DECOMPOSITION OF MATRICES INTO COMMUTATORS OF UNIPOTENT MATRICES OF INDEX 2

XIN HOU†

Abstract. Let \( \mathbb{C} \) be the complex field. Denote by \( \text{SL}_n(\mathbb{C}) \) the group of all complex \( n \times n \) matrices with determinant 1. It is proved that every matrix in \( \text{SL}_n(\mathbb{C}) \) can be decomposed into a product of two commutators of unipotent matrices of index 2. Moreover, two is the smallest such number.

Key words. Unipotent matrix of index 2, Special linear group, Commutator, Complex field.

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1. Introduction. It is an interesting topic to express a matrix in a matrix group as a product of matrices with special nature such as unipotent matrices and involutions. Let \( \text{GL}_n(F) \) be the group of all \( n \times n \) invertible matrices with over a field \( F \). \( \text{SL}_n(F) \) stands for the subgroup of matrices with determinant 1. A unipotent matrix of index \( k \) is a matrix \( A \) satisfying \( (A - I)^k = 0 \). When \( F \) is the complex field \( \mathbb{C} \), Fong and Sourour in [1] investigated the group generated by unipotent matrices and proved that every matrix in the group \( \text{SL}_n(\mathbb{C}) \) is a product of three unipotent matrices (without limitation on the index). In another article [4], Wang and Wu gave a further result that every matrix in the group \( \text{SL}_n(\mathbb{C}) \) is a product of at most four unipotent matrices of index 2.

Denote by \( [X,Y] = XYX^{-1}Y^{-1} \) the commutator of matrices \( X \) and \( Y \). Decomposing matrices into commutators of matrices with special nature is also an interesting topic. In [5], Zheng proved that if \( F \) is the complex number field or the real number field, every matrix \( A \) in \( \text{SL}_n(F) \) is a product of at most two commutators of involutions. In this article, we consider the problem decomposing matrices in \( \text{SL}_n(\mathbb{C}) \) into products of commutators of unipotent matrices of index 2. Our main result is the following theorem.

Theorem 1.1. Every element in the group \( \text{SL}_n(\mathbb{C}) \) can be decomposed into a product of at most two commutators of unipotent matrices of index 2.

Since a commutator of unipotent matrices of index 2 is a product of two conjugate unipotent matrices of index 2, Theorem 1.1 implies Wang and Wu’s concluding result in [4] that every matrix in the group \( \text{SL}_n(\mathbb{C}) \) is a product of at most four unipotent matrices of index 2.

2. Proof of the main result. First, one can easily verify the following remarks.

Remark 2.1. Let \( G \) be a matrix group and let \( k \) be a positive integer number. If \( A \in G \) is a product of \( k \) commutators of unipotent matrices of index 2, then for any element \( B \in G \), \( B^{-1}AB \) is a product of \( k \) commutators of unipotent matrices of index 2 as well.

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†Capital Normal University, Beijing, 100048, People’s Republic of China (houge19870512@126.com). The research of the manuscript was supported by a grant from Scientific Research Project of Beijing Educational Committee (No. KM202110028004).
Let $A$ and $B$ be two matrices which may have different size. Denote by $A \oplus B$ the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

**Remark 2.2.** Let $F$ be a field. If $A \in \text{SL}_m(F)$ is a product of $k$ commutators of unipotent matrices of index 2 and $B \in \text{SL}_n(F)$ is a product of $l$ commutators of unipotent matrices of index 2, then $A \oplus B \in \text{SL}_{m+n}(F)$ is a product of $\max\{k, l\}$ commutators of unipotent matrices of index 2.

Now let us begin with $2 \times 2$ matrices.

**Lemma 2.3.** The matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq -1$,
is a commutator of unipotent matrices of index 2.

**Proof.** If $\lambda = 1$, there is nothing to prove. If $\lambda \neq \pm 1$, for each $a \in \mathbb{C}$,

$\begin{pmatrix} \lambda & a \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$,

are similar (conjugate in $\text{SL}_2(\mathbb{C})$). Choose a complex number $\mu$ such that $\mu^2 = \lambda$. One checks that

$\begin{pmatrix} \lambda & 2(\mu - \mu^{-1}) \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 2\mu & \mu + 1 \\ \mu - 1 & -\mu \end{pmatrix} \begin{pmatrix} 2 & \mu^{-1} \\ -\mu & 0 \end{pmatrix} \begin{pmatrix} 2\mu & \mu + 1 \\ \mu - 1 & -\mu \end{pmatrix}^{-1} \begin{pmatrix} 2 & \mu^{-1} \\ -\mu & 0 \end{pmatrix}^{-1}$,
is a commutator of

$\begin{pmatrix} 2\mu & \mu + 1 \\ \mu - 1 & -\mu \end{pmatrix}$ and $\begin{pmatrix} 2 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}$,

and the two matrices are both unipotent matrices of index 2. By Remark 2.1, we get the conclusion. \[\square\]

**Lemma 2.4.** The matrix $\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$, $a \neq 0$,
is a commutator of unipotent matrices of index 2.

**Proof.** Observe that

$\begin{pmatrix} -1 & 4i \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1-i & -i \\ i & 1+i \end{pmatrix} \begin{pmatrix} 2 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1-i & -i \\ i & 1+i \end{pmatrix}^{-1} \begin{pmatrix} 2 & -i \\ -i & 0 \end{pmatrix}^{-1}$,
is a commutator of

$\begin{pmatrix} 1-i & -i \\ i & 1+i \end{pmatrix}$ and $\begin{pmatrix} 2 & -i \\ -i & 0 \end{pmatrix}$,

and the two matrices are both unipotent matrices of index 2. By Remark 2.1, we get the conclusion. \[\square\]

**Lemma 2.5.** Each $2 \times 2$ matrix $A \in \text{SL}_2(\mathbb{C})$ can be decomposed into a product of at most two commutators of unipotent matrices of index 2. Moreover, two is the smallest such number.

**Proof.** Since $A$ is an element of $\text{SL}_2(\mathbb{C})$, it must be similar to one of the following matrices.

(a) $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq -1$, \[\square\]
33  Decomposition of Matrices into Commutators of Unipotent Matrices of Index 2

\[
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}, \ a \neq 0.
\]

(b) \[
\begin{pmatrix}
-1 & a \\
0 & -1
\end{pmatrix}, \ a \neq 0,
\]

(c) \[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix},
\]

(d) \[
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}, \ a \neq 0.
\]

By Lemma 2.3 and 2.4, both (a) and (b) are commutators of unipotent matrices of index 2. So we need only to prove this lemma for (c) and (d).

If \( A = -I_2 \), then \( A \) can be written as the product of

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
-\lambda^{-1} & 0 \\
0 & -\lambda
\end{pmatrix},
\]

where \( \pm 1 \neq \lambda \in \mathbb{C} \). By Lemma 2.3, the two factors are both commutators of unipotent matrices of index 2. Since a commutator of unipotent matrices of index 2 is just a product of two unipotent matrices of index 2, by Lemma 2.8 of [2], \( A = -I_2 \) is not a commutator of unipotent matrices of index 2. Thus, \( A = -I_2 \) is a product of two commutators of unipotent matrices of index 2 and two is the smallest such number.

If \( A \) is similar to (d) for some nonzero complex number \( a \), then \( A \) can be written as the product of

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\lambda^{-1} & \lambda^{-1}a \\
0 & \lambda
\end{pmatrix},
\]

where \( \pm 1 \neq \lambda \in \mathbb{C} \). By Lemma 2.3, \( A \) can be decomposed into a product of two commutators of unipotent matrices of index 2.

The following factorization theorem, which is given by Sourour, is necessary for the proof of the main result.

**Theorem 2.6 ([3, Theorem 1]).** Let \( A \) be a nonscalar invertible \( n \times n \) matrix over a field \( F \) and let \( b_j \) and \( c_j (1 \leq j \leq n) \) be elements of \( F \) such that \( \prod_{j=1}^{n} b_j c_j = \det A \). There exist \( n \times n \) matrices \( B \) and \( C \) with eigenvalues \( b_1, b_2, \ldots, b_n \), and \( c_1, c_2, \ldots, c_n \), respectively, such that \( A = BC \). Furthermore, \( B \) and \( C \) can be chosen so that \( B \) is lower triangularizable and \( C \) is simultaneously upper triangularizable.

Now let us prove Theorem 1.1 for the nonscalar case.

**Theorem 2.7.** Each nonscalar matrix \( A \in \text{SL}_n(\mathbb{C}) \) can be decomposed into a product of at most two commutators of unipotent matrices of index 2.

**Proof.** If \( n \) is even, let \( n = 2k \). Denote by \( \sigma(A) \) the set of all eigenvalues of \( A \). Let \( a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1} \) be \( n \) different complex numbers. By Theorem 2.6, we can choose matrices \( B \) and \( C \) such that \( \sigma(B) = \sigma(C) = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1}\} \) and \( A = BC \). Therefore, both \( B \) and \( C \) are diagonalizable and similar to \( \text{diag}(a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1}) \). By Lemma 2.3, Remarks 2.1 and 2.2, both \( B \) and \( C \) are commutators of two unipotent matrices of index 2. Thus, \( A \) can be decomposed into a product of at most two commutators of unipotent matrices of index 2.

If \( n \) is odd, let \( n = 2k + 1 \) and \( a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1} \) be \( n - 1 \) different complex numbers. By Theorem 2.6, we can choose matrices \( B \) and \( C \) such that \( \sigma(B) = \sigma(C) = \{1, a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1}\} \)
and $A = BC$. Therefore, both $B$ and $C$ are diagonalizable and similar to diag$(1, a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1})$. By Lemma 2.3, Remarks 2.1 and 2.2, both $B$ and $C$ are commutators of two unipotent matrices of index 2. Thus, $A$ can be decomposed into a product of at most two commutators of unipotent matrices of index 2.

Now let us finish the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Theorem 2.7, the result is true for all nonscalar matrices. So we need only consider the case $A = \lambda I_n$ with $\lambda^n = 1$.

Assume first that $n$ is odd. Write $\lambda I_n$ as

$$\lambda I_n = \text{diag} \left( \lambda, \lambda^2, \ldots, \lambda^{n-1}, 1 \right) \text{diag} \left( 1, \lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda \right).$$

One checks that both $\text{diag} \left( \lambda, \lambda^2, \ldots, \lambda^{n-1}, 1 \right)$ and $\text{diag} \left( 1, \lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda \right)$ are similar to the matrix

$$1 \oplus \text{diag} \left( \lambda, \lambda^{n-1} \right) \oplus \text{diag} \left( \lambda^2, \lambda^{n-2} \right) \oplus \cdots \oplus \text{diag} \left( \lambda^{\frac{n-1}{2}}, \lambda^{\frac{n+1}{2}} \right).$$

For $k \in \{1, 2, \ldots, (n-1)/2\}$, if $\lambda^k = \lambda^{n-k}$, then $\lambda^k = \lambda^{-k} = \pm 1$ since $\lambda^n = 1$. Recall that $n$ is odd. Then $\lambda$ is a primitive unit root of odd order. So either $\lambda^k = \lambda^{n-k} = 1$ or $\lambda^k \neq \lambda^{n-k}$ holds. In both cases, diag$(\lambda^k, \lambda^{n-k})$ is a commutator of unipotent matrices of index 2 by Lemma 2.3. Thus, both $\text{diag} \left( \lambda, \lambda^2, \ldots, \lambda^{n-1}, 1 \right)$ and $\text{diag} \left( 1, \lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda \right)$ are commutators of unipotent matrices of index 2 by Remark 2.2. Hence, $\lambda I_n$ is a product of at most two commutators of unipotent matrices of index 2.

Assume now that $n$ is even, and write $n = 2k$. Let $K(\alpha) = \text{diag}(\alpha, \alpha^{-1})$ for $\alpha \in \mathbb{C}$. By Lemma 2.3, if $\alpha \neq -1$, then $K(\alpha)$ is a commutator of unipotent matrices of index 2. Choose a complex number $a$ such that $a^n \neq 1$ and write $\lambda I_n$ as

$$\lambda I_n = \left( \lambda K(\alpha) \oplus \lambda^2 K(\alpha) \oplus \cdots \oplus \lambda^{2k-1} K(\alpha) \right) \left( K(\alpha^{-1}) \oplus \lambda^{2k-2} K(a^{-1}) \oplus \cdots \oplus \lambda^2 K(a^{-1}) \right).$$

One checks that $\lambda K(\alpha) \oplus \lambda^3 K(\alpha) \oplus \cdots \oplus \lambda^{2k-1} K(\alpha)$ is similar to $K(\lambda a) \oplus K(\lambda^3 a) \oplus \cdots \oplus K(\lambda^{2k-1} a)$ and $K(\alpha^{-1}) \oplus \lambda^{2k-2} K(a^{-1}) \oplus \cdots \oplus \lambda^2 K(a^{-1})$ is similar to $K(\alpha^{-1}) \oplus K(\lambda^{2k-2} a^{-1}) \oplus \cdots \oplus K(\lambda^2 a^{-1})$. Since $a^n \neq 1$, by Lemma 2.3 and Remark 2.2 it follows that both $K(\lambda a) \oplus K(\lambda^3 a) \oplus \cdots \oplus K(\lambda^{2k-1} a)$ and $K(\alpha^{-1}) \oplus K(\lambda^{2k-2} a^{-1}) \oplus \cdots \oplus K(\lambda^2 a^{-1})$ are commutators of unipotent matrices of index 2. Thus, $\lambda I_n$ is a product of at most two commutators of unipotent matrices of index 2 by Remark 2.1.

**REFERENCES**