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A PARAMETERIZED LOWER BOUND FOR THE SMALLEST SINGULAR VALUE*

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Abstract. This paper presents a parameterized lower bound for the smallest singular value of a matrix based on a new Geršgorin-type inclusion region that has been established recently by these authors. The comparison of the new lower bound with known ones is supplemented with a numerical example.

Key words. Eigenvalue, Inclusion region, Smallest singular value, Lower bound.

AMS subject classifications. 15A12, 65F15.

1. Introduction. To estimate matrix singular values is an attractive topic in matrix theory and numerical analysis, especially to give a lower bound for the smallest one. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and let A^* be the conjugate transpose of A. Then the singular values of A are the square roots of the eigenvalues of AA^*. Throughout the paper we use $\sigma_n(A)$ to denote the smallest singular value of A. Denote $N := \{1, 2, \ldots, n\}$. Define, for all $k \in N$,

$$r_k(A) := \sum_{j\in N\setminus\{k\}} |a_{kj}|, \quad c_k(A) := \sum_{j\in N\setminus\{k\}} |a_{jk}|, \quad h_k(A) := \frac{1}{2}\left(r_k(A) + c_k(A)\right).$$

In the past three decades, several useful lower bounds for the smallest singular value of a matrix have been presented in the literature; see Varah [7] and Qi [6]. By using Geršgorin's theorem (see Chapter 6 of [1]), Johnson [4] obtained a lower bound for $\sigma_n(A)$:

$$\sigma_n(A) \ge \min_{k\in N} \{|a_{kk}| - h_k(A)\}. \tag{1.1}$$

Recently, Johnson and Szulc [5] provided several further lower bounds for the

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smallest singular value. Two of them are

$$\sigma_n(A) \geq \frac{1}{2} \min_{k \in N} \left\{ \left(4|a_{kk}|^2 + [r_k(A) - c_k(A)]^2 \right)^{\frac{1}{2}} - 2h_k(A) \right\} \tag{1.2}$$

and

$$\sigma_n(A) \geq \frac{1}{2} \min_{i \neq k} \left\{ |a_{ii}| + |a_{kk}| - \left[(|a_{ii}| - |a_{kk}|)^2 + 4h_i(A)h_k(A) \right]^{\frac{1}{2}} \right\}. \tag{1.3}$$

In this paper, after introducing a new inclusion region for eigenvalues of a matrix in Section 2, we present a parameterized lower bound for the smallest singular value in Section 3. In Section 4, a numerical example is given to compare our results with the known ones.

2. Inclusion regions for eigenvalues. Recently, a new Geršgorin-type inclusion region for eigenvalues of a matrix has been provided by Huang, Zhang, and Shen in [3]. To introduce the result, we first present some notation. Let S be a nonempty subset of N. Denote $\bar{S} := N \backslash S$ and let $\mathcal{P}(N)$ denote the power set of N. For $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, define, for all $i \in N$

$$r_i^S(A) := \sum_{k \in S \backslash \{i\}} |a_{ik}|, \quad r_i^{\bar{S}}(A) := \sum_{k \in \bar{S} \backslash \{i\}} |a_{ik}|,$$

$$c_i^S(A) := \sum_{k \in S \backslash \{i\}} |a_{ki}|, \quad c_i^{\bar{S}}(A) := \sum_{k \in \bar{S} \backslash \{i\}} |a_{ki}|.$$

If S contains a single element, say $S = \{i_0\}$, then we let $r_{i_0}^S(A) = 0$. Similarly $r_{i_0}^{\bar{S}}(A) = 0$ if $\bar{S} = \{i_0\}$. We sometimes use r_i^S (c_i^S, $r_i^{\bar{S}}$, $c_i^{\bar{S}}$) to denote $r_i^S(A)$ ($c_i^S(A)$, $r_i^{\bar{S}}(A)$, $c_i^{\bar{S}}(A)$, respectively). Define, for all $i \in S$ and $j \in \bar{S}$,

$$G_i^S(A) := \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq r_i^S \right\}, \quad G_j^{\bar{S}}(A) := \left\{ z \in \mathbb{C} : |z - a_{jj}| \leq r_j^{\bar{S}} \right\}$$

and

$$G_{i,j}^S(A) := \left\{ z \in \mathbb{C} : z \notin G_i^S(A) \cup G_j^{\bar{S}}(A), \ \left(|z - a_{ii}| - r_i^S \right) \left(|z - a_{jj}| - r_j^{\bar{S}} \right) \leq r_i^{\bar{S}} r_j^S \right\}.$$

Proposition 2.1. ([3]) *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Then all the eigenvalues of A are located in*

$$G^S(A) := \left(\bigcup_{i \in S} G_i^S(A) \right) \cup \left(\bigcup_{j \in \bar{S}} G_j^{\bar{S}}(A) \right) \cup \left(\bigcup_{i \in S, j \in \bar{S}} G_{i,j}^S(A) \right).$$

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This result is interesting, but it can not be applied directly to estimate $\sigma_n(A)$. So we must deduce other inclusion regions. Let $\alpha \in [0,1]$ be given. For $i \in S$, $j \in \bar{S}$, define the following regions in the complex plane:

$$U_{i,j}^S(A) := \left\{ z \in \mathbb{C} : |z - a_{ii}| - r_i^S \leq \left(r_i^{\bar{S}} r_j^S \right)^{\alpha} \right\},$$

$$V_{i,j}^S(A) := \left\{ z \in \mathbb{C} : |z - a_{jj}| - r_j^{\bar{S}} \leq \left(r_i^{\bar{S}} r_j^S \right)^{1-\alpha} \right\},$$

Proposition 2.2. *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Then all the eigenvalues of A are located in*

$$K^S(A) := \left(\bigcup_{i \in S, j \in \bar{S}} U_{i,j}^S(A) \right) \cup \left(\bigcup_{i \in S, j \in \bar{S}} V_{i,j}^S(A) \right).$$

Proof. It is sufficient to show that $G^S(A) \subset K^S(A)$. Note that $G_i^S(A) \subseteq U_{i,j}^S(A)$ and $G_j^{\bar{S}}(A) \subseteq V_{i,j}^S(A)$. Therefore, for any $z \in G^S(A)$, if

$$z \in \bigcup_{i \in S} G_i^S(A) \quad \text{or} \quad z \in \bigcup_{j \in \bar{S}} G_j^{\bar{S}}(A),$$

then $z \in K^S(A)$. Otherwise, there exist $i_0 \in S$ and $j_0 \in \bar{S}$, such that

$$\left(|z - a_{i_0 i_0}| - r_{i_0}^S\right) \left(|z - a_{j_0 j_0}| - r_{j_0}^{\bar{S}}\right) \leq r_{i_0}^{\bar{S}} r_{j_0}^S = \left(r_{i_0}^{\bar{S}} r_{j_0}^S \right)^{\alpha} \left(r_{i_0}^{\bar{S}} r_{j_0}^S \right)^{1-\alpha}$$

which leads to

$$|z - a_{i_0 i_0}| - r_{i_0}^S \leq \left(r_{i_0}^{\bar{S}} r_{j_0}^S \right)^{\alpha} \quad \text{or} \quad |z - a_{j_0 j_0}| - r_{j_0}^{\bar{S}} \leq \left(r_{i_0}^{\bar{S}} r_{j_0}^S \right)^{1-\alpha}.$$

Hence

$$z \in U_{i_0,j_0}^S(A) \cup V_{i_0,j_0}^S(A).$$

And then $z \in K^S(A)$. □

3. Main results. In this section we use the inclusions derived in Section 2 to estimate $\sigma_n(A)$. Denote the Hermitian part of A by

$$H(A) := \tfrac{1}{2}(A + A^*).$$

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Let $\lambda_{\min}(H(A))$ be the smallest eigenvalue of $H(A)$. It is known that $\lambda_{\min}(H(A))$ is a lower bound for $\sigma_n(A)$ [2, p.227]. Moreover, define for all $i \in S$, $j \in \bar{S}$, and $\alpha \in [0,1]$,

$$P_{i,j}^S(A) := |a_{ii}| - \tfrac{1}{2}\left[r_i^S(A) + c_i^S(A)\right] - \left[\tfrac{1}{4}\left(r_i^{\bar{S}}(A) + c_i^{\bar{S}}(A)\right)\left(r_j^S(A) + c_j^S(A)\right)\right]^{\alpha},$$

$$Q_{i,j}^S(A) := |a_{jj}| - \tfrac{1}{2}\left[r_j^{\bar{S}}(A) + c_j^{\bar{S}}(A)\right] - \left[\tfrac{1}{4}\left(r_i^{\bar{S}}(A) + c_i^{\bar{S}}(A)\right)\left(r_j^S(A) + c_j^S(A)\right)\right]^{1-\alpha}.$$

THEOREM 3.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. Then*

$$\sigma_n(A) \geq \min_{i\in S, j\in\bar{S}} \left\{P_{i,j}^S(A), Q_{i,j}^S(A)\right\}. \tag{3.1}$$

Proof. We first define a diagonal matrix $D = diag(e^{i\theta_1}, \ldots, e^{i\theta_n})$, where $e^{i\theta_k}a_{kk} = |a_{kk}|$ if $a_{kk} \neq 0$ and $\theta_k = 0$ if $a_{kk} = 0$, $k \in N$. Since D is unitary, the singular values of DA are the same as those of A. Consequently, we have

$$\sigma_n(A) = \sigma_n(DA) \geq \lambda_{\min}(H(DA)). \tag{3.2}$$

Denote $B = (b_{kl}) := H(DA) = \frac{1}{2}(DA + A^*D^*)$. Thus, $b_{kk} = |a_{kk}|$, for $k \in N$, and

$$b_{kl} = \frac{1}{2}\left(e^{k\theta_k}a_{kl} + \bar{a}_{lk}e^{-k\theta_k}\right), \quad \text{for all} \quad k \neq l, \quad k,l \in N.$$

Since B is a Hermitian matrix, its eigenvalues are all real. Let $\lambda_{\min}(B)$ denote the smallest eigenvalue of B. Then, by using Proposition 2.2, $\lambda_{\min}(B)$ must satisfy at least one of the following conditions

$$\lambda_{\min}(B) \geq |b_{ii}| - r_i^S(B) - \left[r_i^{\bar{S}}(B)r_j^S(B)\right]^{\alpha}, \qquad i \in S,\ j \in \bar{S},$$

$$\lambda_{\min}(B) \geq |b_{jj}| - r_j^{\bar{S}}(B) - \left[r_i^{\bar{S}}(B)r_j^S(B)\right]^{1-\alpha}, \quad i \in S,\ j \in \bar{S}.$$

It follows that

$$\lambda_{\min}(B) \geq \min_{i\in S, j\in\bar{S}} \left\{|b_{ii}| - r_i^S(B) - \left[r_i^{\bar{S}}(B)r_j^S(B)\right]^{\alpha}, \right.$$
$$\left. |b_{jj}| - r_j^{\bar{S}}(B) - \left[r_i^{\bar{S}}(B)r_j^S(B)\right]^{1-\alpha}\right\}.$$

By applying the triangle inequality, we have

$$\begin{aligned}
&|b_{ii}| - r_i^S(B) - \left[r_i^{\bar{S}}(B)r_j^S(B)\right]^{\alpha} \\
&\quad = |a_{ii}| - r_i^S\left(H(DA)\right) - \left[r_i^{\bar{S}}\left(H(DA)\right)r_j^S\left(H(DA)\right)\right]^{\alpha} \\
&\quad \geq |a_{ii}| - \tfrac{1}{2}\left(r_i^S(A) + c_i^S(A)\right) - \left[\tfrac{1}{4}\left(r_i^{\bar{S}}(A) + c_i^{\bar{S}}(A)\right)\left(r_j^S(A) + c_j^S(A)\right)\right]^{\alpha} \\
&\quad = P_{i,j}^S(A).
\end{aligned}$$

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Similarly, one can obtain

$$|b_{ii}| - r_i^{\bar{S}}(B) - \left[r_i^{\bar{S}}(B) r_j^S(B) \right]^{1-\alpha} \geq Q_{i,j}^S(A).$$

Then from (3.2), we have

$$\sigma_n(A) \geq \min_{i \in S, j \in \bar{S}} \left\{ P_{i,j}^S(A), Q_{i,j}^S(A) \right\}. \qquad \square$$

Since the bound (3.1) holds for any nonempty $S \in \mathcal{P}(N)$ and any $\alpha \in [0,1]$, we have obtain the following corollaries:

Corollary 3.2. $\sigma_n(A) \geq \max_{S \in \mathcal{P}(N)} \max_{\alpha \in [0,1]} \min_{i \in S, j \in \bar{S}} \left\{ P_{i,j}^S(A), Q_{i,j}^S(A) \right\}.$

Corollary 3.3. ([4]) $\sigma_n(A) \geq \min_{i \in N} \left\{ |a_{ii}| - \frac{1}{2} \left(r_i(A) + c_i(A) \right) \right\}.$

4. Numerical example. In this section, we give a numerical example to compare our bound (3.1) with known ones.

Example 4.1. *Consider the following matrices*

$$A_1 = \begin{bmatrix} 11 & 5 & 6 \\ 4 & 12 & -5 \\ 3 & 4 & 13 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 18 & 2 & -5 \\ 6 & 15 & 8 \\ -6 & -3 & 17 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 6 & 2 & -1 \\ 2 & 9 & 1 \\ 2 & -2 & -13 \end{bmatrix}.$$

Table 1. Comparison of lower bounds for $\sigma_n(A)$

Matrix	$\sigma_n(A_i)$	S	α	(1.1)	(1.2)	(1.3)	(3.1)
A_1	5.8446	$\{1\}$	0.57	2.0000	2.1803	2.4861	2.5886
A_2	9.6861	$\{2\}$	0.56	5.5000	6.1172	5.7827	5.7989
A_3	4.5433	$\{3\}$	0.10	2.5000	3.6921	2.3028	2.8377

From Table 1, we can see that bounds (1.2), (1.3), (3.1) are not comparable. Note that the tightness of our bounds depend on the choice of S and α in which, unfortunately, we do not find a method such that the derived bounds are optimal. However, these parameters offer the possibility to optimize the estimation. We hope that future research will propose a method to determine the parameters S and α that can give tighter lower bounds.

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