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SPACES OF RANK-2 MATRICES OVER GF(2)*

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Abstract. The possible dimensions of spaces of matrices over GF(2) whose nonzero elements all have rank 2 are investigated.

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Let $\mathcal{M}_{m,n}(F)$ denote the vector space of all $m \times n$ matrices over the field F. In the case that $m = n$ we write $\mathcal{M}_n(F)$. A subspace, $\mathcal{K}$, is called a *rank-k* space if each nonzero entry in $\mathcal{K}$ has rank equal k. We assume throughout that $1 \le k \le m \le n$.

The structure of rank-k spaces has been studied lately by not only matrix theorists but group theorists and algebraic geometers; see [4], [5], [6]. In [3], [7], it was shown that the dimension of a rank-k space is at most $n + m - 2k + 1$, and in [1] that the dimension of a rank-k space is at most $\max(k+1, n-k+1)$, when the field is algebraically closed.

In [2] it was shown that, if $|F| \ge n + 1$ and $n \ge 2k - 1$, then the dimension of a rank-k space is at most n. Thus, if $k = 2$ and F is not the field of two elements, we know that the dimension of a rank-2 space is at most n. In [2] it was also shown that if $n = qk + r$, with $0 \le r < k$ then if F has an extension of degree k and one of degree $k + r$, then there is a rank-k space of dimension n. Thus, for $k = 2$, the only case left to investigate is when $|F| = 2$.

In this paper we shall show that if $m = n = 3$ there is a rank-2 space of dimension $n+1$ over the field of two elements and that if $n \ge 4$ the dimension of a rank-2 space is at most n.

Further, is easily shown that for any field, the dimension of a rank-m space is at most n. Thus, henceforth, we assume that $k = 2$, $3 \le m \le n$ and that $F = \mathcal{Z}_2$, the field of two elements.

EXAMPLE 1. Consider the space of matrices

$$\left\{ \begin{bmatrix} a & c & c \\ d & a+b & c \\ d & d & b \end{bmatrix} | a, b, c, d \in \mathcal{Z}_2 \right\}.$$

It is easily checked that this is a 4 dimensional rank-2 subspace of $\mathcal{M}_3(\mathcal{Z}_2)$.

It follows that for $n = 3$, n is not an upper bound on the dimension of a rank-2 space.

We let I_k denote the identity matrix of order $k \times k$, $O_{k,l}$ the zero matrix of order $k \times l$, and O_k denotes $O_{k,k}$. When the order is obvious from the context, we omit

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the subscripts. We shall use the notation $\rho(A)$ to denote the rank of the matrix A. For increasing sequences, $\alpha \subseteq \{1, 2, \cdots, m\}$, and $\beta \subseteq \{1, 2, \cdots, n\}$, we will let $A[\alpha|\beta]$ denote the submatrix of A on rows α and columns β. That is, $A[i, j|k, l] = \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}$.

THEOREM 2. *If $n \geq 4$, $\mathcal{K}$ is a rank 2 space and $\mathcal{F} = \mathcal{Z}_2$, then* $\dim \mathcal{K} \leq n$.

Proof. Without loss of generality, we may assume that $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} \in \mathcal{K}$. We also suppose that $\dim \mathcal{K} > n$.

Suppose that there is some nonzero $C \in \mathcal{K}$ such that $C = \begin{bmatrix} O_2 & C_2 \\ O & C_4 \end{bmatrix}$. Then $C_4 = O$ and $\rho(C_2) = 2$. Multiplying all elements of $\mathcal{K}$ by appropriate matrices that leave $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$ fixed, we can assume that $\begin{bmatrix} O & I_2 & O \\ O & O & O \end{bmatrix} \in \mathcal{K}$.

Now, since $\dim \mathcal{K} > n$, there exists $A \in \mathcal{K}$ such that $a_{1j} = 0$ for all j and the rank of A is 2. Now, let $B(x, y) = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + y \begin{bmatrix} O & I_2 & O \\ O & O & O \end{bmatrix} + A$. Then, $B(x, y)[1, 2, 3|1, 2, 3] = \begin{bmatrix} x & 0 & y \\ a_{21} & a_{22} + x & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ must have zero determinant for all $x, y \in \mathcal{F}$. That is

$$a_{33}x + a_{33}a_{22}x + a_{23}a_{32}x + a_{31}a_{22}y + a_{31}xy + a_{32}a_{21}y = 0. \tag{1}$$

Recall that in $\mathcal{F} = \mathcal{Z}_2$, $x^2 = x$ for all x. It follows that for $x = 0$ and $y = 1$ we have

$$a_{31}a_{22} + a_{32}a_{21} = 0 \tag{2}$$

and for x = 1 and y = 0 we have

$$a_{33} + a_{33}a_{22} + a_{32}a_{23} = 0.$$

Now, we have

$$y(a_{31}a_{22} + a_{32}a_{21}) + x(a_{33} + a_{33}a_{22} + a_{32}a_{23}) = a_{31}xy$$

from (1) and each term of the left hand side is zero. Thus $a_{31} = 0$.

By considering $B(x, y)[1, 2, r|1, 2, 3]$ as above, we get that $a_{r1} = 0$ for all r.

Similarly, $B(x, y)[1, 2, r|1, 2, 4]$ must have zero determinant for all $r \geq 3$. Thus, $a_{r4}x + a_{22}a_{r4}x + a_{24}a_{r2}x + a_{r2}xy = 0$. If $x = 1$ and $y = 0$ we get $a_{r4} + a_{22}a_{r4} + a_{24}a_{r2} = 0$ and hence $a_{r2}xy = 0$ for all x, y. Thus, $a_{r2} = 0$ for all $r \geq 3$.

Now, $B(x, y)[1, 2, r|1, 3, 4]$ must also have zero determinant. That is, $a_{23}a_{r4}x + a_{24}a_{43}x + a_{21}a_{r4}y + a_{r3}xy = 0$. As above we get that $a_{r3} = 0$ for all $r \geq 3$.

Since $B(x, y)[1, 2, r|2, 3, s]$ must have zero determinant for $r, s \geq 3$, we get $a_{22}a_{rs}y + a_{rs}xy = 0$ for all x, y. As above, we get that $a_{rs} = 0$ for $r, s \geq 3$.

The above contradicts that A has rank 2 since A has only one nonzero row. Thus, there is no matrix $C \in \mathcal{K}$ of the form $C = \begin{bmatrix} O_2 & C_2 \\ O & C_4 \end{bmatrix}$.

Similarly, there is no matrix in $\mathcal{K}$ of the form $\begin{bmatrix} O_2 & O \\ C_3 & C_4 \end{bmatrix}$.

Since $n \geq 4$ and we have supposed that $\dim \mathcal{K} > n$, there is some rank-2 matrix $A \in \mathcal{K}$ of the form $A = \begin{bmatrix} O_2 & A_2 \\ A_3 & A_4 \end{bmatrix}$. From the above, we know that A_2 and A_3 are not zero. Since $\rho(A) \geq \rho(A_2) + \rho(A_3)$, we must have $\rho(A_2) = \rho(A_3) = 1$.

Let R, S and Q be invertible matrices such that

$$RA_2Q = \begin{bmatrix} 1 & \vec{0}^t \\ 0 & \vec{0}^t \end{bmatrix}$$

and

$$SA_3R^{-1} = \begin{bmatrix} \alpha & \beta \\ \vec{0} & \vec{0} \end{bmatrix}.$$

Let $B = (R \oplus S)A(R_{-1} \oplus Q)$. We have two cases: $\alpha = 0$ (and hence $\beta = 1$) or $\alpha = 1$.

Case 1. $\alpha = 1$. In this case let $E = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \oplus I_{n-2}$ and $F = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \oplus I_{m-2}$.

Then $F\left(x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + B\right)E = \begin{bmatrix} x & 0 & 1 & \vec{0}^t \\ 0 & x & 0 & \vec{0}^t \\ 1 & 0 & & \\ \vec{0} & \vec{0} & & A_4 \end{bmatrix} = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + FBE.$

Let $C = FBE$. Now, since $\det\left(x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + C\right)[1, 2, r|1, 2, s]$ must be zero, if $r \geq 3$ and $s \geq 3$, we have $c_{rs} = 0$ for all such $(r, s) \neq (3, 3)$ since the coefficient of x must be zero. For $(r, s) = (3, 3)$ we get $x^2 c_{33} + x = 0$, so $c_{33} = 1$. That is

$$C = \begin{bmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & O \\ 1 & 0 & 1 & \\ & O & & O \end{bmatrix}.$$

With out loss of generality we may assume that $C \in \mathcal{K}$. Since $\dim \mathcal{K} > n \geq 4$, there is some $B \in \mathcal{K}$ which is rank 2 and such that

$$B = \begin{bmatrix} 0 & b_{12} & 0 & b_{14} & \cdots \\ b_{21} & b_{22} & b_{23} & b_{24} & \cdots \\ 0 & b_{32} & 0 & b_{34} & \cdots \\ b_{41} & b_{42} & b_{43} & b_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $G(x,y) = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yC + B$. Since $G(x,y) \in \mathcal{K}$ for all x and y, we must have that $\det G(x,y)[rst|uvw] = 0$ for any increasing sequences (r,s,t) and (u,v,w), and hence, the coefficient of each term in the polynomial must be 0. Thus, we obtain

a) $b_{\alpha\beta} = 0$ for all $\alpha, \beta \geq 4$,
b) $b_{1\beta} = 0$ for all $\beta \geq 4$,
c) $b_{2\beta} = 0$ for all $\beta \geq 4$,
d) $b_{3\beta} = 0$ for all $\beta \geq 4$,
e) $b_{\alpha 1} = 0$ for all $\alpha \geq 4$,
f) $b_{\alpha 2} = 0$ for all $\alpha \geq 4$, and
g) $b_{\alpha 3} = 0$ for all $\alpha \geq 4$;

when we take $[rst|uvw] =$

a) $[23\alpha|23\beta]$,
b) $[123|12\beta]$,
c) $[123|13\beta]$,
d) $[123|23\beta]$,
e) $[12\alpha|123]$,
f) $[13\alpha|123]$, and
g) $[23\alpha|123]$, respectively.

Thus

$$B = \begin{bmatrix} 0 & b_{12} & 0 & \\ b_{21} & b_{22} & b_{23} & O \\ 0 & b_{32} & 0 & \\ & O & & O \end{bmatrix},$$

and hence, $\det G(x,y)[123|123] = x(b_{23}b_{32}) + y(b_{12}b_{23} + b_{22} + b_{21}b_{32} + b_{12}b_{21}) + xy(b_{22})$. Thus $b_{22} = 0$ and $b_{23}b_{32} = 0$. By the symmetry of $x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yC$ we may assume that $b_{23} = 0$. Since B must have rank 2, we have that $b_{21} = 1$ and from the coefficient of y, and that B must have rank 2, we get that $b_{32} = b_{12} = 1$. So,

$$B = \begin{bmatrix} 0 & 1 & 0 & \\ 1 & 0 & 0 & O \\ 0 & 1 & 0 & \\ & O & & O \end{bmatrix}.$$

Now, since $\dim \mathcal{K} \geq 4$ we must have $F \in \mathcal{K}$ of rank 2 such that

$$F = \begin{bmatrix} 0 & 0 & f_{13} & f_{14} & \cdots \\ 0 & f_{22} & f_{23} & f_{24} & \cdots \\ 0 & f_{32} & f_{33} & f_{34} & \cdots \\ f_{41} & f_{42} & f_{43} & f_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For $H(x,y,z) = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yC + zB + F$, by considering the minors on $[rst|uvw]$ as above, we obtain

a) $f_{\alpha\beta} = 0$ for all $\alpha, \beta \geq 4$,
b) $f_{1\beta} = f_{2\beta} = 0$ for all $\beta \geq 4$,
c) $f_{3\beta} = 0$ for all $\beta \geq 4$,
d) $f_{\alpha 1} = f_{\alpha 2} = 0$ for all $\alpha \geq 4$, and
e) $f_{\alpha 3} = 0$ for all $\alpha \geq 4$;

when we take $[rst|uvw] =$
a) $[23\alpha|23\beta]$,
b) $[123|12\beta]$,
c) $[123|13\beta]$,
d) $[12\alpha|123]$, and
e) $[23\alpha|123]$, respectively.

Now, $\det H(x,y,z)[123|123] = x(f_{33} + f_{22}f_{33} + f_{32}f_{23}) + y(f_{22} + f_{22}f_{13}) + z(f_{13} + f_{13}f_{32} + f_{33}) + xy(f_{22} + f_{13}) + xz(f_{23}) + yz(f_{32} + f_{23})$. Thus $f_{23} = f_{32} = 0$ and $f_{22} = f_{13} = f_{33}$. Since the rank of F must be 2, we have that

$$F = \begin{bmatrix} 0 & 0 & 1 & \\ 0 & 1 & 0 & O \\ 0 & 0 & 1 & \\ & O & & O \end{bmatrix}.$$

But then,

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} + C + F = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 0 & 0 & O \\ 1 & 0 & 0 & \\ & O & & O \end{bmatrix},$$

a rank 1 matrix, a contradiction.

Case 2. $\alpha = 0$. Here we must have $\beta = 1$ so that

$$B = \begin{bmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & O \\ 0 & 1 & 0 & \\ & O & & O \end{bmatrix}.$$

(Note that if $b_{ij} \neq 0$ for any (i,j) with $i,j \geq 3$ then $x \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} + B$ has rank 3 or more for some x.)

Now, since $\dim \mathcal{K} \geq 4$. we have some matrix $E \in \mathcal{K}$ such that

$$E = \begin{bmatrix} 0 & e_{12} & 0 & e_{14} & \cdots \\ e_{21} & 0 & e_{23} & e_{24} & \cdots \\ e_{31} & 0 & e_{33} & e_{34} & \cdots \\ e_{41} & e_{42} & e_{43} & e_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $C(x,y) = x\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yB + E$. Then, since $\det C(x,y)[r,s,t|u,v,w] = 0$ for all strictly increasing (r,s,t) and (u,v,w), each term in the polynomial must be zero, i.e., each coefficient in the polynomial expansion must be zero. Thus, we get

a) $e_{33} = e_{21} = 0$ and $e_{31} = e_{23}$,
b) $e_{\alpha 3} = 0$ for $\alpha \geq 4$,
c) $e_{\alpha 1} = 0$ for $\alpha \geq 4$,
d) $e_{2\beta} = 0$ for $\beta \geq 4$,
e) $e_{3\beta} = 0$ for $\beta \geq 4$, and
f) $e_{\alpha\beta} = 0$ for $\alpha, \beta \geq 4$;

when considering $\det C(x,y)[r,s,t|u,v,w] = 0$ for $[r,s,t|u,v,w] =$

a) $[1,2,3|1,2,3]$,
b) $[1,3,\alpha|1,2,3]$,
c) $[1,2,\alpha|1,2,3]$,
d) $[1,2,3|1,2,\beta]$,
e) $[1,2,3|2,3,\beta]$, and,
f) $[1,3,\alpha|1,2,\beta]$, respectively.

Thus,

$$E = \begin{bmatrix} 0 & e_{12} & 0 & e_{14} & \cdots \\ 0 & 0 & e_{23} & 0 & \cdots \\ e_{23} & 0 & 0 & 0 & \cdots \\ 0 & e_{42} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Subcase 1. $e_{23} = 1$. In this case, since the rank of E must be 2, and $e_{31} = e_{23}$, we have that $e_{1i} = 0$ and $e_{j2} = 0$ for all $i, j \geq 4$, and that $e_{12} = 0$ by considering that $\det E[123|13i] = 0$ and $\det E[13j|123] = 0$, and $\det E[123|123] = 0$ respectively. Thus

$$E = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} & O \\ O & O \end{bmatrix}.$$

Now, since $\dim \mathcal{K} > 4$, there is some nonzero $F \in \mathcal{K}$ such that

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & \cdots \\ f_{21} & 0 & 0 & f_{24} & \cdots \\ f_{31} & 0 & 0 & f_{34} & \cdots \\ f_{41} & f_{42} & f_{43} & f_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $G(x,y,z) = x\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yB + zE + F$. As above we get

a) $f_{\alpha\beta} = 0$ for all $\alpha, \beta \geq 4$,

b) $f_{13} = f_{31} = f_{11} = f_{12} = f_{21} = 0$,
c) $f_{3\beta} = f_{2\beta} = f_{1\beta} = 0$ for $\beta \geq 4$, and
d) $f_{\alpha 3} = f_{\alpha 2} = f_{\alpha 1} = 0$ for $\alpha \geq 4$;

when considering that $\det G(x,y,z)[r,s,t|u,v,w]$ must be zero for $[r,s,t|u,v,w] =$

a) $[23\alpha|23\beta]$,
b) $[123|123]$,
c) $[123|12\beta]$, and
d) $[12\alpha|123]$, respectively.

But then, $F = O$, a contradiction.

Subcase 2. $e_{23} = 0$. In this case, since the rank of E is 2, we have that $e_{1i} = 1$ for some $i \geq 4$ and $e_{j2} = 1$ for some $j \geq 4$. Here, there are invertible matrices U and V such that $UEV = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} & O \\ O & O \end{bmatrix}$, and $U \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} V = \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$ and $UBV = B$. Thus we may assume that $E = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} & O \\ O & O \end{bmatrix}$.

Now, let $G(x,y,z) = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yB + zE$. Then,

$$G(x,y,z) = \begin{bmatrix} x & 0 & y & z & 0 & \cdots \\ 0 & x & 0 & 0 & 0 & \cdots \\ 0 & y & 0 & 0 & 0 & \cdots \\ 0 & z & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $\dim \mathcal{K} > n$, there exists $H \in \mathcal{K}$, $H \neq 0$ such that $h_{1j} = 0$ for all j. Let $K(x,y,z) = G(x,y,z) + H$. We get

a) $h_{\alpha\beta} = 0$ for $\alpha \geq 3$ and $\beta \geq 4$,
b) $h_{2\beta} = 0$ for $\beta \geq 4$,
c) $h_{\alpha 1} = 0$, for $\alpha \geq 4$,
d) $h_{\alpha 3} = 0$, for $\alpha \geq 4$,
e) $h_{33} = h_{23} = 0$, and
f) $h_{31} = h_{21} = 0$;

when we consider that $\det H(x,y,z)[\gamma|\eta] = 0$ for $[\gamma|\eta] =$

a) $[12\alpha|23\beta]$,
b) $[123|12\beta]$,
c) $[12\alpha|123]$,
d) $[13\alpha|123]$,
e) $[123|234]$, and

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f) [123|234], respectively.

That is $H = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & h_{22} & 0 & \cdots \\ 0 & h_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$. But the rank of H is 1 since $H \neq 0$, a contradiction since $H \in \mathcal{K}$ and $\mathcal{K}$ is a rank 2 space.

We have thus obtained a contradiction in each case, and hence our supposition that $\dim \mathcal{K} > n$ is false, and the theorem is proved. $\square$

COROLLARY 3. *Over any field F, for $m, n \geq 3$, the dimension of any rank-2 subspace of $\mathcal{M}_{m,n}(F)$ is at most n, except when $m = n = 3$ and $F = \mathcal{Z}_2$, in which case, the dimension is at most 4.*

Proof. By the above theorem and comments, unless $m = n = 3$ and $F = \mathcal{Z}_2$, the dimension of any rank-2 space is at most n. If $\mathcal{K}$ is a rank-2 subspace of $\mathcal{M}_3(\mathcal{Z}_2)$, define $\mathcal{K}^+ = \{[A \quad \vec{0}] | A \in \mathcal{K}\}$. Then $\mathcal{K}^+$ is a rank-2 subspace of $\mathcal{M}_{3,4}(\mathcal{Z}_2)$ and hence the dimension is at most 4. Clearly $\mathcal{K}$ and $\mathcal{K}^+$ are isomorphic so that the dimension of $\mathcal{K}$ is at most 4 also. $\square$

Another example of a 4 dimensional rank-2 subspace of $\mathcal{M}_3(\mathcal{Z}_2)$ which is not equivalent to the one in Example 1 is given below.

EXAMPLE 4. Consider the space of matrices

$$\left\{ \begin{bmatrix} a & 0 & c \\ d & a+b & 0 \\ 0 & c+d & b \end{bmatrix} | a, b, c, d \in \mathcal{Z}_2 \right\}.$$

It is easily checked that this is a 4 dimensional rank-2 subspace of $\mathcal{M}_3(\mathcal{Z}_2)$.

CONJECTURE 5. *Over any field, F, the dimension of a rank-k subspace of $\mathcal{M}_{m,n}(F)$ is at most n unless $m = n = 3$, $k = 2$ and $F = \mathcal{Z}_2$.*

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