Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

ELA

ON A NEW CLASS OF STRUCTURED MATRICES RELATED TO THE DISCRETE SKEW-SELF-ADJOINT DIRAC SYSTEMS*

B. FRITZSCHE†, B. KIRSTEIN†, AND A.L. SAKHNOVICH‡

Abstract. A new class of the structured matrices related to the discrete skew-self-adjoint Dirac systems is introduced. The corresponding matrix identities and inversion procedure are treated. Analogs of the Schur coefficients and of the Christoffel-Darboux formula are studied. It is shown that the structured matrices from this class are always positive-definite, and applications for an inverse problem for the discrete skew-self-adjoint Dirac system are obtained.

Key words. Structured matrices, Matrix identity, Schur coefficients, Christoffel-Darboux formula, Transfer matrix function, Discrete skew-self-adjoint Dirac system, Weyl function, Inverse problem.

AMS subject classifications. 15A09, 15A24, 39A12.

1. Introduction.

It is well-known that Toeplitz and block Toeplitz matrices are closely related to a discrete system of equations, namely to Szegö recurrence. This connection have been actively studied during the last decades. See, for instance, [1]–[5], [12, 25] and numerous references therein. The connections between block Toeplitz matrices and Weyl theory for the self-adjoint discrete Dirac system were treated in [11]. (See [26] for the Weyl theory of the discrete analog of the Schrödinger equation.) The Weyl theory for the skew-self-adjoint discrete Dirac system

$$W_{k+1}(\lambda) - W_k(\lambda) = -\frac{i}{\lambda} C_k W_k(\lambda), \quad C_k = C_k^* = C_k^{-1}, \quad k = 0, 1, \dots \tag{1.1}$$

was developed in [14, 18]. Here C_k are $2p \times 2p$ matrix functions. When $p = 1$, system (1.1) is an auxiliary linear system for the isotropic Heisenberg magnet model. Explicit solutions of the inverse problem were constructed in [14]. A general procedure to construct the solutions of the inverse problem for system (1.1) was given in [18], using a new class of structured matrices S, which satisfy the matrix identity

$$AS - SA^* = i\Pi\Pi^*. \tag{1.2}$$

*Received by the editors May 19, 2008. Accepted for publication September 11, 2008. Handling Editor: Harm Bart.

†Fakultät für Mathematik und Informatik, Mathematisches Institut, Universität Leipzig, Johannisgasse 26, D-04103 Leipzig, Germany (fritzsche@math.uni-leipzig.de, kirstein@math.uni-leipzig.de).

‡Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria (al_sakhnov@yahoo.com). Supported by the Austrian Science Fund (FWF) under Grant no. Y330.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

Here, S and A are $(n+1)p \times (n+1)p$ matrices and Π is an $(n+1)p \times 2p$ matrix. The block matrix A has the form

$$A := A(n) = \left\{a_{j-k}\right\}_{k,j=0}^{n}, \quad a_r = \begin{cases} 0 & \text{for} \quad r>0 \\ \dfrac{i}{2} I_p & \text{for} \quad r=0 \\ i\, I_p & \text{for} \quad r<0 \end{cases}, \tag{1.3}$$

where I_p is the $p \times p$ identity matrix. The matrix $\Pi = [\Phi_1 \;\; \Phi_2]$ consists of two block columns of the form

$$\Phi_1 = \begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} \alpha_0 \\ \alpha_0 + \alpha_1 \\ \vdots \\ \alpha_0 + \alpha_1 + \cdots + \alpha_n \end{bmatrix}. \tag{1.4}$$

Definition 1.1. *The class of the block matrices S determined by the matrix identity (1.2) and formulas (1.3) and (1.4) is denoted by Ω_n.*

Notice that the blocks α_k in [18] are Taylor coefficients of the Weyl functions and that the matrices C_n $(0 \le n \le l)$ in (1.1) are easily recovered from the expressions $\Pi(n)^* S(n)^{-1} \Pi(n)$ $(0 \le n \le l)$ (see Theorem 3.4 of [18]). In this way, the structure of the matrices S determined by the matrix identity (1.2) and formulas (1.3) and (1.4), their inversion and conditions of invertibility prove essential. Recall that the self-adjoint block Toeplitz matrices satisfy [15]–[17] the identity $AS - SA^* = i\Pi J \Pi^*$ ($J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}$), which is close to (1.2)–(1.4). We refer also to [20]–[24] and references therein for the general method of the operator identities. The analogs of various results on the Toeplitz matrices and j-theory from [6]–[11] can be obtained for the class Ω_n, too.

2. Structure of the matrices from Ω_n. Consider first the block matrix $S = \left\{s_{kj}\right\}_{k,j=0}^{n}$ with the $p \times p$ entries s_{kj}, which satisfies the identity

$$AS - SA^* = iQ, \quad Q = \left\{q_{kj}\right\}_{k,j=0}^{n}. \tag{2.1}$$

One can easily see that the equality

$$q_{kj} = s_{kj} + \sum_{r=0}^{k-1} s_{rj} + \sum_{r=0}^{j-1} s_{kr} \tag{2.2}$$

follows from (2.1). Sometimes we add comma between the indices and write $s_{k,j}$. Putting $s_{-1,j} = s_{k,-1} = q_{-1,j} = q_{k,-1} = 0$, from (2.2) we have

$$s_{k+1,j+1} - s_{kj} = q_{kj} + q_{k+1,j+1} - q_{k+1,j} - q_{k,j+1}, \quad -1 \le k, j \le n-1. \tag{2.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

Now, putting $Q = i\Pi\Pi^*$ and taking into account (2.3), we get the structure of S.

PROPOSITION 2.1. *Let $S \in \Omega_n$. Then we have*

$$s_{k+1,j+1} - s_{kj} = \alpha_{k+1}\alpha_{j+1}^* \quad (-1 \leq k, j \leq n-1), \tag{2.4}$$

excluding the case when $k = -1$ and $j = -1$ simultaneously. For that case, we have

$$s_{00} = I_p + \alpha_0\alpha_0^*. \tag{2.5}$$

Notice that for the block Toeplitz matrix, the equalities $s_{k+1,j+1} - s_{kj} = 0$ $(0 \leq k, j \leq n-1)$ hold. Therefore, Toeplitz and block Toeplitz matrices can be used to study certain homogeneous processes and appear as a result of discretization of homogeneous equations. From this point of view, the matrix $S \in \Omega_n$ is perturbed by the simplest inhomogeneity.

The authors are grateful to the referee for the next interesting remark.

REMARK 2.2. *From (1.2)–(1.4) we get another useful identity, namely,*

$$S - NSN^* = \widehat{\Pi}\widehat{\Pi}^*, \tag{2.6}$$

where

$$N = \{\delta_{k-j-1}I_p\}_{k,j=0}^{n} = \begin{bmatrix} 0 & & & 0 \\ I_p & & & 0 \\ & \ddots & & \vdots \\ & & I_p & 0 \end{bmatrix}, \quad \widehat{\Pi} = \begin{bmatrix} I_p & \alpha_0 \\ 0 & \alpha_1 \\ \vdots & \vdots \\ 0 & \alpha_n \end{bmatrix}. \tag{2.7}$$

Indeed, it is easy to see that $(I_{(n+1)p} - N)A = \frac{i}{2}(I_{(n+1)p} + N)$. Hence, the identity

$$i(S - NSN^*) = i(I_{(n+1)p} - N)\Pi\Pi^*(I_{(n+1)p} - N^*)$$

follows from (1.2). By (2.7), we have $(I_{(n+1)p} - N)\Pi = \widehat{\Pi}$, and so (2.6) is valid. Relations (2.4) and (2.5) are immediate from (2.6).

PROPOSITION 2.3. *Let $S = \left\{s_{kj}\right\}_{k,j=0}^{n} \in \Omega_n$. Then S is positive and, moreover, $S \geq I_{(n+1)p}$. We have $S > I_{(n+1)p}$ if and only if $\det\alpha_0 \neq 0$.*

Proof. From (2.5) it follows that $S(0) = s_{00} \geq I_p$ and that $S(0) > I_p$, when $\det\alpha_0 \neq 0$. The necessity of $\det\alpha_0 \neq 0$, for the inequality $S > I_{(n+1)p}$ to be true, follows from (2.5), too. We shall prove that $S \geq I_{(n+1)p}$ and that $S > I_{(n+1)p}$, when $\det\alpha_0 \neq 0$, by induction.

Suppose that $S(r-1) = \left\{s_{kj}\right\}_{k,j=0}^{r-1} \geq I_{rp}$ $(r \geq 1)$. According to (2.6), we can

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

present $S(r) = \left\{s_{kj}\right\}_{k,j=0}^{r}$ in the form $S(r) = S_1 + S_2$,

$$S_1 := \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \cdots & \alpha_r^* \end{bmatrix}, \quad S_2 := \begin{bmatrix} I_p & 0 \\ 0 & S(r-1) \end{bmatrix}. \tag{2.8}$$

By the assumption of induction, it is immediate that $S(r) \geq S_2 \geq I_{(r+1)p}$. Hence, we get $S = S(n) \geq I_{(n+1)p}$.

Suppose that $\det \alpha_0 \neq 0$ and $S(r-1) > I_{(n+1)p}$. Let $S(r)f = f$ $(f \in BC^{(r+1)p})$, i.e., let $f^*\big(S(r) - I_{(r+1)p}\big)f = 0$. By (2.8), we have $S_1 \geq 0$, and by the assumption of induction, we have $S_2 - I_{(r+1)p} \geq 0$. So, it follows from $f^*\big(S(r) - I_{(r+1)p}\big)f = 0$ that $f^*S_1 f = 0$ and $f^*\big(S_2 - I_{(r+1)p}\big)f = 0$. Hence, as $\alpha_0\alpha_0^* > 0$ and $S(r-1) > I_{rp}$, we derive $f = 0$. In other words, $S(r)f = f$ implies $f = 0$, that is, $\det\big(S(r) - I_{(r+1)p}\big) \neq 0$. From $\det\big(S(r) - I_{(r+1)p}\big) \neq 0$ and $S(r) \geq I_{(r+1)p}$, we get $S(r) > 0$. So, the condition $\det \alpha_0 \neq 0$ implies $S(n) > I_{(n+1)p}$ by induction. □

Remark 2.4. *Using formula (2.5) and representations $S(r) = S_1(r) + S_2(r)$ $(0 < r \leq n)$, where $S_1(r)$ and $S_2(r)$ are given by (2.8), one easily gets*

$$\begin{aligned} S = I_{(n+1)p} &+ \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \cdots & \alpha_n^* \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} \begin{bmatrix} 0 & \alpha_0^* & \cdots & \alpha_{n-1}^* \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & \alpha_0^* \end{bmatrix} \\ &= I_{(n+1)p} + V_\alpha V_\alpha^*, \quad V_\alpha := \begin{bmatrix} \alpha_0 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_0 \end{bmatrix}. \end{aligned} \tag{2.9}$$

Here, V_α is a triangular block Toeplitz matrix, and formula (2.9) is another way to prove Proposition 2.3. Further, we will be interested in a block triangular factorization of the matrix S itself, namely, $S = V_-^{-1}(V_-^*)^{-1}$, where V_- is a lower triangular matrix.

Similar to the block Toeplitz case (see [13] and references therein) the matrices $S \in \Omega_n$ admit the matrix identity of the form $A_1 S - S A_1 = Q_1$, where Q_1 is of low

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

rank, $A_1 := \{\delta_{k-j+1} I_p\}_{k,j=0}^n = N^*$ and N is given in (2.7). The next proposition follows easily from (2.4).

Proposition 2.5. *Let $S \in \Omega_n$. Then we have*

$$A_1 S - S A_1 = y_1 y_2^* + y_3 y_4^* + y_5 y_6^*, \quad A_1^* S - S A_1^* = -(y_2 y_1^* + y_4 y_3^* + y_6 y_5^*), \tag{2.10}$$

where

$$y_1 = \begin{bmatrix} s_{10} \\ s_{20} \\ \vdots \\ s_{n0} \\ 0 \end{bmatrix}, \quad y_3 = -\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \end{bmatrix}, \quad y_5 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ 0 \end{bmatrix}, \quad y_6 = \begin{bmatrix} 0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \tag{2.11}$$

$$y_2^* = \begin{bmatrix} I_p & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad y_4^* = \begin{bmatrix} 0 & s_{n0} & s_{n1} & \cdots & s_{n,n-1} \end{bmatrix}. \tag{2.12}$$

Differently than the block Toeplitz matrix case, the rank of $A_1 S - S A_1$ is in general situation larger than the rank of $AS - SA^*$, where A is given by (1.3). (To see this compare (1.2)–(1.4) and (2.10)–(2.12).)

3. Transfer matrix function and Weyl functions. Introduce the $(r+1)p \times (n+1)p$ matrix

$$P_k := \begin{bmatrix} I_{(r+1)p} & 0 \end{bmatrix}, \quad r \leq n. \tag{3.1}$$

It follows from (1.3) that $P_r A(n) = A(r) P_r$. Hence, using (1.2) we derive

$$A(r)S(r) - S(r)A(r)^* = i\Pi(r)\Pi(r)^*, \quad \Pi(r) := P_r \Pi. \tag{3.2}$$

As $S > 0$, it admits a block triangular factorization

$$S = V_-^{-1}(V_-^*)^{-1}, \tag{3.3}$$

where $V_-^{\pm 1}$ are block lower triangular matrices. It is immediate from (3.3) that

$$S(r) = V_-(r)^{-1}\big(V_-(r)^*\big)^{-1}, \quad V_-(r) := P_r V_- P_r^*. \tag{3.4}$$

Recall that S-node [21, 23, 24] is the triple $\Big(A(r),\, S(r),\, \Pi(r)\Big)$ that satisfies the matrix identity (3.2) (see also [21, 23, 24] for a more general definition of the S-node). Following [21, 23, 24], introduce the transfer matrix function corresponding to the S-node:

$$w_A(r,\lambda) = I_{2p} - i\Pi(r)^* S(r)^{-1}\big(A(r) - \lambda I_{(r+1)p}\big)^{-1}\Pi(r). \tag{3.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

In particular, taking into account (3.4) and (3.5), we get

$$w_A(0,\lambda) = I_{2p} - \frac{2i}{i-2\lambda}\beta(0)^*\beta(0), \quad \beta(0) = V_-(0)\Pi(0). \tag{3.6}$$

By the factorization theorem 4 from [21] (see also [23, p. 188]), we have

$$\begin{aligned} w_A(r,\lambda) = \Big(I_{2p} - i\Pi(r)^* S(r)^{-1} P^* \big(PA(r)P^* - \lambda I_p\big)^{-1} \big(PS(r)^{-1}P^*\big)^{-1} \\ \times PS(r)^{-1}\Pi(r)\Big) w_A(r-1,\lambda), \quad P = [\,0 \ \cdots \ 0 \ \ I_p\,]. \end{aligned} \tag{3.7}$$

According to (1.3), we obtain

$$\big(PA(r)P^* - \lambda I_p\big)^{-1} = (\frac{i}{2} - \lambda)^{-1} I_p. \tag{3.8}$$

Using (3.4), we derive

$$PS(r)^{-1}P^* = (V_-(r))_{rr}^*(V_-(r))_{rr}, \quad PS(r)^{-1}\Pi(r) = (V_-(r))_{rr}^* PV_-(r)\Pi(r), \tag{3.9}$$

where $(V_-(r))_{rr}$ is the block entry of $V_-(r)$ (the entry from the r-th block row and the r-th block column). In view of (3.8) and (3.9), we rewrite (3.7) in the form

$$w_A(r,\lambda) = \Big(I_{2p} - \frac{2i}{i-2\lambda}\beta(r)^*\beta(r)\Big) w_A(r-1,\lambda), \tag{3.10}$$

$$\beta(r) = PV_-(r)\Pi(r) = (V_-\Pi)_r, \quad 0 < r \le n. \tag{3.11}$$

Here, $(V_-\Pi)_r$ is the r-th $p \times 2p$ block of the block column vector $V_-\Pi$. Moreover, according to (3.9) and definitions (3.6), (3.11) of β, we have

$$\begin{aligned} \Big(PS(r)^{-1}P^*\Big)^{-\frac{1}{2}} PS(r)^{-1}\Pi(r) = u(r)\beta(r), \\ u(r) := \Big(PS(r)^{-1}P^*\Big)^{-\frac{1}{2}} (V_-(r))_{rr}^*, \quad u(r)^*u(r) = I_p. \end{aligned} \tag{3.12}$$

As u is unitary, the properties of $\Big(PS(r)^{-1}P^*\Big)^{-\frac{1}{2}} PS(r)^{-1}\Pi(r)$ proved in [18, p. 2098] imply the next proposition.

Proposition 3.1. *Let $S \in \Omega_n$ and let $\beta(k)$ $(0 \le k \le n)$ be given by (3.3), (3.4), (3.6) and (3.11). Then we have*

$$\begin{cases} \beta(k)\beta(k)^* = I_p \quad (0 \le k \le n), \\ \det\, \beta(k-1)\beta(k)^* \ne 0 \quad (0 < k \le n), \\ \det\, \beta_1(0) \ne 0, \end{cases} \tag{3.13}$$

where $\beta_1(k)$, $\beta_2(k)$ are $p \times p$ blocks of $\beta(k)$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

ELA

REMARK 3.2. *Notice that the lower triangular factor V_- is not defined by S uniquely. Hence, the matrices $\beta(k)$ are not defined uniquely, too. Nevertheless, in view of (3.12), the matrices $\beta(k)^*\beta(k)$ are uniquely defined, which suffices for our considerations.*

When $p = 1$ and $C_k \neq \pm I_2$, the matrices $C_k = C_k^* = C_k^{-1}$ (i.e., the potential of the system (1.1)) can be presented in the form $C_k = I_2 - 2\beta(k)^*\beta(k)$, where $\beta(k)\beta(k)^* = 1$. Therefore, it is assumed in [18] for the system (1.1) on the interval $0 \leq k \leq n$, that

$$C_k = I_{2p} - 2\beta(k)^*\beta(k), \tag{3.14}$$

where $\beta(k)$ are $p \times 2p$ matrices and (3.13) holds. Relation (3.14) implies $C_k = U_k j U_k^*$, where $j = \begin{bmatrix} -I_p & 0 \\ 0 & I_p \end{bmatrix}$ and U_k are unitary $2p \times 2p$ matrices. The equalities $C_k = C_k^* = C_k^{-1}$ follow. Consider the fundamental solution $W_r(\lambda)$ of the system (1.1) normalized by $W_0(\lambda) = I_{2p}$. Using (3.6) and (3.10), one easily derives

$$W_{r+1}(\lambda) = \left(\frac{\lambda - i}{\lambda}\right)^{r+1} w_A\left(r, \frac{\lambda}{2}\right), \quad 0 \leq r \leq n. \tag{3.15}$$

Similar to the continuous case, the Weyl functions of the system (1.1) are defined via Möbius (linear-fractional) transformation

$$\varphi(\lambda) = \big(\mathcal{W}_{11}(\lambda)R(\lambda) + \mathcal{W}_{12}(\lambda)Q(\lambda)\big)\big(\mathcal{W}_{21}(\lambda)R(\lambda) + \mathcal{W}_{22}(\lambda)Q(\lambda)\big)^{-1}, \tag{3.16}$$

where $\mathcal{W}_{ij}$ are $p \times p$ blocks of $\mathcal{W}$ and

$$\mathcal{W}(\lambda) = \{\mathcal{W}_{ij}(\lambda)\}_{i,j=1}^2 := W_{n+1}(\overline{\lambda})^*. \tag{3.17}$$

Here, R and Q are any $p \times p$ matrix functions analytic in the neighborhood of $\lambda = i$ and such that

$$\det\Big(\mathcal{W}_{21}(i)R(i) + \mathcal{W}_{22}(i)Q(i)\Big) \neq 0. \tag{3.18}$$

One can easily verify that such pairs always exist (see [18, p. 2090]). A matrix function $\varphi(\lambda)$ of order p, analytic at $\lambda = i$, generates a matrix $S \in \Omega_n$ via the Taylor coefficients

$$\varphi\Big(i\frac{1+z}{1-z}\Big) = -(\alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n) + O(z^{n+1}) \quad (z \to 0) \tag{3.19}$$

and identity (1.2). By Theorem 3.7 in [18], such φ is a Weyl function of some system (1.1) if and only if S is invertible. Now, from Proposition 2.3 it follows that $S > 0$, and the next proposition is immediate.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

Proposition 3.3. *Any $p \times p$ matrix function φ, which is analytic at $\lambda = i$, is a Weyl function of some system (1.1) on the interval $0 \le k \le n$, such that (3.13) and (3.14) hold.*

Moreover, from the proof of the statement (ii) of Theorem 3.7 in [18], the Corollary 3.6 in [18] and our Proposition 3.3, we get:

Proposition 3.4. *Let the $p \times p$ matrix function φ be analytic at $\lambda = i$ and admit expansion (3.19). Then φ is a Weyl function of the system (1.1) $(0 \le k \le n)$, where C_k are defined by the formulas (1.2)–(1.4), $\Pi = [\Phi_1 \quad \Phi_2]$, (3.3), (3.11) and (3.14). Moreover, any Weyl function of this system admits expansion (3.19).*

4. Schur coefficients and Christoffel-Darboux formula. The sequence $\{\alpha_k\}_{k=0}^n$ uniquely determines via formulas (1.2)–(1.4) or (1.3), (1.4), (2.4) and (2.5) the S-node $\big(A,\, S,\, \Pi\big)$. Then, using (3.3), (3.11) and (3.14), we uniquely recover the system (1.1) $(0 \le k \le n)$, or equivalently, we recover the sequence $\{\beta_k^*\beta_k\}_{k=0}^n$, such that (3.13) holds. By Proposition 3.4, one can use Weyl functions of this system to obtain the sequence $\{\alpha_k\}_{k=0}^n$.

Remark 4.1. *Thus, there are one to one correspondences between the sequences $\{\alpha_k\}_{k=0}^n$, the S-nodes $\big(A,\, S,\, \Pi\big)$ satisfying (1.2), the systems (1.1) $(0 \le k \le n)$ with C_k of the form (3.14) and the sequences $\{\beta_k^*\beta_k\}_{k=0}^n$, such that (3.13) holds.*

Next, we consider a correspondence between $\{\beta_k^*\beta_k\}_{k=0}^n$ and some $p \times p$ matrices $\{\rho_k\}_{k=0}^n$ $(\|\rho_k\| \le 1)$. Notice that $0 \le \beta_1(k)\beta_1(k)^* \le I_p$, and suppose that these inequalities are strict:

$$0 < \beta_1(k)\beta_1(k)^* < I_p \quad (0 \le k \le n). \tag{4.1}$$

In view of the first relation in (3.13) and inequalities (4.1), we have $\det \beta_1(k) \neq 0$ and $\det \beta_2(k) \neq 0$. So, we can put

$$\rho_k := \Big(\beta_2(k)^*\beta_2(k)\Big)^{-\frac{1}{2}} \beta_2(k)^*\beta_1(k). \tag{4.2}$$

It follows from (4.2) that

$$\begin{aligned}\rho_k\rho_k^* &= \Big(\beta_2(k)^*\beta_2(k)\Big)^{-\frac{1}{2}} \beta_2(k)^*\big(I_p - \beta_2(k)\beta_2(k)^*\big)\beta_2(k)\Big(\beta_2(k)^*\beta_2(k)\Big)^{-\frac{1}{2}} \\ &= I_p - \beta_2(k)^*\beta_2(k).\end{aligned} \tag{4.3}$$

By (4.2) and (4.3), we obtain

$$[\rho_k \quad (I_p - \rho_k\rho_k^*)^{\frac{1}{2}}] = u_k\beta(k), \quad \|\rho_k\| < 1, \tag{4.4}$$

where

$$u_k := \Big(\beta_2(k)^*\beta_2(k)\Big)^{-\frac{1}{2}} \beta_2(k)^*, \quad u_k u_k^* = I_p, \tag{4.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

i.e., u_k is unitary.

REMARK 4.2. *Under condition (4.1), according to (4.4) and (4.5), the sequence* $\{\beta_k^*\beta_k\}_{k=0}^n$ *is uniquely recovered from the sequence* $\{\rho_k\}_{k=0}^n$ $(\|\rho_k\| \le 1)$*:*

$$\beta_k^*\beta_k = \begin{bmatrix} \rho_k^* \\ (I_p - \rho_k\rho_k^*)^{\frac{1}{2}} \end{bmatrix} [\rho_k \quad (I_p - \rho_k\rho_k^*)^{\frac{1}{2}}]. \tag{4.6}$$

By Remark 4.1 this means that the S-node can be recovered from the sequence $\{\rho_k\}_{k=0}^n$. *Therefore, similar to the Toeplitz case, we call ρ_k the Schur coefficients of the S-node* $\big(A,\, S,\, \Pi\big)$.

Besides Schur coefficients, we obtain an analog of the Christoffel-Darboux formula.

PROPOSITION 4.3. *Let $S \in \Omega_n$, let $w_A(r, \lambda)$ be introduced by (3.5) for $r \ge 0$ and put $w_A(-1, \lambda) = I_{2p}$. Then we have*

$$\begin{aligned} &\sum_{k=-1}^{n-1} w_A(k,\mu)^*\beta(k+1)^*\beta(k+1)w_A(k,\lambda) \\ &\qquad = \frac{(2\lambda - i)(2\overline{\mu} + i)}{4i(\overline{\mu} - \lambda)}\Big(w_A(n,\mu)^*w_A(n,\lambda) - I_{2p}\Big). \end{aligned} \tag{4.7}$$

Proof. From (3.10) it follows that

$$\begin{aligned} &w_A(k+1,\mu)^*w_A(k+1,\lambda) - w_A(k,\mu)^*w_A(k,\lambda) = \\ &w_A(k,\mu)^*\left(\left(I_{2p} - \frac{2i}{2\overline{\mu}+i}\beta(k+1)^*\beta(k+1)\right)\right. \\ &\left.\times \left(I_{2p} + \frac{2i}{2\lambda - i}\beta(k+1)^*\beta(k+1)\right) - I_{2p}\right) w_A(k,\lambda). \end{aligned} \tag{4.8}$$

Using $\beta(k)\beta(k)^* = I_p$, we rewrite (4.8) in the form

$$\begin{aligned} &w_A(k+1,\mu)^*w_A(k+1,\lambda) - w_A(k,\mu)^*w_A(k,\lambda) \\ &= \frac{4i(\overline{\mu} - \lambda)}{(2\lambda - i)(2\overline{\mu} + i)} w_A(k,\mu)^*\beta(k+1)^*\beta(k+1)w_A(k,\lambda). \end{aligned} \tag{4.9}$$

Equality (4.7) follows from (4.9). □

5. Inversion of $S \in \Omega_n$. To recover the system (1.1) from $\{\alpha_k\}_{k=0}^n$, it is convenient to use formula (3.11). The matrices $V_-(r)$ $(r \ge 0)$ in this formula can be constructed recursively.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

Proposition 5.1. *Let $S = V_-^{-1}(V_-^*)^{-1} \in \Omega_n$. Then $V_-(r+1)$ $(0 \le r < n)$ can be constructed by the formula*

$$V_-(r+1) = \begin{bmatrix} V_-(r) & 0 \\ -t(r)S_{21}(r)V_-(r)^*V_-(r) & t(r) \end{bmatrix}, \tag{5.1}$$

where $S_{21}(r) = [s_{r+1,0} \quad s_{r+1,1} \quad \dots \quad s_{r+1,r}]$,

$$t(r) = \Big(s_{r+1,r+1} - S_{21}(r)V_-(r)^*V_-(r)S_{21}(r)^*\Big)^{-\frac{1}{2}}. \tag{5.2}$$

Proof. To prove the proposition it suffices to assume that $V_-(r)$ satisfies (3.4) and prove $S(r+1) = V_-(r+1)^{-1}(V_-(r+1)^*)^{-1}$. In view of Proposition 2.3 and (3.4), we have $s_{r+1,r+1} - S_{21}(r)V_-(r)^*V_-(r)S_{21}(r)^* > 0$, i.e., formula (5.2) is well defined. Now, it is easily checked that $S(r+1)^{-1} = V_-(r+1)^*V_-(r+1)$ (see formula (2.7) in [17]). □

Put $T = \{t_{kj}\}_{k,j=0}^n = S^{-1}$,

$$\widehat{Q} = \{\widehat{q}_{kj}\}_{k,j=0}^n = T\Pi\Pi^*T, \quad X = T\Phi_1, \quad Y = T\Phi_2, \tag{5.3}$$

where t_{kj} and $\widehat{q}_{kj}$ are $p \times p$ blocks of T and $\widehat{Q}$, respectively. Similar to [15, 16, 20, 22] and references therein, we get the next proposition.

Proposition 5.2. *Let $S \in \Omega_n$. Then $T = S^{-1}$ is recovered from X and Y by the formula*

$$t_{kj} = \widehat{q}_{kj} + \widehat{q}_{k+1,j+1} - \widehat{q}_{k+1,j} - \widehat{q}_{k,j+1} + t_{k+1,j+1}, \tag{5.4}$$

or, equivalently, by the formula

$$t_{kj} = \widehat{q}_{kj} + 2\sum_{r=1}^{n-k} \widehat{q}_{k+r,j+r} - \sum_{r=1}^{n-k} \widehat{q}_{k+r,j+r-1} - \sum_{r=1}^{n-k+1} \widehat{q}_{k+r-1,j+r}, \tag{5.5}$$

where we fix $t_{kj} = 0$ and $\widehat{q}_{kj} = 0$ for $k > n$ or $j > n$, and

$$\widehat{Q} = XX^* + YY^*. \tag{5.6}$$

The block vectors X and Y are connected by the relations

$$\sum_{r=0}^{n}(X_r - X_r^*) = 0, \quad \sum_{r=0}^{n-k} X_{n-r} = \sum_{r=0}^{n-k} \widehat{q}_{k+r,r} \quad (k \ge 0),$$
$$\sum_{r=0}^{n-k} X_{n-r}^* = \sum_{r=0}^{n-k} \widehat{q}_{r,k+r} \quad (k > 0). \tag{5.7}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

Proof. From the identity (1.2) and formula (5.3), it follows that

$$TA - A^*T = i\widehat{Q}, \tag{5.8}$$

where $\widehat{Q}$ satisfies (5.6). The identity $TA - A^*T = i\widehat{Q}$ yields (5.4), which, in its turn, implies (5.5).

To derive (5.7), we rewrite (5.8) in the form

$$\begin{aligned}&\big(A^* - \lambda I_{(n+1)p}\big)^{-1}T - T\big(A - \lambda I_{(n+1)p}\big)^{-1}\\ &= i\big(A^* - \lambda I_{(n+1)p}\big)^{-1}\widehat{Q}\big(A - \lambda I_{(n+1)p}\big)^{-1},\end{aligned} \tag{5.9}$$

and multiply both sides of (5.9) by Φ_1 from the right and by Φ_1^* from the left. Taking into account (5.3), we get

$$\begin{aligned}&\Phi_1^*\big(A^* - \lambda I_{(n+1)p}\big)^{-1}X - X^*\big(A - \lambda I_{(n+1)p}\big)^{-1}\Phi_1\\ &= i\Phi_1^*\big(A^* - \lambda I_{(n+1)p}\big)^{-1}\widehat{Q}\big(A - \lambda I_{(n+1)p}\big)^{-1}\Phi_1.\end{aligned} \tag{5.10}$$

It is easily checked (see formula (1.10) in [17]) that

$$\begin{aligned}\big(A - \lambda I_{(n+1)p}\big)^{-1}\Phi_1 &= \Big(\frac{i}{2} - \lambda\Big)^{-1}\mathrm{col}[\, I_p \;\; \zeta^{-1}I_p \;\; \cdots \;\; \zeta^{-n} \;\; I_p\,],\\ \Phi_1^*\big(A^* - \lambda I_{(n+1)p}\big)^{-1} &= -\Big(\frac{i}{2} + \lambda\Big)^{-1}[\, I_p \;\; \zeta I_p \;\; \cdots \;\; \zeta^n I_p\,],\end{aligned} \tag{5.11}$$

where col means column,

$$\zeta = \frac{\lambda - \frac{i}{2}}{\lambda + \frac{i}{2}}, \quad \frac{i}{2} - \lambda = \frac{i\zeta}{\zeta - 1}, \quad -\frac{i}{2} - \lambda = \frac{i}{\zeta - 1}. \tag{5.12}$$

Notice that we have

$$\Phi_1^*T\Phi_1 = \Phi_1^*X = X^*\Phi_1, \tag{5.13}$$

which implies the first equality in (5.7) . Multiply both sides of (5.10) by $\lambda^2 + \frac{1}{4}$ and use (5.11), (5.12) and the first equality in (5.7) to rewrite the result in the form

$$\begin{aligned}&\frac{i}{\zeta - 1}\Big([\,(\zeta - 1)I_p \;\; (\zeta^2 - 1)I_p \;\; \cdots \;\; (\zeta^n - 1)I_p\,]X\\ &+X^*\mathrm{col}[\,0 \;\; \zeta^{-1}(\zeta - 1)I_p \;\; \cdots \;\; \zeta^{-n}(\zeta^n - 1)I_p\,]\Big)\\ &= i[\, I_p \;\; \zeta I_p \;\; \cdots \;\; \zeta^n I_p\,]\widehat{Q}\,\mathrm{col}[\, I_p \;\; \zeta^{-1}I_p \;\; \cdots \;\; \zeta^{-n}I_p\,].\end{aligned} \tag{5.14}$$

The equalities for the coefficients corresponding to the same degrees of ζ on the left-hand side and on the right-hand side of (5.14) imply the second and the third relations in (5.7) . ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

6. Factorization and similarity conditions. The block matrix

$$K = \begin{bmatrix} K_0 \\ K_1 \\ \vdots \\ K_n \end{bmatrix}, \tag{6.1}$$

where K_j are $p \times (n+1)p$ matrices of the form

$$K_j = i\beta(j)[\,\beta(0)^* \;\; \beta(1)^* \;\; \cdots \;\; \beta(j-1)^* \;\; \beta(j)^*/2 \;\;\; 0 \;\; \cdots \;\; 0\,], \tag{6.2}$$

plays an essential role in [18]. From the proof of Theorem 3.4 in [18] the following result is immediate.

PROPOSITION 6.1. *Let a $(n+1)p \times (n+1)p$ matrix K be given by formulas (6.1) and (6.2), and let conditions (3.13) hold. Then K is similar to A:*

$$K = V_- A V_-^{-1}, \tag{6.3}$$

where $V_-^{\pm 1}$ are block lower triangular matrices.

Proposition 6.1 is a discrete analog of the theorem on similarity to the integration operator [19].

REMARK 6.2. *Note that V_-^{-1} can be chosen so that*

$$V_-^{-1} \begin{bmatrix} \beta_1(0) \\ \vdots \\ \beta_1(n) \end{bmatrix} = \Phi_1. \tag{6.4}$$

Moreover, V_-^{-1} is a factor of S, i.e., $S = V_-^{-1}\big(V_-^*\big)^{-1} \in \Omega_n$. Any matrix $S \in \Omega_n$ can be obtained in this way.

An analogue of Proposition 6.1 for the self-adjoint discrete Dirac system and block Toeplitz matrices S follows from the proof of Theorem 5.2 in [11].

PROPOSITION 6.3. *Let a $(n+1)p \times (n+1)p$ matrix K be given by formulas (6.1) and*

$$K_j = i\beta(j)J[\,\beta(0)^* \;\; \cdots \;\; \beta(j-1)^* \;\; \beta(j)^*/2 \;\; 0 \;\; \cdots \;\; 0\,], \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \tag{6.5}$$

where $\beta(k)$ are $p \times 2p$ matrices. Let conditions $\beta(k)J\beta(k)^* = I_p$ $(0 \le k \le n)$ hold. Then K is similar to A: $K = V_- A V_-^{-1}$, where $V_-^{\pm 1}$ are block lower triangular matrices. Moreover, V_- can be chosen so that $S = V_-^{-1}\big(V_-^*\big)^{-1}$ is a block Toeplitz matrix.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

ELA

Acknowledgment. The authors are grateful to Professor W. Schempp and GALA project for the opportunity to meet in Grossbothen and discuss this paper.

REFERENCES

[1] D. Alpay and I. Gohberg. Connections between the Carathodory-Toeplitz and the Nehari extension problems: the discrete scalar case. *Integral Equations Operator Theory*, 37:125–142, 2000.

[2] Ph. Delsarte, Y. Genin, and Y. Kamp. Orthogonal polynomial matrices on the unit circles. *IEEE Trans. Circuits and Systems*, CAS-25:149–160, 1978.

[3] Ph. Delsarte, Y. Genin, and Y. Kamp. Schur parametrization of positive definite block-Toeplitz systems. *SIAM J. Appl. Math.*, 36:34–46, 1979.

[4] V.K. Dubovoj, B. Fritzsche, and B. Kirstein. *Matricial version of the classical Schur problem.* Teubner-Texte zur Mathematik [Teubner Texts in Mathematics] 129, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1992.

[5] H. Dym. Hermitian block Toeplitz matrices, orthogonal polynomials, reproducing kernel Pontryagin spaces, interpolation and extension. *Oper. Theory Adv. Appl.*, 34:79-135, 1988.

[6] B. Fritzsche and B. Kirstein. An extension problem for non-negative Hermitian block Toeplitz matrices. *Math. Nachr.*, Part I, 130:121–135, 1987; Part II, 131:287–297, 1987; Part III, 135:319–341, 1988; Part IV, 143:329–354, 1989; Part V, 144:283–308, 1989.

[7] B. Fritzsche and B. Kirstein. On the Weyl matrix balls associated with nondegenerate matrix-valued Carathodory functions. *Z. Anal. Anwendungen*, 12:239–261, 1993.

[8] B. Fritzsche, B. Kirstein, and M. Mosch. On block completion problems for Arov-normalized j_{qq}-J_q-elementary factors. *Linear Algebra Appl.*, 346:273–291, 2002.

[9] B. Fritzsche, B. Kirstein, and K. Müller. An analysis of the block structure of certain subclasses of j_{qq}-inner functions. *Z. Anal. Anwendungen*, 17:459–478, 1998.

[10] B. Fritzsche, B. Kirstein, and A.L. Sakhnovich. Completion problems and scattering problems for Dirac type differential equations with singularities. *J. Math. Anal. Appl.*, 317:510–525, 2006.

[11] B. Fritzsche, B. Kirstein, I. Roitberg, and A.L. Sakhnovich. Weyl matrix functions and inverse problems for discrete Dirac-type self-adjoint systems: explicit and general solutions. *Operators and Matrices*, 2:201–231, 2008.

[12] L. Golinskii and P. Nevai. Szegö difference equations, transfer matrices and orthogonal polynomials on the unit circle. *Comm. Math. Phys.*, 223:223–259, 2001.

[13] G. Heinig and K. Rost. *Algebraic methods for Toeplitz-like matrices and operators.* Oper. Theory Adv. Appl., 13, Birkhäuser Verlag, Basel, 1984.

[14] M.A. Kaashoek and A.L. Sakhnovich. Discrete skew self-adjoint canonical system and the isotropic Heisenberg magnet model. *J. Functional Anal.*, 228:207–233, 2005.

[15] A.L. Sakhnovich. A certain method of inverting Toeplitz matrices. *Mat. Issled.*, 8:180–186, 1973.

[16] A.L. Sakhnovich. On the continuation of the block Toeplitz matrices. *Functional Analysis* (Uljanovsk), 14:116–127, 1980.

[17] A.L. Sakhnovich. Toeplitz matrices with an exponential growth of entries and the first Szegö limit theorem. *J. Functional Anal.*, 171:449–482, 2000.

[18] A.L. Sakhnovich. Skew-self-adjoint discrete and continuous Dirac-type systems: inverse problems and Borg-Marchenko theorems. *Inverse Problems*, 22:2083–2101, 2006.

[19] L.A. Sakhnovich. Spectral analysis of Volterra's operators defined in the space of vector-functions $L^2_m(0,l)$. *Ukr. Mat. Zh.*, 16:259–268, 1964.

[20] L.A. Sakhnovich. An integral equation with a kernel dependent on the difference of the arguments. *Mat. Issled.*, 8:138–146, 1973.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 473-486, September 2008

ELA

[21] L.A. Sakhnovich. On the factorization of the transfer matrix function. *Sov. Math. Dokl.*, 17:203–207, 1976.

[22] L.A. Sakhnovich. *Integral equations with difference kernels on finite intervals.* Oper. Theory Adv. Appl., 84, Birkhäuser Verlag, Basel, 1996.

[23] L.A. Sakhnovich. *Interpolation theory and its applications.* Mathematics and its Applications, 428, Kluwer Academic Publishers, Dordrecht, 1997.

[24] L.A. Sakhnovich. *Spectral theory of canonical differential systems, method of operator identities.* Oper. Theory Adv. Appl., 107, Birkhäuser Verlag, Basel, 1999.

[25] B. Simon. *Orthogonal polynomials on the unit circle*, Parts 1/2. Colloquium Publications, 51/54, American Mathematical Society, Providence, 2005.

[26] G. Teschl. *Jacobi operators and completely integrable nonlinear lattices.* Mathematical Surveys and Monographs, 72, American Mathematical Society, Providence, 2000.