Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 518-531, November 2008

SHARP LOWER BOUNDS FOR THE DIMENSION OF LINEARIZATIONS OF MATRIX POLYNOMIALS*

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Abstract. A standard way of dealing with matrix polynomial eigenvalue problems is to use linearizations. Byers, Mehrmann and Xu have recently defined and studied linearizations of dimensions smaller than the classical ones. In this paper, lower bounds are provided for the dimensions of linearizations and strong linearizations of a given $m \times n$ matrix polynomial, and particular linearizations are constructed for which these bounds are attained. It is also proven that strong linearizations of an $n \times n$ regular matrix polynomial of degree ℓ must have dimension $n\ell \times n\ell$.

Key words. Matrix polynomials, Matrix pencils, Linearizations, Dimension.

AMS subject classifications. 15A18, 15A21, 15A22, 65F15.

1. Introduction.

We will say that a matrix polynomial of degree $\ell \geq 1$

$$P(\lambda) = \lambda^\ell A_\ell + \lambda^{\ell-1} A_{\ell-1} + \cdots + \lambda A_1 + A_0, \tag{1.1}$$

where $A_0, A_1, \ldots, A_\ell \in \mathbb{C}^{m\times n}$ and $A_\ell \neq 0$, is *regular* if $m = n$ and $\det P(\lambda)$ is not identically zero as a polynomial in λ. We will say that $P(\lambda)$ is *singular* otherwise. A *linearization* of $P(\lambda)$ is a *matrix pencil* $L(\lambda) = \lambda X + Y$ such that there exist unimodular matrix polynomials, i.e., matrix polynomials with constant nonzero determinant, of appropriate dimensions, $E(\lambda)$ and $F(\lambda)$, such that

$$E(\lambda)L(\lambda)F(\lambda) = \left[\begin{array}{c|c} P(\lambda) & 0 \\ \hline 0 & I_s \end{array}\right], \tag{1.2}$$

where I_s denotes the $s \times s$ identity matrix. Classically $s = (\ell - 1)\min\{m, n\}$, but recently linearizations of smaller dimension have been considered [3]. In fact, we will see that I_s is not necessarily present in (1.2) for some polynomials, a situation that

*Received by the editors August 22, 2008. Accepted for publication October 23, 2008. Handling Editor: Joao Filipe Queiro.

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Electronic Journal of Linear Algebra ISSN 1081-3810
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corresponds to $s = 0$. The classical value $s = (\ell - 1)\min\{m, n\}$ is obtained, for instance, in the linearizations given by the first and second companion forms of $P(\lambda)$ [8].

Linearizations of regular matrix polynomials have been widely used in contexts like the Polynomial Eigenvalue Problem (see [1, 9, 10, 12, 13, 15] and the references therein) or the solution of Linear Differential Algebraic Equations (see [14] and the references therein). On the other hand, though linearizations of singular matrix polynomials have not been formally defined, they appear in different problems of Control Theory (see [2, 16] and the references therein). In the classical text on regular matrix polynomials [8] linearizations play a relevant role in the study of the spectral properties of the matrix polynomials. This reference is also a good introduction to some of the mentioned applications.

The advantage of linearizations is that they allow us to reduce a problem of higher degree to a problem of degree one, without affecting the spectral information of the *finite eigenvalues* of the matrix polynomial. This spectral information is contained in the finite *elementary divisors* of the polynomial [6, Ch. VI]. In most situations, it is also relevant to preserve the spectral information of the *infinite eigenvalue*. This information is given by the elementary divisors corresponding to the zero eigenvalue in the *reversal polynomial* $P^\sharp(\lambda) = \lambda^\ell P(1/\lambda)$, which are usually called *infinite elementary divisors*. Unfortunately, linearizations of a matrix polynomial do not necessarily have the same infinite elementary divisors as the polynomial. In order to preserve the infinite elementary divisors one has to work with *strong linearizations* [7]. A matrix pencil $L(\lambda)$ is a strong linearization of $P(\lambda)$ if it is a linearization of $P(\lambda)$ and the reversal pencil $L^\sharp(\lambda)$ is also a linearization of the reversal polynomial $P^\sharp(\lambda)$. Therefore, strong linearizations have the same finite and infinite elementary divisors as the matrix polynomial. Note, however, that linearizations of singular polynomials do not necessarily preserve the *singular structure*, contained in the so-called *minimal indices* [5]. In fact, the most classical linearizations of a given matrix polynomial, known as the *first* and the *second companion forms*, do not have the same minimal indices as the matrix polynomial, although they are related in a simple way [4].

The main disadvantage of linearizations is that they increase the dimension of the problem. In all the references mentioned above, with the exception of [3], it is imposed by definition that the dimension of linearizations is such that the dimension of the identity block in the right hand side of (1.2) is $s = (\ell - 1)\min\{m, n\}$, but it is natural to ask whether or not linearizations with lower dimension exist and, if they exist, to ask for the minimum dimension of any linearization of a given matrix polynomial $P(\lambda)$. The solution of this problem is trivial if $P(\lambda)$ is regular with no infinite eigenvalues, i.e., if the leading $n \times n$ coefficient matrix A_ℓ is nonsingular, because by equating the determinants of both sides in (1.2) one immediately sees

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A publication of the International Linear Algebra Society
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that the dimension of $L(\lambda)$ must be at least $n\ell \times n\ell$, i.e., $s \geq (\ell - 1)n$, since the degree of the determinant of the right hand side is exactly $n\ell$. The solution of the problem is not so obvious if $P(\lambda)$ has infinite eigenvalues or/and it is singular, and linearizations with $s < (\ell - 1)\min\{m, n\}$ have been recently considered in [3]. The purpose of this paper is to show that linearizations and strong linearizations with $s < (\ell - 1)\min\{m, n\}$ of an $m \times n$ matrix polynomial $P(\lambda)$ may exist when $P(\lambda)$ has infinite eigenvalues or/and it is singular, and, more important, to provide sharp lower bounds for their dimensions. These lower bounds will depend on the normal rank of $P(\lambda)$, the degree of the greatest common divisor of the non-identically zero minors with largest dimension of the polynomial and the degree of the zero root in the greatest common divisor of the non-identically zero minors with largest dimension of the dual polynomial (these last two quantities will be called, respectively, the *finite* and *infinite degree of* $P(\lambda)$). We will also see that if $P(\lambda)$ is $n \times n$ and regular then every strong linearization of $P(\lambda)$ has dimension exactly $n\ell \times n\ell$.

The paper is organized as follows. Basic definitions and notations are summarized in Section 2. The main results are presented in Section 3. A brief discussion on the largest possible dimension of linearizations and strong linearizations is included in Section 4. Finally, the conclusions and the future work motivated by our results are discussed in Section 5.

2. Basic definitions and notation. In this brief section, we introduce the basic definitions that will be used in the remainder of the paper. The *normal rank* of a matrix polynomial $P(\lambda)$, denoted by $\operatorname{nrank} P(\lambda)$, is the dimension of the largest minor of $P(\lambda)$ that is not identically zero as a polynomial in λ. Note that $\operatorname{nrank} P(\lambda) = \operatorname{nrank} P^{\sharp}(\lambda)$, and that if $P(\lambda)$ is $n \times n$ and regular then $\operatorname{nrank} P(\lambda) = n$. Throughout this paper we will deal with matrix polynomials $P(\lambda)$ with degree $\ell \geq 1$. For the sake of brevity we will omit the condition $\ell \geq 1$ in the statement of our results. Note that the idea of linearization is only interesting if $\ell > 1$. We will also assume that the degree of $P(\lambda)$ is exactly ℓ, that is, $A_\ell \neq 0$ in (1.1). Recently, Lancaster [11] has proposed a more general approach with the so-called *extended degree*, which allows some of the leading matrix coefficients of the polynomial to be zero. This approach introduces some extra infinite elementary divisors, so we have preferred to impose the usual condition $A_\ell \neq 0$, even though Lancaster's approach may allow us to simplify some of the proofs.

We will say that two matrix polynomials $P_1(\lambda)$ and $P_2(\lambda)$ with the same dimension are *equivalent* if there exist two unimodular matrix polynomials $R(\lambda)$ and $S(\lambda)$ such that $R(\lambda)P_1(\lambda)S(\lambda) = P_2(\lambda)$ [6, Ch. VI]. Note that this is an equivalence relation because the inverse of a unimodular matrix polynomial is also a unimodular matrix polynomial. An $m \times n$ matrix polynomial $P(\lambda)$ is always equivalent to its *Smith*

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normal form

$$\Delta_P(\lambda) = \begin{bmatrix} d_1(\lambda) & & & & & \\ & \ddots & & & & \\ & & d_r(\lambda) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, \tag{2.1}$$

where the diagonal entries $d_1(\lambda), \ldots, d_r(\lambda)$ are nonzero monic polynomials such that $d_i(\lambda)$ divides $d_{i+1}(\lambda)$, and they are called the *invariant factors* of $P(\lambda)$. Notice that the value r in (2.1) is the normal rank of $P(\lambda)$. The product $d_1(\lambda) \cdots d_r(\lambda)$ is the greatest common divisor of all the $r \times r$ minors of $P(\lambda)$ [6, Ch. VI]. If we decompose each invariant factor $d_i(\lambda)$, for $i = 1, \ldots, r$, as a product of powers of irreducible factors

$$d_i(\lambda) = (\lambda - \lambda_1)^{\alpha_{i1}} \cdots (\lambda - \lambda_q)^{\alpha_{iq}}\,, \tag{2.2}$$

where $\alpha_{ij} \geq 0$ for all i, j, and $\lambda_k \neq \lambda_l$ for $k \neq l$, then the factors $(\lambda - \lambda_j)^{\alpha_{ij}}$ with $\alpha_{ij} > 0$ are the *finite elementary divisors* of $P(\lambda)$. Analogously, the *infinite elementary divisors* of $P(\lambda)$ are the finite elementary divisors of the reversal polynomial $P^\sharp(\lambda)$ whose root is equal to zero, i.e., the ones of the form λ^{β_i} with $\beta_i > 0$ for $P^\sharp(\lambda)$.

Next, we introduce in Definition 1 two concepts that are essential in Theorem 3.6, which is the main result of this paper.

Definition 1. *Let $P(\lambda)$ be a matrix polynomial with normal rank equal to r. The* finite degree *of $P(\lambda)$ is the degree of the greatest common divisor of all the $r \times r$ minors of $P(\lambda)$. The* infinite degree *of $P(\lambda)$ is the multiplicity of the zero root in the greatest common divisor of all the $r \times r$ minors of $P^\sharp(\lambda)$.*

In terms of the Smith normal form (2.1) of $P(\lambda)$, the finite degree of $P(\lambda)$ is the degree of the product $d_1(\lambda) \cdots d_r(\lambda)$. On the other hand, the infinite degree of $P(\lambda)$ is the sum of the degrees of the infinite elementary divisors of $P(\lambda)$. In particular, if $P(\lambda)$ is regular, then the finite degree of $P(\lambda)$ is the degree of $\det P(\lambda)$, and the infinite degree of $P(\lambda)$ is the multiplicity of the zero root of $\det P^\sharp(\lambda)$. We will see in Section 3 that the least possible dimension of any strong linearization of $P(\lambda)$ depends on the finite and infinite degrees of $P(\lambda)$, while the least possible dimension of any linearization of $P(\lambda)$ depends only on the finite degree of $P(\lambda)$.

Lemma 2.1 establishes that the degrees of the elementary divisors of $P(\lambda)$ associated with nonzero finite eigenvalues are equal to the degrees of the corresponding elementary divisors of $P^\sharp(\lambda)$. This result will be used in the proof of Theorem 3.6. Its simple proof is omitted.

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A publication of the International Linear Algebra Society
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LEMMA 2.1. *Let $P(\lambda)$ be an $m\times n$ matrix polynomial with normal rank equal to r, and $P^\sharp(\lambda)$ be its reversal polynomial. Let $0 \neq \mu \in \mathbb{C}$ and α be a positive integer. Then $(\lambda-\mu)^\alpha$ is an elementary divisor of $P(\lambda)$ if and only if $(\lambda - \frac{1}{\mu})^\alpha$ is an elementary divisor of $P^\sharp(\lambda)$. As a consequence, if $d_1(\lambda), \ldots, d_r(\lambda)$ are the invariant factors of $P(\lambda)$ and $\widetilde{d}_1(\lambda), \ldots, \widetilde{d}_r(\lambda)$ are the invariant factors of $P^\sharp(\lambda)$, then*

$$d_i(\lambda) = \lambda^{\alpha_{i1}}\,(\lambda-\lambda_2)^{\alpha_{i2}} \cdots (\lambda-\lambda_q)^{\alpha_{iq}} \quad (\lambda_2, \ldots, \lambda_q \neq 0)$$

implies

$$\widetilde{d}_i(\lambda) = \lambda^{\beta_{i1}}\,\left(\lambda - \frac{1}{\lambda_2}\right)^{\alpha_{i2}} \cdots \left(\lambda - \frac{1}{\lambda_q}\right)^{\alpha_{iq}}.$$

Observe that in general $\alpha_{i1} \neq \beta_{i1}$, and that we are assuming that $\lambda_1 = 0$ in (2.2).

Let us define the following elementary pencils that will be frequently used. A *Jordan block* of dimension k associated with $\lambda_i \in \mathbb{C}$ is the $k\times k$ matrix pencil

$$J_{k,\lambda_i}(\lambda) = \begin{bmatrix} \lambda-\lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda-\lambda_i \end{bmatrix},$$

and an *infinite Jordan block* of dimension k is the $k\times k$ matrix pencil

$$N_k(\lambda) = \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix}.$$

A *right singular block of dimension* $k\times(k+1)$ [6, Ch. XII] is the pencil

$$S_k(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix}.$$

We follow the usual convention that if $S_0(\lambda)$ (resp., $S_0^T(\lambda)$) appears in a direct sum, i.e., in a block diagonal matrix, then a zero column (resp., a zero row) of the proper dimension is included in the matrix resulting from the direct sum. For instance,

$$S_0(\lambda) \oplus N_2(\lambda) \oplus S_0^T(\lambda) = \begin{bmatrix} 0 & 1 & \lambda \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Finally, $0_{p\times q}$ denotes the $p \times q$ matrix whose entries are equal to zero. We also follow the convention that if $0_{0\times q}$ (resp., $0_{p\times 0}$) appears in a direct sum then q zero columns (resp., p zero rows) are included in the matrix.

3. Minimal dimensions of linearizations. We have seen in the introduction that if the matrix polynomial (1.1) has a nonsingular $n \times n$ leading coefficient matrix A_ℓ, i.e., if $P(\lambda)$ is regular without infinite elementary divisors, then the dimension of any linearization of $P(\lambda)$ must be at least $n\ell \times n\ell$. The situation is completely different when A_ℓ is singular, and, even in the case when $P(\lambda)$ is regular, linearizations of smaller dimension may exist. Consider, for instance, $P(\lambda) = \left[\begin{smallmatrix} \lambda^2 & -\lambda \\ \lambda & 0 \end{smallmatrix}\right]$ and $L(\lambda) = \left[\begin{smallmatrix} 0 & -\lambda \\ \lambda & 0 \end{smallmatrix}\right]$, which is clearly a linearization of $P(\lambda)$ because $L(\lambda)$ is obtained from $P(\lambda)$ by adding to the first column λ times the second one. However, $L(\lambda)$ is not a strong linearization of $P(\lambda)$ because $\det P^\sharp(\lambda) = \lambda^2$ and $\det L^\sharp(\lambda) = 1$.

In this section we will find the minimum possible value of s in (1.2) both for linearizations and strong linearizations. In addition, we will prove that every strong linearization of a regular $n \times n$ matrix polynomial with degree ℓ has dimension $n\ell \times n\ell$.

3.1. Strong linearizations of regular polynomials. If $L(\lambda)$ is a strong linearization of a matrix polynomial $P(\lambda)$, then the finite and the infinite degrees of $L(\lambda)$ are equal to those of $P(\lambda)$. We will prove that this property forces any strong linearization of an $n \times n$ regular matrix polynomial with degree ℓ to have dimension $n\ell \times n\ell$. For this purpose we need first to prove Lemma 3.1.

LEMMA 3.1. *Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial of degree ℓ with finite degree α and infinite degree β. Then $\alpha + \beta = n\ell$.*

Proof. Set $\det P(\lambda) = a_0\lambda^\alpha + a_1\lambda^{\alpha-1} + \cdots + a_\alpha$. By the definition of the reversal polynomial, we have

$$\det P^\sharp(\lambda) = \lambda^{n\ell} \det P(1/\lambda) = \lambda^{n\ell}(a_0(1/\lambda)^\alpha + a_1(1/\lambda)^{\alpha-1} + \cdots + a_\alpha) = a_0\lambda^{n\ell-\alpha} + \cdots ,$$

where the dots at the end of the right hand side correspond to terms with higher degree. This means that $\beta = n\ell - \alpha$, and the result is proved. □

THEOREM 3.2. *Let $L(\lambda)$ be a strong linearization of an $n \times n$ regular matrix polynomial of degree ℓ. Then $L(\lambda)$ has dimension $n\ell \times n\ell$.*

Proof. Let α and β be the finite and infinite degrees of $P(\lambda)$, respectively. Since $L(\lambda)$ is a strong linearization, there exist unimodular matrices $R(\lambda), S(\lambda), \widetilde{R}(\lambda), \widetilde{S}(\lambda)$ such that

$$R(\lambda)L(\lambda)S(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_s \end{bmatrix} \quad \text{and} \quad \widetilde{R}(\lambda)L^\sharp(\lambda)\widetilde{S}(\lambda) = \begin{bmatrix} P^\sharp(\lambda) & 0 \\ 0 & I_s \end{bmatrix}$$

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A publication of the International Linear Algebra Society
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for a certain s. Then $\det L(\lambda) = c \det P(\lambda)$ and $\det L^\sharp(\lambda) = \widetilde{c} \det P^\sharp(\lambda)$ for some nonzero constants $c, \widetilde{c}$. These equalities imply, in the first place, that the degrees of $\det L(\lambda)$ and $\det P(\lambda)$ are both equal to α and, in the second place, that if $\det P^\sharp(\lambda) = \lambda^\beta q(\lambda)$ with $q(0) \neq 0$, then also $\det L^\sharp(\lambda) = \lambda^\beta \widetilde{q}(\alpha)$ with $\widetilde{q}(0) \neq 0$. Therefore, the finite and infinite degrees of $L(\lambda)$ are also α and β. By Lemma 3.1 applied on $P(\lambda)$, we have that $\alpha + \beta = n\ell$, and by Lemma 3.1 applied on $L(\lambda)$, we have that the dimension of $L(\lambda)$ is $\alpha + \beta = n\ell$, because the degree of $L(\lambda)$ is one.[1] $\square$

We have obtained that strong linearizations of regular matrix polynomials cannot have smaller dimension than the usual ones [8], but for singular matrix polynomials this may happen, as the following example shows.

Example 1. *Let us consider the following* 2×2 *singular matrix polynomial with degree* 2*,*

$$P(\lambda) = \begin{bmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{bmatrix},$$

and the 2×2 *matrix pencil*

$$L(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & 0 \end{bmatrix}.$$

This pencil is a linearization of $P(\lambda)$*, as it can be seen from the identity*

$$\begin{bmatrix} 0 & 1 \\ 1 & -\lambda \end{bmatrix} P(\lambda) = L(\lambda).$$

Moreover, $L(\lambda)$ *is a strong linearization of* $P(\lambda)$ *because*

$$\begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix} P^\sharp(\lambda) = L^\sharp(\lambda).$$

Example 1 presents an extreme case where the strong linearization has the same dimension as the polynomial. Note that, by definition, linearizations with smaller dimension than the polynomial do not exist. As a consequence of our main result, Theorem 3.6, it is easy to derive precise conditions for this extreme situation to hold. In order to prove Theorem 3.6, we need to state some technical lemmas.

[1] The degree of $L(\lambda)$ cannot be zero, because, as it was mentioned in Sections 1 and 2, we are assuming $\ell \geq 1$. This implies by Lemma 3.1 that $\alpha + \beta > 0$, and according to our definitions, a pencil with zero degree has both the finite and the infinite degrees equal to zero.

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A publication of the International Linear Algebra Society
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3.2. Technical lemmas. We will use the symbol $\sim$ to denote equivalence of matrix polynomials by unimodular matrices as it was defined in Section 2.

LEMMA 3.3. *Let $\lambda_1, \ldots, \lambda_k$ be complex numbers with $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the following $k \times k$ matrix polynomials are equivalent:*

1.

$$\begin{bmatrix} (\lambda-\lambda_i)^k & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \sim J_{k,\lambda_i}(\lambda) = \begin{bmatrix} \lambda-\lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda-\lambda_i \end{bmatrix}.$$

2.

$$\begin{bmatrix} (\lambda-\lambda_1)^{f_1}\cdots(\lambda-\lambda_k)^{f_k} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \sim \begin{bmatrix} (\lambda-\lambda_1)^{f_1} & & \\ & \ddots & \\ & & (\lambda-\lambda_k)^{f_k} \end{bmatrix},$$

where $f_1, f_2, \ldots, f_k$ are positive integer numbers.

3. *If $\lambda_i \neq 0$, then*

$$J^{\sharp}_{k,\lambda_i}(\lambda) \sim \begin{bmatrix} \left(\lambda - \dfrac{1}{\lambda_i}\right)^k & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

4. $J^{\sharp}_{k,0}(\lambda) = N_k(\lambda) \sim I_k$.
5. *In addition, the following $k \times (k+1)$ matrix polynomials are equivalent:*

$$S^{\sharp}_k(\lambda) \sim S_k(\lambda) \sim [I_k \; 0_{k\times 1}].$$

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Proof. The proof of the lemma is elementary by using, for instance, the ideas in [6, Ch. VI, Sec. 6]. We only prove the third claim. Note that

$$J^\sharp_{k,\lambda_i}(\lambda) = \begin{bmatrix} 1-\lambda\lambda_i & \lambda & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda \\ & & & 1-\lambda\lambda_i \end{bmatrix}$$

$$\sim \begin{bmatrix} \lambda - \frac{1}{\lambda_i} & \frac{\lambda}{\lambda_i} & & \\ & \ddots & \ddots & \\ & & \ddots & \frac{\lambda}{\lambda_i} \\ & & & \lambda - \frac{1}{\lambda_i} \end{bmatrix} \equiv T_k(\lambda).$$

Let us denote by $D_p(\lambda)$ the greatest common divisor of all the minors of dimension p in $T_k(\lambda)$. So, $D_k(\lambda) = \left(\lambda - \frac{1}{\lambda_i}\right)^k$. In addition, the $(k-1)\times(k-1)$ minor of $T_k(\lambda)$ complementary of the $(k,1)$ entry is $m_{k1} = (\lambda/\lambda_i)^{k-1}$, and the minor complementary of the $(1,1)$ entry is $m_{11} = (\lambda - 1/\lambda_i)^{k-1}$. Note that both minors have no common roots, and therefore, $D_{k-1}(\lambda) = 1$. By using the definition of invariant factors in [6, Ch. VI, Sec. 3], we get that $T_k(\lambda)$ has $k-1$ invariant factors equal to 1 and one equal to $\left(\lambda - \frac{1}{\lambda_i}\right)^k$, and the result is proved by [6, Cor. 1, p. 141]. ◻

Lemma 3.4. *Let p, q and t be positive integers such that $p \geq t$ and $q \geq t$, and at least one of these inequalities is strict. Then the following statements hold:*

1. *There exists a $p \times q$ matrix pencil $T(\lambda)$ of degree 1, with normal rank equal to t, and having neither finite nor infinite elementary divisors.*
2. *Therefore,*

$$T(\lambda) \sim I_t \oplus 0_{(p-t)\times(q-t)} \quad \textit{and} \quad T^\sharp(\lambda) \sim I_t \oplus 0_{(p-t)\times(q-t)}.$$

Proof. The pencil $T(\lambda)$ satisfying the conditions of the statement may be not unique. We will show how to construct one of them. Let us consider three cases:

1. If $p = q > t$, then $T(\lambda) = S_t(\lambda) \oplus 0_{(p-t)\times(q-(t+1))}$.
2. If $p > q \geq t$, then $T(\lambda) = S_t(\lambda)^T \oplus 0_{(p-(t+1))\times(q-t)}$.
3. If $q > p \geq t$, then $T(\lambda) = S_t(\lambda) \oplus 0_{(p-t)\times(q-(t+1))}$.

The rest of the proof is a direct consequence of the fifth claim of Lemma 3.3. ◻

Lemma 3.5 below will be used in the proof of our main result, and besides it allows us to see that the minimal dimensions of linearizations found in Theorem 3.6 are less than or equal to the dimension in (1.2) with the usual value $s = (\ell - 1)\min\{m, n\}$.

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LEMMA 3.5. *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree ℓ with normal rank r, finite degree α and infinite degree β. Then $\alpha + \beta \leq r\ell$.*

Proof. Let $Z(\lambda)$ be a square $r \times r$ submatrix of $P(\lambda)$ with nonzero determinant, and $\widetilde{Z}(\lambda)$ be the submatrix of $P^\sharp(\lambda)$ corresponding to the same rows and columns as those of $Z(\lambda)$. The $r \times r$ matrix polynomial $Z(\lambda)$ is regular. Let ℓ_z be its degree ($\ell_z \leq \ell$), and α_z and β_z be its finite and infinite degrees. By Lemma 3.1, $\alpha_z + \beta_z = r\,\ell_z$. Note that $\widetilde{Z}(\lambda) = \lambda^{\ell - \ell_z} Z^\sharp(\lambda)$. Let us denote by μ the multiplicity of the zero root in $\det \widetilde{Z}(\lambda)$ and by μ' the multiplicity of the zero root in $\det Z^\sharp(\lambda)$. Then $\alpha \leq \alpha_z$ and $\beta \leq \mu = \mu' + r(\ell - \ell_z) = \beta_z + r(\ell - \ell_z)$. So, $\alpha + \beta \leq \alpha_z + \beta_z + r(\ell - \ell_z) = r\ell$. $\square$

3.3. The main result. Now we are in the position to state and prove the main result of this paper.

THEOREM 3.6. *Let $P(\lambda)$ be an $m \times n$ matrix polynomial with normal rank r, finite degree α and infinite degree β. Then the following statements hold:*

1. *There exists a linearization of $P(\lambda)$ with dimension*

$$(\max\{\alpha, r\} + m - r) \times (\max\{\alpha, r\} + n - r), \tag{3.1}$$

and there are no linearizations of $P(\lambda)$ with dimension smaller than (3.1).

2. *There exists a strong linearization of $P(\lambda)$ with dimension*

$$(\max\{\alpha + \beta, r\} + m - r) \times (\max\{\alpha + \beta, r\} + n - r), \tag{3.2}$$

and there are no strong linearizations of $P(\lambda)$ with dimension smaller than (3.2).

Proof. For brevity, we will only prove the second claim, i.e., the result on the strong linearizations, because the first claim is simpler and similar. We will present at the end of this proof some brief comments on the proof of (3.1).

The matrix polynomial $P(\lambda)$ is equivalent to its Smith normal form (2.1) and also the reversal polynomial $P^\sharp(\lambda)$ is equivalent to the Smith normal form of $P^\sharp(\lambda)$, denoted by $\Delta_{P^\sharp}(\lambda)$. Decompose the invariant factors of $P(\lambda)$ as a product of the elementary divisors as in (2.2). Let us decompose in a similar way the invariant factors of $P^\sharp(\lambda)$, $\widetilde{d}_1(\lambda), \ldots, \widetilde{d}_r(\lambda)$, in the form

$$\widetilde{d}_i(\lambda) = \lambda^{\beta_i} q_i(\lambda), \quad \text{for } i = 1, \ldots, r,$$

with $q_i(0) \neq 0$. Notice that with the notation of (2.2), $\alpha = \sum_{ij} \alpha_{ij}$ and $\beta = \sum_{i=1}^r \beta_i$. Recall also the result in Lemma 2.1. Now, we will consider two cases:

1. Case $\alpha + \beta \geq r$. Then the block diagonal matrix pencil

$$L(\lambda) = \bigoplus_{i=1, \alpha_{i1} > 0}^{r} J_{\alpha_{i1}, \lambda_1}(\lambda) \cdots \bigoplus_{i=1, \alpha_{iq} > 0}^{r} J_{\alpha_{iq}, \lambda_q}(\lambda) \bigoplus_{i=1, \beta_i > 0}^{r} N_{\beta_i}(\lambda) \bigoplus 0_{(m-r) \times (n-r)}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 518-531, November 2008

has dimension (3.2). Now, by using the equivalences of Lemma 3.3, it is easy to see that

$$L(\lambda) \sim \begin{bmatrix} \Delta_P(\lambda) & 0 \\ 0 & I_s \end{bmatrix} \quad \text{and} \quad L^\sharp(\lambda) \sim \begin{bmatrix} \Delta_{P^\sharp}(\lambda) & 0 \\ 0 & I_s \end{bmatrix},$$

with $s = \alpha + \beta - r$. Hence, $L(\lambda)$ is a strong linearization of $P(\lambda)$ of dimension (3.2).

2. Case $\alpha+\beta < r$. Define the numbers $p \equiv m-(\alpha+\beta) > 0$, $q \equiv n-(\alpha+\beta) > 0$, and $t \equiv r - (\alpha + \beta) > 0$. Note that $p \geq t$ and $q \geq t$, and at least one of these inequalities is strict, because, otherwise $m = n = r$, which implies that $P(\lambda)$ is regular and $\alpha + \beta = n\ell \geq r$, by Lemma 3.1. This is a contradiction. Therefore, p, q, and t satisfy the conditions of Lemma 3.4, and the corresponding $p \times q$ pencil $T(\lambda)$ exists. Then the block diagonal matrix pencil

$$L(\lambda) = \bigoplus_{i=1,\alpha_{i1}>0}^{r} J_{\alpha_{i1},\lambda_1}(\lambda) \cdots \bigoplus_{i=1,\alpha_{iq}>0}^{r} J_{\alpha_{iq},\lambda_q}(\lambda) \bigoplus_{i=1,\beta_i>0}^{r} N_{\beta_i}(\lambda) \bigoplus T(\lambda),$$

has dimension (3.2), and, by using Lemmas 3.3 and 3.4, it can be seen that

$$L(\lambda) \sim \Delta_P(\lambda) \quad \text{and} \quad L^\sharp(\lambda) \sim \Delta_{P^\sharp}(\lambda).$$

Hence, $L(\lambda)$ is a strong linearization of $P(\lambda)$ of dimension (3.2).

Notice that in the particular case where $\alpha = 0$ and $\beta_1 = \cdots = \beta_r = 1$ the pencil $L(\lambda) = I_r \bigoplus 0_{(m-r)\times(n-r)}$ defined in the first item is not a strong linearization because $L^\sharp(\lambda) = L(\lambda)$, with no elementary divisors, so the infinite degree of $L(\lambda)$ is zero in this case (and the one of $P(\lambda)$ is r). But this case is impossible, because a matrix polynomial $P(\lambda)$ with this conditions satisfies $P^\sharp(0) = 0$, that is $A_\ell = 0$, in contradiction with our initial assumptions.

We still have to prove that there are no strong linearizations of $P(\lambda)$ with dimension smaller than (3.2). For this purpose, let $L(\lambda)$ be a strong linearization of $P(\lambda)$. We may assume that $\alpha + \beta > r$ because, otherwise, the result is a consequence of the definition of linearization. The finite and infinite degrees of $L(\lambda)$ are equal to those of $P(\lambda)$ because they are determined by the elementary divisors. Let s be the dimension of the identity matrix in (1.2). Then $L(\lambda)$ is of dimension $(m + s) \times (n + s)$. Since the normal rank is invariant under equivalence, $\operatorname{nrank} L(\lambda) = \operatorname{nrank} \left[\begin{smallmatrix} P(\lambda) & 0 \\ 0 & I_s \end{smallmatrix}\right] = r + s$, and Lemma 3.5 applied on $L(\lambda)$ implies that $\alpha + \beta \leq r + s$, i.e., $\alpha + \beta - r \leq s$. Note that the degree of $L(\lambda)$ cannot be zero, because a pencil with zero degree has its finite and infinite degrees equal to zero, and in our case $\alpha + \beta > r \geq 0$.

The proof of the result for linearizations that are not necessarily strong follows the same steps, but it is simpler because the reversal linearization is not involved in

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A publication of the International Linear Algebra Society
Volume 17, pp. 518-531, November 2008

the argument. In this case, it is easy to see that the block diagonal matrix pencil

$$L(\lambda) = \bigoplus_{i=1,\alpha_{i1}>0}^{r} J_{\alpha_{i1},\lambda_1}(\lambda) \cdots \bigoplus_{i=1,\alpha_{iq}>0}^{r} J_{\alpha_{iq},\lambda_q}(\lambda) \bigoplus 0_{(m-r)\times(n-r)} \bigoplus I_{r-\alpha}\,,$$

where the last identity block is only present if $r-\alpha>0$, has dimension (3.1) and is a linearization of $P(\lambda)$. □

Notice that the degree of $P(\lambda)$ is irrelevant in the statement of Theorem 3.6. Besides, note that we have constructed in the proof of Theorem 3.6 linearizations with the minimal possible dimensions (3.1) and (3.2), but that these are not necessarily the unique linearizations with these dimensions.

4. Remarks on the largest possible dimension of linearizations. Theorem 3.6 establishes strict lower bounds for the dimension of linearizations. In the case of $P(\lambda)$ being an $n\times n$ regular matrix polynomial we know that $r=n$ and, from Lemma 3.1, that $\alpha+\beta=n\ell$, then (3.2) implies that $n\ell\times n\ell$ is the minimum possible dimension of a strong linearization of $P(\lambda)$. However, we know from Theorem 3.2 that $n\ell\times n\ell$ is, in fact, the exact dimension of any strong linearization of $P(\lambda)$. This raises the following natural theoretical question: can we extend Theorem 3.6 to obtain upper bounds of the dimension of linearizations? The answer is obviously negative in the case of linearizations that are not strong, because if $L(\lambda)$ is a linearization of $P(\lambda)$, then $L(\lambda)\oplus I_q$ is also a linearization of $P(\lambda)$ for any possible value of q. So, there exist linearizations of any dimension $(m+s)\times(n+s)$ larger than (3.1). This argument does not hold for strong linearizations because I_q may produce extra infinite elementary divisors. However, in the next theorem, we show that for singular matrix polynomials, there exist strong linearizations of any dimension larger than (3.2).

THEOREM 4.1. *Let $P(\lambda)$ be an $m\times n$ matrix polynomial with normal rank r, finite degree α and infinite degree β. Then the following statements hold:*

1. *There exists a linearization of $P(\lambda)$ with dimension $k_1\times k_2$ for any*

$$k_1 \geq (\max\{\alpha,r\}+m-r) \quad \textit{and} \quad k_2 \geq (\max\{\alpha,r\}+n-r)$$

satisfying $m+k_2=n+k_1$.

2. *If, in addition, $P(\lambda)$ is singular, then there exists a strong linearization of $P(\lambda)$ with dimension $k_1\times k_2$ for any*

$$k_1 \geq (\max\{\alpha+\beta,r\}+m-r) \quad \textit{and} \quad k_2 \geq (\max\{\alpha+\beta,r\}+n-r) \tag{4.1}$$

satisfying $m+k_2=n+k_1$.

(Note that the condition $m+k_2=n+k_1$ is imposed by the definition of linearization.)

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A publication of the International Linear Algebra Society
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Proof. We will only sketch the proof of the claim for strong linearizations of singular matrix polynomials, because the first claim is trivial. For the sake of brevity, we will set $d = \alpha + \beta$ along the proof. Apart from this, we use the same notation as in the proof of Theorem 3.6.

We consider the case where the inequalities in (4.1) are strict, because the equalities correspond to the linearizations with minimum dimensions found in the proof of Theorem 3.6. The linearization we look for will have the form

$$L(\lambda) = \bigoplus_{i=1,\alpha_{i1}>0}^{r} J_{\alpha_{i1},\lambda_1}(\lambda) \cdots \bigoplus_{i=1,\alpha_{iq}>0}^{r} J_{\alpha_{iq},\lambda_q}(\lambda) \bigoplus_{i=1,\beta_i>0}^{r} N_{\beta_i}(\lambda) \bigoplus H(\lambda),$$

where $H(\lambda)$ is a singular $(k_1 - d) \times (k_2 - d)$ matrix pencil which is to be determined. Note that $k_1 - d > 0$ and $k_2 - d > 0$. If $H(\lambda)$ has neither finite nor infinite elementary divisors and it has normal rank equal to $k_1 - d - (m - r)$ (which is equal to $k_2 - d - (n - r)$), then the Smith form of $L(\lambda)$ is

$$\Delta_L(\lambda) = \begin{bmatrix} \Delta_P(\lambda) & \\ & I_s \end{bmatrix},$$

and the Smith form of $L^\sharp(\lambda)$ is

$$\Delta_{L^\sharp}(\lambda) = \begin{bmatrix} \Delta_{P^\sharp}(\lambda) & \\ & I_s \end{bmatrix},$$

where $s = k_1 - m = k_2 - n$. Then $L(\lambda)$ would be a $k_1 \times k_2$ strong linearization of $P(\lambda)$. Hence, it is enough to show that it is always possible to find a matrix pencil $H(\lambda)$ with dimension $(k_1 - d) \times (k_2 - d)$ and normal rank equal to $k_1 - d - (m - r) = k_2 - d - (n - r)$ having neither finite nor infinite elementary divisors. The existence of the pencil $H(\lambda)$ follows from Lemma 3.4 with $p = k_1 - d > 0$, $q = k_2 - d > 0$ and $t = k_1 - d - (m - r) = k_2 - d - (n - r) > 0$. This last inequality is a consequence of the fact that the inequalities in (4.1) are strict. The fact that $P(\lambda)$ is singular implies that at least one of the inequalities $p \geq t$ or $q \geq t$ is strict. ☐

5. Conclusions and future work. We have found sharp lower bounds for the dimensions of linearizations of a given matrix polynomial $P(\lambda)$ in terms of the normal rank and the finite degree of the polynomial. In the case of strong linearizations, the infinite degree of the polynomial is also involved in the lower bound. The proof of these lower bounds requires to construct linearizations with minimum possible dimension. However, the linearizations that we have constructed are based on the eigenvalues and the elementary divisors of $P(\lambda)$ that are precisely the information that one wants to get when a linearization of $P(\lambda)$ is used. Therefore, the linearizations that we have constructed are not useful in practice, although they have been very useful to

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 518-531, November 2008

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prove the results that we have presented. It remains as an open problem to devise practical methods for constructing linearizations with minimum possible dimension, and also to construct structured linearizations with minimum dimension for structured polynomials. In the important case of $P(\lambda)$ being a regular $n \times n$ matrix polynomial with degree $\ell \geq 1$, we have proved that all its strong linearizations have dimension $n\ell \times n\ell$.

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