Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 694-697, October 2020.

AN ELEMENTARY PROOF OF MIRSKY'S LOW RANK APPROXIMATION THEOREM*

CHI-KWONG LI† AND GILBERT STRANG‡

Abstract. An elementary proof is given for Mirsky's result on best low rank approximations of a given matrix with respect to all unitarily invariant norms.

Key words. Unitarily invariant norm, Singular value, Best approximation.

AMS subject classifications. 15A60.

1. Introduction. Let $M_{m,n}$ be the set of $m\times n$ matrices over $\mathbb{F}$, where $\mathbb{F}$ is the real field or the complex field. Denote by $\boldsymbol{x}^*$ and A^* the conjugate transpose of a vector $\boldsymbol{x}\in\mathbb{F}^n$ and a matrix $A\in M_{m,n}$. They reduce to the transposes of $\boldsymbol{x}$ and A if their entries are real. It is well-known that every $A\in M_{m,n}$ admits a singular value decomposition

$$A=\sum_{j=1}^{r}\sigma_j\boldsymbol{u}_j\boldsymbol{v}_j^*=[\boldsymbol{u}_1\cdots\boldsymbol{u}_r]\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}[\boldsymbol{v}_1\cdots\boldsymbol{v}_r]^*,$$

where r is the rank of A, $\{\boldsymbol{u}_1,\dots,\boldsymbol{u}_r\}\in\mathbb{C}^m$ and $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_r\}\in\mathbb{C}^n$ are orthonormal families, and $\sigma_1\ge\cdots\ge\sigma_r>0$ are the nonzero singular values of A; e.g., see [5, 7]. By this result, one may use r positive numbers $\sigma_1,\dots,\sigma_r$, and $[\boldsymbol{u}_1\cdots\boldsymbol{u}_r]\in M_{m,r}$ and $[\boldsymbol{v}_1\cdots\boldsymbol{v}_r]\in M_{n,r}$ to represent the $m\times n$ matrix A. If m,n are large and r is small, the singular value decomposition provides efficient means to store or transmit data encoded in the matrix A. In case the matrix A has a high rank, one may find a suitable low rank approximation of A within an acceptable error bound condition that can be stored or transmitted efficiently. The singular value decomposition allows us to construct the best low rank approximation for A by the following result of Mirsky [5, Theorem 3], which is an extension of the result of Schmidt [6, §18, Das Approximations Theorem]; see also [1].

THEOREM 1.1. *Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}$. Suppose $A\in M_{m,n}$ has singular value decomposition $A=\sum_{j=1}^{r}\sigma_j\boldsymbol{u}_j\boldsymbol{v}_j^*$. If $k\le r$, then the matrix $A_k=\sum_{j=1}^{k}\sigma_j\boldsymbol{u}_j\boldsymbol{v}_j^*$ satisfies*

$$\|A-A_k\|\le\|A-B\|\quad\text{for any } B\in M_{m,n} \text{ with rank at most } k.$$

Recall that a norm on $M_{m,n}$ is unitarily invariant if $\|UAV\|=\|A\|$ for any $A\in M_{m,n}$, $U\in\mathcal{U}_m$ and $V\in\mathcal{U}_n$, where $\mathcal{U}_N=\{A\in M_N : A^*A=I_N\}$ is the group of unitary matrices in the complex case and the group of orthogonal matrices in the real case. Note that the nonzero singular values of A are just the positive square roots of the nonzero eigenvalues of A^*A so that the singular values of A and UAV are always

*Received by the editors on July 28, 2020. Accepted for publication on August 9, 2020. Handling Editor: Michael Tsatsomeros. Corresponding Author: Chi-Kwong Li.

†Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA (ckli@math.wm.edu). Supported by the Simons Foundation Grant 351047.

‡Department of Mathematics, MIT, Cambridge, MA 02139, USA (gilstrang@gmail.com).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 694-697, October 2020.

the same for any $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$. Thus, $\|A\|$ only depends on the singular values of A. Examples of unitarily invariant norms include the spectral norm and the Frobenius norm defined by

$$\|A\|_2 = \max\{|Ax| : \boldsymbol{x} \in \mathbb{F}^n,\ |\boldsymbol{x}| = 1\} \quad \text{and} \quad \|A\|_F = (\operatorname{tr} A^*A)^{1/2}, \quad \text{respectively,}$$

where $|\boldsymbol{u}| = (\sum_{j=1}^n |u_j|^2)^{1/2}$ for $\boldsymbol{u} = (u_1, \dots, u_n)^t \in \mathbb{F}^n$. If A has nonzero singular values $\sigma_1 \geq \cdots \geq \sigma_r$, then $\|A\|_2 = \sigma_1$ and $\|A\|_F = (\sum_{j=1}^r \sigma_j^2)^{1/2}$.

One may see [1, 2, 3, 4, 5, 7] and their references for the wide applications of Theorem 1.1. The original and subsequent proofs of Theorem 1.1 used symmetric gauge functions, Weyl inequalities, and the Ky Fan dominance theorem [2, 4, 5]. In [7], simple proofs of Theorem 1.1 for the special cases of the spectral norm and the Frobenius norm were given. In the next section, we will give a self-contained elementary proof of Mirsky's result that only uses the condition for a homogeneous system of linear equations $B\boldsymbol{z} = 0$ to have a nonzero solution and the fact that matrices $X, Y \in M_{m,n}$ with the same singular values satisfy $\|X\| = \|Y\|$ for any unitarily invariant norm $\|\cdot\|$.

2. An elementary proof of Theorem 1.1. Suppose $\|\cdot\|$ is a unitarily invariant norm on $M_{m,n}$, and $A \in M_{m,n}$ has singular value decomposition $A = \sum_{j=1}^r \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^*$. Suppose $B \in M_{m,n}$ with rank at most $k \leq r$, and $C = A - B$ has singular value decomposition

$$C = \sum_{j=1}^{\ell} \xi_j \boldsymbol{x}_j \boldsymbol{y}_j^*$$

with $\xi_1 \geq \cdots \geq \xi_\ell \geq 0$ and orthonormal sets $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_\ell\} \subseteq \mathbb{F}^m, \{\boldsymbol{y}_1, \dots, \boldsymbol{y}_\ell\} \subseteq \mathbb{F}^n$. Let $\sigma_j = 0$ for $j > r$. First, we show that

$$\xi_j \geq \sigma_{k+j} \quad \text{for } j = 1, \dots, \ell. \tag{2.1}$$

Extend $\{\boldsymbol{y}_1, \dots, \boldsymbol{y}_\ell\}$ to an orthonormal basis $\{\boldsymbol{y}_1, \dots, \boldsymbol{y}_n\}$ for $\mathbb{F}^n$. Then every unit vector $\boldsymbol{y} \in \mathbb{F}^n$ can be written as $\boldsymbol{y} = \sum_{j=1}^n a_j \boldsymbol{y}_j$ for a unit vector $(a_1, \dots, a_n)^t \in \mathbb{F}^n$ so that

$$|C\boldsymbol{y}| = \Big| C\Big(\sum_{j=1}^n a_j \boldsymbol{y}_j\Big)\Big| = \Big|\sum_{j=1}^{\ell} a_j \xi_j \boldsymbol{x}_j\Big| = \Big(\sum_{j=1}^{\ell} |a_j \xi_j|^2\Big)^{1/2}.$$

Thus, $\xi_1 = |C\boldsymbol{y}_1| = \max\{|C\boldsymbol{v}| : \boldsymbol{v} \in \mathbb{F}^n,\ |\boldsymbol{v}| = 1\}$; for $j = 2, \dots, \ell$,

$$\xi_j = |C\boldsymbol{y}_j| = \max\left\{|C\boldsymbol{v}| : \boldsymbol{v} \in \mathbb{F}^n,\ \boldsymbol{y}_1^*\boldsymbol{v} = \cdots = \boldsymbol{y}_{j-1}^*\boldsymbol{v} = 0,\ |\boldsymbol{v}| = 1\right\}. \tag{2.2}$$

Now, B has rank at most k and so does the matrix $B[\boldsymbol{v}_1|\cdots|\boldsymbol{v}_{k+1}] \in M_{n,k+1}$. Hence, there is a unit vector $\boldsymbol{z}_1 = (a_1, \dots, a_{k+1})^t \in \mathbb{F}^m$ satisfying $B[\boldsymbol{v}_1|\cdots|\boldsymbol{v}_{k+1}]\boldsymbol{z}_1 = 0$. Consider the unit vector $\widetilde{\boldsymbol{z}} = a_1\boldsymbol{v}_1 + \cdots + a_{k+1}\boldsymbol{v}_{k+1} \in \mathbb{F}^n$. We have

$$\begin{aligned}
\xi_1 = \|A - B\|_2 \geq |(A-B)\widetilde{\boldsymbol{z}}| = |A\widetilde{\boldsymbol{z}}| &= |A(a_1\boldsymbol{v}_1 + \cdots + a_{k+1}\boldsymbol{v}_{k+1})| \\
= \Big|\sum_{j=1}^{k+1} a_j\sigma_j\boldsymbol{u}_j\Big| &= \Big\{\sum_{j=1}^{k+1} |a_j\sigma_j|^2\Big\}^{1/2} \geq \sigma_{k+1}\Big\{\sum_{j=1}^{k+1} |a_j|^2\Big\}^{1/2} = \sigma_{k+1}.
\end{aligned}$$

For $j > 1$, append row $\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \dots, \boldsymbol{y}_{j-1}^*$ to the matrix B to get $B_j \in M_{m+j-1,n}$. Then B_j has rank at most $k + j - 1$ and so does the matrix $B_j[\boldsymbol{v}_1|\cdots|\boldsymbol{v}_{k+j}] \in M_{m+j-1,k+j}$. Hence, there is a unit vector

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 694-697, October 2020.

$\boldsymbol{z}_j = (b_1, \ldots, b_{k+j})^t \in \mathbb{F}^{k+j}$ satisfying $B_j[\boldsymbol{v}_1|\cdots|\boldsymbol{v}_{k+j}]\boldsymbol{z}_j = 0$. As a result, the unit vector $\widetilde{\boldsymbol{z}}_j = b_1\boldsymbol{v}_1 + \cdots + b_{k+j}\boldsymbol{v}_{k+j} \in \mathbb{F}^n$ satisfies $B\widetilde{\boldsymbol{z}}_j = 0 \in \mathbb{F}^m$ and $\boldsymbol{y}_i^*\widetilde{\boldsymbol{z}}_j = 0$ for $i = 1, \ldots, j-1$. By (2.2),

$$\begin{aligned}\xi_j &\geq |(A-B)\widetilde{\boldsymbol{z}}_j| = |A\widetilde{\boldsymbol{z}}_j| = |A(b_1\boldsymbol{v}_1 + \cdots + b_{k+j}\boldsymbol{v}_{k+j})| \\ &= \Big|\sum_{j=1}^{k+j} b_j\sigma_j\boldsymbol{u}_j\Big| = \Big\{\sum_{j=1}^{k+j}|b_j\sigma_j|^2\Big\}^{1/2} \geq \sigma_{k+j}\Big\{\sum_{j=1}^{k+j}|b_j|^2\Big\}^{1/2} = \sigma_{k+j}.\end{aligned}$$

The proof of (2.1) is complete.

To prove Theorem 1.1, we construct matrices $C_1, \ldots, C_\ell$ in $M_{m,n}$ such that

$$\|A - A_k\| \leq \|C_1\| \leq \cdots \leq \|C_\ell\| = \|A - B\|. \tag{2.3}$$

Let $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ be the standard basis for $M_{m,n}$. We continue to assume $\sigma_j = 0$ if $j > r$, and set $D = \sum_{j=1}^{\ell}\sigma_{k+j}E_{jj}$ so that $\|D\| = \|A - A_k\|$. By (2.1), for every $j = 1, \ldots, \ell$, there is $t_j \in [0,1]$ such that $t_j\xi_j = \sigma_{k+j}$. Let $C_1 = \xi_1E_{11} + \sum_{j=2}^{\ell}\sigma_{k+j}E_{jj}$ and $\widetilde{C}_1 = -\xi_1E_{11} + \sum_{j=2}^{\ell}\sigma_{k+j}E_{jj}$. Then both C_1 and $\widetilde{C}_1$ have singular values $\xi_1, \sigma_{k+2}, \ldots, \sigma_\ell$, and $D = \frac{1+t_1}{2}C_1 + \frac{1-t_1}{2}\widetilde{C}_1$:

$$\|D\| = \Big\|\frac{1+t_1}{2}C_1 + \frac{1-t_1}{2}\widetilde{C}_1\Big\| \leq \frac{1+t_1}{2}\|C_1\| + \frac{1-t_1}{2}\|\widetilde{C}_1\| = \|C_1\|.$$

Now, let $C_2 = \xi_1E_{11} + \xi_2E_{22} + \sum_{j=3}^{\ell}\sigma_{k+j}E_{jj}$ and $\widetilde{C}_2 = \xi_1E_{11} - \xi_2E_{22} + \sum_{j=3}^{\ell}\sigma_{k+j}E_{jj}$. Then both C_2 and $\widetilde{C}_2$ have singular values $\xi_1, \xi_2, \sigma_{k+3}, \ldots, \sigma_\ell$, and

$$\|C_1\| = \Big\|\frac{1+t_2}{2}C_2 + \frac{1-t_2}{2}\widetilde{C}_2\Big\| \leq \frac{1+t_2}{2}\|C_2\| + \frac{1-t_2}{2}\|\widetilde{C}_2\| = \|C_2\|.$$

Repeating this argument ℓ times, we get $C_1, \ldots, C_\ell$, where C_ℓ has singular values $\xi_1, \ldots, \xi_\ell$ and

$$\|A - A_k\| = \|D\| \leq \|C_1\| \leq \cdots \leq \|C_\ell\| = \|C\| = \|A - B\|.$$

Thus, (2.3) holds.

Notes added in proof. As observed by the referee, inequalities in (2.1) are just special cases of Weyl inequalities asserting that for any $X, Y \in M_{m,n}$ and $i + j < \min\{m, n\}$, we have $\sigma_i(X) + \sigma_j(Y) \geq \sigma_{i+j-1}(X+Y)$, where $\sigma_1(Z) \geq \sigma_2(Z) \geq \cdots$ are the singular values of $Z \in M_{m,n}$. Applying this result to our matrices B with rank at most k and $C = A - B$, one gets

$$\sigma_{k+j}(A) = \sigma_{k+j}(C+B) \leq \sigma_j(C) + \sigma_{k+1}(B) = \sigma_j(C),$$

which yields (2.1). The Weyl inequalities can be proved using subspace intersection properties; e.g., see [2, Theorem 3.3.16 (a)]. As mentioned in the introduction, one may prove Theorem 1.1 using singular value inequalities and the Ky Fan dominance theorem, e.g., see [2, p. 215].

Acknowledgments. We thank the referee for alerting us that the special case of Theorem 1.1 for the Frobenius norm was published in [6], and rediscovered in [1]. We also thank the referee for some helpful suggestions.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 694-697, October 2020.

REFERENCES

[1] C. Eckart and G. Young. The approximation of one matrix by another of lower rank. *Psychometrika*, 1:211−218, 1936.
[2] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.
[3] C.K. Li and N.K. Tsing. On unitarily invariant norms and related results. *Linear and Multilinear Algebra*, 20:107−119, 1987.
[4] A.W. Marshall, I. Olkin, and B.C. Arnold. *Inequalities: Theory of Majorization and Its Applications*, second edition. Springer, New York, 2011.
[5] L. Mirsky. Symmetric gauge functions and unitarily invariant norms. *Quarterly Journal of Mathematics*, 11:50−59, 1960.
[6] E. Schmidt. Zur Theorie der linearen und nichtlinearen Integralgleichungen. *Mathematische Annalen*, 63:433−476, 1907.
[7] G. Strang. *Linear Algebra and Learning from Data*. Wellesley-Cambridge Press, 2019.