ldots,U_{3,j}=4(j-1)(j-2),\ldots,4(n-1)(n-2))$.

Finally, we see that

$$\begin{aligned} 8j^2-24j+24 &= A_{4,j} \\ &= \textstyle\sum_{l=1}^{4} L_{4,l}\cdot U_{l,j} \\ &= 3U_{1,j}-4U_{2,j}+\tfrac{13}{8}U_{3,j}+U_{4,j} \\ &= 3j^2-4(j-1)^2+\tfrac{13}{8}\cdot 4(j-1)(j-2)+U_{4,j} \\ &\qquad \text{(by (4.24), (4.25) and (4.26))} \end{aligned}$$

and hence, we get

$$U_{4,j} = 5\cdot\frac{(j-2)(j-3)}{2} = 5\binom{j-2}{2}.$$

Therefore, it follows that $\mathrm{R}_4(U(n)) = (0,0,0,5,15,\ldots,U_{4,j}=5\binom{j-2}{2},\ldots,5\binom{n-2}{2})$. Moreover, if we put $T_{\breve{\mathcal{S}},\beta}(n-3) = (t_{i,j})_{4\le i,j\le n}$, then we easily see that

$$t_{i,j} = \begin{cases} 0 & i>j, \\ 5\binom{j-i+2}{2} & i\le j. \end{cases}$$

We will conclude this subsection with a short discussion about the computational complexity of the problem. As we mentioned in the beginning of this subsection, computing $\det A$, where $A = A_{\chi^{(k)}\mathcal{F}}(n)$, is more complicated for $k\ge 3$.

Let $R\langle i,j,c\rangle$ denote the row operation of adding to row i, c times row j. Starting with a matrix $A = A_{t,\mathcal{F}}$ for an arbitrary sequence t, we work our way by performing $R\langle n-1,n,-1\rangle$, then $R\langle n-2,n-1,-1\rangle$, continuing through $R\langle 1,2,-1\rangle$, we obtain a matrix whose first column is $(F(1),F(0),F(1),F(2),\ldots,F(n-2))^T$, the first row is still t, and the remaining $(n-1)\times(n-1)$ block is the matrix A with the last row and last column removed. We continue to do the same thing on the $(2,3,\ldots,n)\times(2,3,\ldots,n)$ block, in order to push A further to the lower left corner. The first column is affected of course. Continuing in this way, we end up with a Töeplitz

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matrix with first column $(F(1), F(0), F(-1), \ldots, F(2-n))^T$, and first row t. Our problem is now reduced to the evaluation of the determinant of a Töeplitz matrix. The miracle that happens for $k = 0, 1, 2$ is that the sequence uniting the first column and row, i.e., $F(2-n), \ldots, F(-1), F(0), F(1), 2^k, 3^k, \ldots, n^k$ is a union of "overlapping" recursive sequences. This means that if we extend the sequence $2^k, 3^k, \ldots$ to the left, by prepending k elements to the left, (k being the length of the recursion), we will just have to prepend the elements $F(0), F(-1)$, etc. This does not happen any more for $k = 3$. In fact, this is very important, because if this happens we can perform row operations by taking successive differences. Doing this in the correct order will annihilate the sequence $t = \chi^{(k)}$, bringing us to a lower triangular matrix. Actually it will be triangular up to a lower left triangular $(k+1) \times (k+1)$ block, due to edge effects. The element in the diagonal will be a constant Fibonacci number, independent of n (since except for the lower $k+1$ rows, the matrix will still be Töeplitz). This explains the geometric behavior of the determinants.

4.4. On the matrices $A_{\xi^{(a)},\mathcal{F}}(n)$ and $A_{\gamma^{(d)},\mathcal{F}}(n)$. In this subsection we first consider the matrices of type $A_{\xi^{(a)},\mathcal{F}}(n)$, where $\xi^{(a)}$ is a geometric sequence and show that in every case the determinant of $A_{\xi^{(a)},\mathcal{F}}(n)$ is 1.

THEOREM 4.8. *Let $A = A_{\xi^{(a)},\mathcal{F}}(n)$. Then we have* $\det A = 1$.

Proof. We apply again LU-factorization method. We claim that

$$A = L \cdot U,$$

where

$$L = (L_{i,j})_{1 \le i,j \le n} = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \hat{\mathcal{F}} & A_{\alpha,\tilde{\mathcal{F}}}(n-1) \end{array}\right],$$

with $\hat{\mathcal{F}} = (F(2), F(3), \ldots, F(n))^T$ and $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_i)_{i \ge 1}$ with

$$\tilde{\mathcal{F}}_i = F(i+2) - 1 - (F(i+1) - 1)a, \quad (i \ge 1),$$

and where $U = (U_{i,j})_{1 \le i,j \le n}$ with

$$U_{i,j} = \begin{cases} 0 & i > j \\ a^k & j = i + k \ (k \ge 0). \end{cases}$$

The matrix L is a lower triangular matrix and U is an upper triangular one with 1's on their diagonals. It is obvious that the claim immediately implies the theorem.

By definition, $\mathrm{R}_1(L) = (1, 0, 0, 0, \ldots, 0)$, $\mathrm{C}_1(L) = (F(1), F(2), \ldots, F(n))^T$, $\mathrm{C}_2(L) = (0, 1, 2-a, 4-2a, 7-4a, \ldots, F(i+1) - 1 - (F(i) - 1)a, \ldots, F(n+1) - 1 - (F(n) - 1)a)^T$ and

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \tag{4.27}$$

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for $2 \leq i \leq n$ and $3 \leq j \leq n$.

As before, here we also compute the $(i,j)-$entry of $L \cdot U$, that is $(L \cdot U)_{i,j} = \sum_{k=1}^{n} L_{i,k} U_{k,j}$. In fact, we must prove that

$$\mathrm{R}_1(L \cdot U) = (1, a, a^2, \ldots, a^{n-1}), \quad \mathrm{C}_1(L \cdot U) = (F(1), F(2), \ldots, F(n))^T,$$

$$\mathrm{C}_2(L \cdot U) = (a, 1+a, 2+a, \ldots, F(i+1) - 1 + a, \ldots, F(n+1) - 1 + a)^T,$$

and

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j},$$

for $2 \leq i \leq n$ and $3 \leq j \leq n$.

First, suppose that $i = 1$. Then

$$(L \cdot U)_{1,j} = \sum_{m=1}^{n} L_{1,m} U_{m,j} = L_{1,1} U_{1,j} = a^{j-1},$$

and so $\mathrm{R}_1(L \cdot U) = (1, a, a^2, \ldots, a^{n-1})$.

Next, suppose that $j = 1$ or 2. If $j = 1$, then we have

$$(L \cdot U)_{i,1} = \sum_{m=1}^{n} L_{i,m} U_{m,1} = L_{i,1} U_{1,1} = F(i),$$

from which we deduce that $\mathrm{C}_1(L \cdot U) = (F(1), F(2), \ldots, F(n))^T$. Similarly, if $j = 2$, then we have

$$(L \cdot U)_{1,2} = 1 \times a = a,$$

and for $i > 1$

$$(L \cdot U)_{i,2} = \sum_{k=1}^{n} L_{i,k} U_{k,2}$$

$$= L_{i,1} U_{1,2} + L_{i,2} U_{2,2} = aF(i) + F(i+1) - 1 - (F(i) - 1)a = F(i+1) - 1 + a,$$

and so we get $\mathrm{C}_2(L \cdot U) = (a, 1+a, 2+a, \ldots, F(i+1) - 1 + a, \ldots, F(n+1) - 1 + a)^T$.

Finally, we assume that $2 \leq i \leq n$ and $3 \leq j \leq n$. Now, we claim that

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}. \tag{4.28}$$

Here, we verify (4.28) by a direct calculation. We split the proof into two cases $i = 2$ and $i > 2$.

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Case 1. $i = 2$. Here, by an easy computation we obtain

$$(L \cdot U)_{2,j} = \sum_{m=1}^{2} L_{2,m} U_{m,j-1} = a^{j-1} + a^{j-2}$$

and

$$(L \cdot U)_{1,j-1} + (L \cdot U)_{1,j} = \sum_{m=1}^{1} L_{1,m} U_{m,j-1} + \sum_{m=1}^{1} L_{1,m} U_{m,j} = a^{j-2} + a^{j-1},$$

and so

$$(L \cdot U)_{2,j} = (L \cdot U)_{1,j-1} + (L \cdot U)_{1,j},$$

as required.

Case 2. $i > 2$. In this case we have

$$(L \cdot U)_{i-1,j-1} = \sum_{m=1}^{j-1} L_{i-1,m} U_{m,j-1} = \sum_{m=1}^{j-1} a^{j-1-m} L_{i-1,m} = \sum_{m=2}^{j} a^{j-m} L_{i-1,m-1},$$

and

$$(L \cdot U)_{i-1,j} = \sum_{m=1}^{j} L_{i-1,m} U_{m,j} = \sum_{m=1}^{j} a^{j-m} L_{i-1,m}.$$

Therefore, by (4.27) we obtain

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}$$

$$= a^{j-2} L_{i-1,1} + a^{j-1} L_{i-1,1} + a^{j-2} L_{i-1,2} + \sum_{m=3}^{j} a^{j-m} (L_{i-1,m-1} + L_{i-1,m})$$

$$= a^{j-2} F(i-1) + a^{j-1} F(i-1) + a^{j-2} [F(i) - 1 - (F(i-1) - 1)a] + \sum_{m=3}^{j} a^{j-m} L_{i,m}$$

$$= a^{j-1} F(i-1) + a^{j-2} [F(i-1) + F(i) - 1 - (F(i-1) - 1)a] + \sum_{m=3}^{j} a^{j-m} L_{i,m}$$

$$= a^{j-1}(F(i-1)+F(i-2))+a^{j-2}[F(i+1)-1-(F(i-1)+F(i-2)-1)a]+\sum_{m=3}^{j} a^{j-m} L_{i,m}$$

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$$= a^{j-1}F(i) + a^{j-2}[F(i+1) - 1 - (F(i)-1)a] + \sum_{m=3}^{j} a^{j-m} L_{i,m}$$

$$= (L \cdot U)_{i,j},$$

as desired. The proof of the theorem is now complete. □

In what follows, we concentrate our attention on arithmetic sequence $\gamma^{(d)}$ and we consider the matrices $A_{\gamma^{(d)},\mathcal{F}}(n)$. The following theorem contains an interesting result which is simple at the same time, and so we leave its proof to the reader.

THEOREM 4.9. *Let $A = A_{\gamma^{(d)},\mathcal{F}}(n)$ and $D(n) = \det A$. Then we have*

$$D(n) = d^{n-1}D(1) + d^{n-2}D(2) + \cdots + d^2 D(n-2) + D(n-1),$$

or equivalently

$$D(n) = (1+d)D(n-1) + d(d-1)D(n-2).$$

In particular, when $d = 1$ and $n \geq 2$, we obtain $\det A = 2^{n-2}$. *(Note that in this case we have $A = A_{\chi^{(1)},\mathcal{F}}(n)$.)*

4.5. On the matrices $A_{\mathcal{F},\chi^{(k)}}(n)$. In this part, we pay attention to the matrices of kind $A_{\mathcal{F},\chi^{(k)}}(n)$ and try to find their determinants. For the case $k = 1$, we have the interesting result contained in the following theorem, but the general case seems to be a hard problem.

THEOREM 4.10. *Let $A(n) = A_{\mathcal{F},\chi}(n)$. Then, we have*

$$\det A(n) = \begin{cases} 0 & n \text{ is even}, \\ 1 & n \text{ is odd}. \end{cases}$$

Proof. Let $A(n) = (a_{i,j})_{1 \leq i,j \leq n}$. First of all, by the structure of $A(n)$, we notice that

$$a_{i-1,n} = a_{i,n-1} \qquad \text{for all } \ 1 < i < n. \tag{4.29}$$

and

$$a_{n-2,n} = a_{n-1,n-1}. \tag{4.30}$$

Assume first that n is even. We will show that the *column rank of* $A(n) < n$, in other words we show that

$$\mathrm{C}_1(A(n)) + \mathrm{C}_3(A(n)) + \cdots + \mathrm{C}_{n-1}(A(n)) = \mathrm{C}_n(A(n)),$$

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or equivalently

$$a_{i,1} + a_{i,3} + a_{i,5} + \ldots + a_{i,n-1} = a_{i,n}, \qquad 1 \le i \le n. \tag{4.31}$$

It is obvious that the above claim immediately implies $\det A(n) = 0$.

We verify the claim (4.31) for $1 \le i \le n-1$ and $i = n$, separately. First, we suppose that $1 \le i \le n-1$ and apply induction on i. If $i = 1$, then we have

$$a_{1,1} + a_{1,3} + a_{1,5} + \cdots + a_{1,n-1} = F(1) + F(3) + F(5) + \cdots + F(n-1) = F(n) = a_{1,n}.$$

Also, if $i = 2$, we obtain

$$a_{2,1} + a_{2,3} + a_{2,5} + \cdots + a_{2,n-1} = 2 + F(4) + F(6) + \cdots + F(n) = F(n+1) = a_{2,n}.$$

Suppose now that $i \ge 3$. By induction hypothesis, we have

$$a_{i-1,1} + a_{i-1,3} + \cdots + a_{i-1,n-1} = a_{i-1,n}; \tag{4.32}$$

or equivalently

$$1 + a_{i-2,1} + (a_{i-2,2} + a_{i-2,3}) + (a_{i-2,4} + a_{i-2,5}) + \cdots + (a_{i-2,n-2} + a_{i-2,n-1}) = a_{i-1,n};$$

from which we deduce that

$$1 + a_{i-1,2} + a_{i-1,4} + a_{i-1,6} + \cdots + a_{i-1,n-2} + a_{i-2,n-1} = a_{i-1,n}. \tag{4.33}$$

Therefore, we obtain

$$\begin{aligned}
a_{i,n} &= a_{i-1,n-1} + a_{i-1,n} \quad \text{(by definition)} \\
&= a_{i-2,n} + a_{i-1,n} \quad \text{(by (4.29))} \\
&= 1 + a_{i-1,2} + a_{i-1,4} + a_{i-1,6} + \cdots + a_{i-1,n-2} + a_{i-2,n-1} + a_{i-2,n} \quad \text{(by (4.33))} \\
&= 1 + a_{i-1,2} + a_{i-1,4} + a_{i-1,6} + \cdots + a_{i-1,n-2} + a_{i-1,n} \quad \text{(by definition)} \\
&= 1 + a_{i-1,1} + a_{i-1,2} + a_{i-1,3} + a_{i-1,4} + \cdots + a_{i-1,n-2} + a_{i-1,n-1} \quad \text{(by (4.32))} \\
&= (1 + a_{i-1,1}) + (a_{i-1,2} + a_{i-1,3}) + (a_{i-1,4} + a_{i-1,5}) + \cdots + (a_{i-1,n-2} + a_{i-1,n-1}) \\
&= a_{i,1} + a_{i,3} + a_{i,5} + \cdots + a_{i,n-1},
\end{aligned}$$

as desired. In fact, we proved that

$$a_{i,1} + a_{i,3} + a_{i,5} + \ldots + a_{i,n-1} = a_{i,n}, \qquad 1 \le i \le n-1. \tag{4.34}$$

Now, we suppose that $i = n$. In this case we have

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$$
\begin{aligned}
a_{n,n} &= a_{n-1,n-1} + a_{n-1,n} \quad \text{(by definition)} \\
&= a_{n-2,n} + a_{n-1,n} \quad \text{(by (4.30))} \\
&= a_{n-1,1} + a_{n-1,3} + a_{n-1,5} + \ldots + a_{n-1,n-1} + a_{n-2,n} \quad \text{(by (4.34) for } i = n-1) \\
&= 1 + a_{n-2,1} + a_{n-2,2} + a_{n-2,3} + \cdots + a_{n-2,n-2} + a_{n-2,n-1} + a_{n-2,n} \quad \text{(by definition)} \\
&= 1 + a_{n-1,2} + a_{n-1,4} + a_{n-1,6} + \cdots + a_{n-1,n-2} + a_{n-1,n} \quad \text{(by definition)} \\
&= (1 + a_{n-1,1}) + (a_{n-1,2} + a_{n-1,3}) + (a_{n-1,4} + a_{n-1,5}) + \cdots + (a_{n-1,n-2} + a_{n-1,n-1}) \\
&\quad \text{(by (4.34) for } i = n-1) \\
&= a_{n,1} + a_{n,3} + a_{n,5} + \cdots + a_{n,n-1},
\end{aligned}
$$

which completes the proof of (4.31).

Assume next that n is odd. For $n = 3, 5$ the result is straightforward. Let $n \geq 7$. We introduce the upper triangular square matrix

$$
B(n) = \left[\begin{array}{c|cc} I_{n-2\times n-2} & \mathbf{0} & \mathbf{0} \\ \hline \nu_1 & 1 & 0 \\ \nu_2 & 0 & 1 \end{array}\right]^T
$$

of order n, where $\nu_1 = (\underbrace{-1, 0, -1, 0, \ldots, -1, 0}_{2-\text{periodic}}, -1)$ and $\nu_2 = (\underbrace{-1, 0, -1, 0, \ldots, -1, 0}_{2-\text{periodic}}, -2)$.

It is not difficult to see that

$$
A(n) \cdot B(n) = \left[\begin{array}{c|cc} A(n-2) & B & C \\ \hline \mathbf{0} & 0 & 1 \\ \mathbf{0} & \text{-}1 & * \end{array}\right],
$$

where $B = \mathrm{C}_{n-1}(A(n)_{[n,n-1]})$, $C = \mathrm{C}_n(A(n)_{[n,n-1]})$ and where $*$ is an entry being obtained by multiplying row n of $B(n)$ and column n of $A(n)$, and the reader himself can check it similar to the previous case. This clearly implies that $\det A(n) = \det A(n-2)$, and so $\det A(n) = 1$, as desired. $\square$

4.6. On a generalized Pascal triangle. In this final subsection, we consider the generalized Pascal triangle matrices $P_{\mathcal{F},\chi}(n)$ with $n \geq 2$, and obtain their determinants. In fact, we prove the following theorem.

THEOREM 4.11. *Let* $P = P_{\mathcal{F},\chi}(n)$ *with* $n \geq 2$. *Then we have* $\det P = 2^{n-2}$.

Proof. As before, we use LU-factorization method. We claim that

$$
P = L \cdot U,
$$

where

$$
L = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \hat{\mathcal{F}} & A_{\alpha,\mathcal{F}}(n-1), \end{array}\right],
$$

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with $\hat{\mathcal{F}} = (F(2), F(3), \ldots, F(n))^T$ and where $U = (U_{i,j})_{1 \le i,j \le n}$ with

$$U_{i,j} = \begin{cases} 0 & i > j, \\ j & i = 1, \\ U_{i-1,j-1} + U_{i,j-1} & i \ne 3\ ,\ 1 < i \le j, \\ U_{3,j-1} - U_{2,j-2} + 2U_{2,j-1} & i = 3\ ,\ 3 \le j. \end{cases}$$

Since the proof of $P = L \cdot U$ is quite similar to the previous ones, we stop here.

Notice that, the matrix L is a lower triangular one with 1's on the diagonal, whereas U is an upper triangular matrix with diagonal entries $1, 1, 2, 2, 2, \ldots, 2$. It is obvious that the claim immediately implies the theorem. $\square$

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