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GENERAL PRESERVERS OF QUASI-COMMUTATIVITY ON HERMITIAN MATRICES*

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Abstract. Let H_n be the set of all $n \times n$ hermitian matrices over $\mathbb{C}$, $n \geq 3$. It is said that $A, B \in H_n$ quasi-commute if there exists a nonzero $\xi \in \mathbb{C}$ such that $AB = \xi BA$. Bijective not necessarily linear maps on hermitian matrices which preserve quasi-commutativity in both directions are classified.

Key words. General preserver, Hermitian matrices, Quasi-Commutativity.

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1. Introduction.

Let H_n be the real vector space of $n \times n$ complex hermitian matrices with the usual involution $A^* = \bar{A}^t$. Note that A^* can be defined also for $A \in M_{m\times n}$, i.e., a rectangular complex $m \times n$ matrix. A hermitian matrix P is called a projection if $P = P^2$. Recall that all eigenvalues of hermitian matrices are real, so the set Diag of all diagonal hermitian matrices equals the set of all real diagonal matrices.

We say that $A, B \in H_n$ quasi-commute if there exists a nonzero $\xi \in \mathbb{C}$ such that $AB = \xi BA$. Note that there is a simple geometric interpretation of this relation: A and B quasi-commute if and only if AB and BA are linearly dependent and both products are either zero or else both are nonzero. We remark that, in case of hermitian matrices, we have a special phenomenon. Namely, two hermitian matrices A, B quasi-commute if and only if they commute ($AB = BA$) or anti-commute ($AB = -BA$); see for example [1, Theorem 1.1]. Given a subset $\Omega \subset H_n$, we define its quasi-commutant by

$$\Omega^{\#} = \{X \in H_n : \ \ X \text{ quasi-commutes with every } A \in \Omega\}$$

and we write $A^{\#} = \{A\}^{\#}$. It follows from [1, Theorem 1.1] that

$$A^{\#} = \{X \in H_n : XA = AX\} \cup \{X \in H_n : XA = -AX\}.$$

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We remark that quasi-commutativity has important applications in quantum mechanics. We refer the reader to [1] for more information. Furthermore, transformations on quantum structures which preserve some relation or operation are usually called symmetries in physics and have been studied by different authors [2]. From a mathematical point of view, maps preserving given algebraic property are called preserves and are extensively studied. Linear maps that preserve quasi-commutativity were already characterized by Molnár [6], Radjavi and Šemrl [7]. In our recent paper [3], we classified nonlinear bijective preservers of quasi-commutativity in both directions on the whole matrix algebra of $n \times n$ complex matrices. Since in quantum mechanics self-adjoint operators are important, we continued with our study and classified such maps also on H_n.

For a bijection $\Phi : H_n \to H_n$ it is easy to see that it preserves quasi-commutativity in both directions if and only if $\Phi(X^\#) = \Phi(X)^\#$ for every $X \in H_n$. Hence, we introduce an equivalence relation $A \simeq B$ whenever $A^\# = B^\#$ and denote the equivalence class of A by $[A]$. If $\Phi(X) \in [X]$, i.e., $\Phi(X)^\# = X^\#$ for every hermitian X, then it follows that Φ preserves quasi-commutativity in both directions. In particular, this shows that Φ can be characterized only up to equivalence classes.

The other simple examples of bijections which also preserve quasi-commutativity in both directions are the maps $X \mapsto X^t$ and $X \mapsto UXU^*$ for some unitary U, i.e., $UU^* = \mathrm{Id}$. We will prove in our theorem that every map which preserves quasi-commutativity in both directions is a composition of above three simple types.

Theorem 1.1. *Let $\Phi\colon H_n \to H_n$, $n \geq 3$, be a bijective map such that A quasi-commutes with B if and only if $\Phi(A)$ quasi-commutes with $\Phi(B)$. Then either $\Phi(X) \in [UXU^*]$ for every X or $\Phi(X) \in [UX^tU^*]$ for every X, where U is unitary.*

We remark that in the case of a matrix algebra M_n, bijections which preserve quasi-commutativity in both directions do not have a nice structure on all of M_n, as is the case with hermitian matrices; see [3].

Example 1.2. Suppose $A = \mathrm{Id}_k \oplus -\mathrm{Id}_{n-k}$ and $P = \mathrm{Id}_k \oplus 0_{n-k}$. Then it is easily seen that

$$A^\# = (H_k \oplus H_{n-k}) \cup \left\{ \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} : X \in M_{k\times(n-k)} \right\}$$

and

$$P^\# = H_k \oplus H_{n-k}.$$

Observe that $A^\#$ is not a linear subspace. Observe also that $P^\# \subseteq A^\#$.

A matrix A is minimal if $A^\# \supseteq B^\#$ implies $A^\# = B^\#$. Similarly a non-scalar matrix A is maximal if $A^\# \subseteq B^\#$ implies $A^\# = B^\#$ for every non-scalar matrix B.

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Example 1.3. If $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $|d_i| \neq |d_j|$, $i \neq j$, then $D^\#$ is equal to the set of all hermitian diagonal matrices.

Namely, by [1], $X = (x_{ij})_{ij} \in D^\#$ precisely when $XD = DX$ or when $XD = -DX$. Comparing the absolute values of (ij) entry on the both sides gives $|x_{ij} d_j| = |d_i x_{ij}|$. When $i \neq j$, the only solution is $|x_{ij}| = 0$, hence X is diagonal. Conversely, any diagonal X clearly quasi-commutes with D.

2. Proofs.

2.1. Preliminary lemmas.

Lemma 2.1. *Let D be a diagonal matrix. If $D^\# \subseteq A^\#$ then A is also diagonal.*

Proof. As usual, let E_{ij} be the matrix with 1 at the (i, j)-th position and zeros elsewhere. Assume $A = (a_{ij})$ is not diagonal. Then there exists $a_{i_0 j_0} \neq 0$ with $i_0 \neq j_0$. Since (i_0, j_0)-th entry of $E_{i_0 i_0} A$ is $a_{i_0 j_0}$, but (i_0, j_0)-th entry of $AE_{i_0 i_0}$ is zero, we see that $E_{i_0 i_0} \notin A^\#$. However, D is diagonal and therefore $E_{i_0 i_0} \in D^\# \setminus A^\#$, a contradiction. ∎

Lemma 2.2. *Let $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $B = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ be diagonal matrices. Then $A^\# = B^\#$ if and only if all of the following three conditions are satisfied for every indices i, j, k, $i \neq j$:*
(i) $\lambda_i = \lambda_j$ if and only if $\mu_i = \mu_j$,
(ii) $\lambda_i = -\lambda_j$ if and only if $\mu_i = -\mu_j$,
(iii) $\lambda_i = -\lambda_j \neq 0$ and $\lambda_k = 0$ if and only if $\mu_i = -\mu_j \neq 0$ and $\mu_k = 0$.

Proof. Assume first that $A^\# = B^\#$. (i) If $\lambda_i = \lambda_j$, then $E_{ij} + E_{ji} + E_{ii} \in A^\# = B^\#$, hence $\mu_i = \mu_j$. Similarly $\mu_i = \mu_j$ implies $\lambda_i = \lambda_j$. (ii) If $\lambda_i = -\lambda_j$ and $\lambda_i \neq 0$, then $E_{ij} + E_{ji} + E_{ii} \notin A^\# = B^\#$, hence $\mu_i \neq \mu_j$, however $E_{ij} + E_{ji} \in A^\# = B^\#$, hence $\mu_i = -\mu_j$. The case $\lambda_i = -\lambda_j = 0$ reduces to (i). Similarly $\mu_i = -\mu_j$ implies $\lambda_i = -\lambda_j$. (iii) If $\lambda_i = -\lambda_j \neq 0$ and $\lambda_k = 0$ then by (i)–(ii), $\mu_i = -\mu_j \neq 0$ and since $E_{ij} + E_{ji} + E_{kk} \in A^\# = B^\#$, we obtain $\mu_k = 0$. In the same way $\mu_i = -\mu_j \neq 0$, $\mu_k = 0$ implies $\lambda_i = -\lambda_j \neq 0$, $\lambda_k = 0$.

Second, assume (i)–(iii) hold. We distinguish two options. To begin with, suppose there exist indices $i_0 \neq j_0$ such that $\lambda_{i_0} = -\lambda_{j_0} \neq 0$. Let $X = \left(x_{ij}\right)_{ij} \in A^\#$. Then either $AX = XA$ or $AX = -XA$. Comparing the (i, j)-th entries we obtain $\lambda_i x_{ij} = \lambda_j x_{ij}$ for every i, j or $\lambda_i x_{ij} = -\lambda_j x_{ij}$ for every i, j. In the former case we easily obtain by (i) that $\mu_i x_{ij} = \mu_j x_{ij}$ for every i, j, hence $X \in B^\#$. In the latter case, if $x_{ij} = 0$ then clearly $\mu_i x_{ij} = -\mu_j x_{ij}$, if $x_{ij} \neq 0$ and $i \neq j$, then also $\mu_i x_{ij} = -\mu_j x_{ij}$ by (ii), and if $x_{kk} \neq 0$ then $\lambda_k = 0$ and by (iii) also $\mu_k = 0$, hence $\mu_k x_{kk} = -\mu_k x_{kk}$. Therefore $X \in B^\#$ also in this case, so $A^\# \subseteq B^\#$. Lastly, suppose for no indices $i \neq j$ we have $\lambda_i = -\lambda_j \neq 0$. Then $X \in A^\#$ implies $AX = XA$ or $AX = -XA$. However,

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as $\lambda_i \neq -\lambda_j$ when $\lambda_i \neq 0$, the latter is equivalent to $AX = XA$. So, from (i) we easily deduce $BX = XB$. Hence, $A^\# \subseteq B^\#$. The inclusion $B^\# \subseteq A^\#$ follows in the same way. $\square$

COROLLARY 2.3. *A matrix is equivalent to E_{ii} if and only if it equals $\lambda E_{ii} + \mu(\mathrm{Id} - E_{ii})$ for some real numbers λ, μ with $|\lambda| \neq |\mu|$.*

Proof. We prove only the nontrivial implication. Pick any matrix A with $A^\# = E_{ii}^\#$. Then, by Lemma 2.1, A is also diagonal. The rest follows from Lemma 2.2. $\square$

LEMMA 2.4. *A matrix A is minimal if and only if there exists a unitary matrix U and a real diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $|d_i| \neq |d_j|$, $i \neq j$, such that $A = UDU^*$.*

Proof. Suppose A is minimal. Hermitian matrices are unitarily diagonalizable, therefore there exists a unitary matrix U and a real diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ such that $A = UDU^*$. Assume erroneously that $|d_i| = |d_j|$ for some $i \neq j$. Consider a matrix $A_0 = U\,\mathrm{diag}(1, \ldots, n)U^*$. By Example 1.3, $A_0^\# = U\,\mathrm{Diag}\,U^*$, and clearly $U\,\mathrm{Diag}\,U^* \subseteq A^\#$. However, $U(E_{ij} + E_{ji})U^* \in A^\# \setminus A_0^\#$, which contradicts minimality of A.

To prove the opposite direction, assume $A = U\,\mathrm{diag}(d_1, \ldots, d_n)U^*$ for some unitary U and real scalars d_i with $|d_i| \neq |d_j|$, $i \neq j$. Without loss of generality we may assume $U = \mathrm{Id}$. Suppose $B^\# \subseteq A^\#$. Then, $B \in B^\# \subseteq A^\#$ and Example 1.3 gives that B is diagonal. Moreover, $A^\# = \mathrm{Diag}$, and clearly, each diagonal matrix quasi-commutes with B. Hence, $A^\# \subseteq B^\#$, so A is minimal. $\square$

Note that all minimal diagonal matrices are equivalent.

LEMMA 2.5. *If B is an immediate successor of a minimal diagonal matrix D, then $B = \mathrm{diag}(b_1, \ldots, b_n)$ is invertible and there exists i_0, j_0 such that $b_{i_0} = -b_{j_0}$ while $|b_i| \neq |b_j|$ for $i \neq j$ and $i \in \{1, \ldots, n\} \setminus \{i_0, j_0\}$.*

Proof. Since all minimal diagonal matrices are equivalent we may assume that $D = \mathrm{diag}(1, \ldots, n)$. By Lemma 2.1, $D^\# \subseteq B^\#$ implies $B = \mathrm{diag}(b_1, \ldots, b_n)$ is also diagonal. Since B is not minimal it follows by Lemma 2.4 that $|b_{i_0}| = |b_{j_0}|$ for some $i_0 \neq j_0$. Assume erroneously that B is not as stated in the lemma. Then B has at least one of the following properties: (i) $|b_{i_0}| = |b_{j_0}| = |b_{k_0}|$ for some $k_0 \notin \{i_0, j_0\}$, or (ii) $|b_{k_0}| = |b_{l_0}|$, $k_0 \neq l_0$, $k_0, l_0 \notin \{i_0, j_0\}$, or (iii) $b_{k_0} = 0$, $k_0 \notin \{i_0, j_0\}$, or (iv) $b_{i_0} = b_{j_0}$.

Let us define a diagonal matrix $C = D - (i_0 + j_0)E_{i_0 i_0}$. Observe that $C^\# = \mathrm{Diag} \cup \{\lambda E_{i_0 j_0} + \overline{\lambda} E_{j_0 i_0} : \lambda \in \mathbb{C}\}$, hence $D^\# \subsetneq C^\# \subseteq B^\#$. If B has the property (i) then $E_{i_0 k_0} + E_{k_0 i_0} \in B^\# \setminus C^\#$, if (ii) then $E_{k_0 l_0} + E_{l_0 k_0} \in B^\# \setminus C^\#$, if (iii) then $E_{i_0 j_0} + E_{j_0 i_0} + E_{k_0 k_0} \in B^\# \setminus C^\#$, and if (iv) then $E_{i_0 j_0} + E_{j_0 i_0} + E_{i_0 i_0} \in B^\# \setminus C^\#$.

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In any of these cases B is not an immediate successor. $\square$

COROLLARY 2.6. *If A is an immediate successor of a minimal diagonal matrix, then $A^{\#} = \mathrm{Diag} \cup \{\lambda E_{ij} + \overline{\lambda} E_{ji} : \lambda \in \mathbb{C}\}$ for some indices $i \neq j$.*

LEMMA 2.7. *A non-scalar diagonal hermitian matrix is maximal if and only if absolute values of all its eigenvalues are equal and nonzero.*

Proof. Let non-scalar $M = \mathrm{diag}(d_1, \dots, d_n)$, where $d_i \in \{-\alpha, \alpha\}$ for every i and some nonzero $\alpha \in \mathbb{R}$. Assume $B^{\#} \supseteq M^{\#}$. By Lemma 2.1, $B = \mathrm{diag}(b_1, \dots, b_n)$ is diagonal. If $|b_i| \neq |b_j|$ then $E_{ij} + E_{ji} \notin B^{\#}$, but $E_{ij} + E_{ji} \in M^{\#}$, a contradiction. Hence $b_i \in \{-b, b\}$ for some $b \in \mathbb{R}$ and every i. Suppose $d_i = d_j$ for $i \neq j$. Then $E_{ii} + E_{ij} + E_{ji} \in M^{\#} \subseteq B^{\#}$, and consequently $b_i = b_j$. Therefore if $d_i \neq d_j$ and $b_i = b_j$ for some $i \neq j$ then B is a scalar matrix. Otherwise, if $d_i \neq d_j$ implies $b_i \neq b_j$, then $B = \pm(b/\alpha)M$ and $B^{\#} = M^{\#}$.

Conversely, let $D = \mathrm{diag}(d_1, \dots, d_n) \in H_n$. We need to show that, unless $|d_i| = |d_j|$ for every indices i, j, a matrix D is not maximal. Assume $|d_{i_0}| \neq |d_{j_0}|$ for some indices i_0, j_0. We define a non-scalar matrix $M = \mathrm{diag}(m_1, \dots, m_n)$, where $m_{i_0} = 1$ for every i with $|d_i| = |d_{i_0}|$ and $m_i = -1$ otherwise. It is easy to see that every matrix which quasi-commutes with D actually commutes with M. So $D^{\#} \subseteq M^{\#}$. Since $E_{i_0 j_0} + E_{j_0 i_0} \in M^{\#}$, but $E_{i_0 j_0} + E_{j_0 i_0} \notin D^{\#}$, we see that D cannot be maximal. $\square$

In the next lemma the projections are classified up to equivalence in terms of quasi-commutativity.

LEMMA 2.8. *Suppose $A \in \mathrm{Diag}$ is non-maximal and non-scalar. Then A is equivalent to a projection if and only if A is an immediate predecessor of some maximal matrix, and, up to equivalence, there exists precisely one maximal matrix which A connects to.*

Proof. Suppose A is not equivalent to a projection. We will prove that when A is invertible, it connects to two nonequivalent maximal matrices, and when A is singular, there exists at least one maximal matrix which A connects to, but which is not its immediate predecessor.

Now, since A is not a scalar matrix, it must have at least two eigenvalues. Actually, it must have more than two eigenvalues because $\alpha \,\mathrm{Id}_r \oplus (-\alpha)\,\mathrm{Id}_{n-r}$, $\alpha \neq 0$ is maximal by Lemma 2.7, and $\alpha \,\mathrm{Id}_r \oplus \beta \,\mathrm{Id}_{n-r}$, $\alpha \neq \pm\beta$, is equivalent to a projection $\mathrm{Id}_r \oplus 0_{n-r}$. Without loss of generality, we may assume that $A = \alpha_1 \mathrm{Id}_{n_1} \oplus \dots \oplus \alpha_k \mathrm{Id}_{n_k}$, $\alpha_i \neq \alpha_j$ for $i \neq j$, and if there are two nonzero eigenvalues with the same absolute value, that $\alpha_1 = -\alpha_2$. When A is singular, we may also assume that $\alpha_k = 0$.

We now construct two diagonal matrices B and C, where $B = \beta_1 \mathrm{Id}_{n_1} \oplus \dots \oplus \beta_k \mathrm{Id}_{n_k}$, $C = (-\beta_1)\,\mathrm{Id}_{n_1} \oplus (-\beta_2)\,\mathrm{Id}_{n_2} \oplus \beta_3 \mathrm{Id}_{n_3} \oplus \dots \oplus \beta_k \mathrm{Id}_{n_k}$, and $\beta_i \in \{-1, 0, 1\}$

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are recursively defined as follows. Start with $\beta_1 = 1$. Assume $\beta_1, \ldots, \beta_i$, $i < k$ are already defined. If $i+1 = k$ and $\alpha_k = 0$ let $\beta_k = 0$. If $\alpha_{i+1} = -\alpha_j$ for some $j \le i$, let $\beta_{i+1} = -\beta_j$. Otherwise, $\beta_{i+1} = -\beta_i$. Observe that $\beta_1 = -\beta_2 \neq 0$, hence B and C are non-scalar.

We now consider two cases separately. Firstly, if A is invertible, then B and C are maximal and moreover $\beta_1 = 1$, $\beta_2 = -1$, $\beta_3 = 1$. It follows that $E_{11} + E_{1(n_1+n_2+1)} + E_{(n_1+n_2+1)1} \in B^{\#} \setminus C^{\#}$, so B and C are nonequivalent, maximal matrices and it is easy to see that $A^{\#} \subseteq B^{\#}$ and $A^{\#} \subseteq C^{\#}$.

Secondly, if A is singular, then $\beta_k = 0$. Again, it is easy to see that $A^{\#} \subseteq B^{\#}$, and also $B^{\#} \subsetneq P^{\#}$, where $P = \mathrm{Id}_{n_1} \oplus \ldots \oplus \mathrm{Id}_{n_{k-1}} \oplus 0_{n_k}$ is a projection, which connects to a maximal matrix $M = \mathrm{Id}_{n_1} \oplus \ldots \oplus \mathrm{Id}_{n_{k-1}} \oplus (-\mathrm{Id}_{n_k})$. Therefore, A is not an immediate predecessor of at least one maximal matrix.

Let us prove the other implication. We can assume that A is a projection and $A = \mathrm{Id}_r \oplus 0_{n-r}$. It is easy to see that A connects to $M_0 = (\mathrm{Id}_r \oplus -\mathrm{Id}_{n-r})$, which is maximal. It remains to show that, up to equivalence, A connects to no other maximal matrix and that A is an immediate predecessor of M_0. Indeed, let M be any maximal matrix such that $A^{\#} \subseteq M^{\#}$. By Lemma 2.1, $M = \mathrm{diag}(m_1, \ldots, m_n)$ is diagonal. For $i \le r$, note that $E_{11} + E_{1i} + E_{i1} \in A^{\#} \subseteq M^{\#}$, which forces $m_1 = m_i$, for every $i \le r$. Likewise we see that $m_j = m_n$ for every $j > r$. Moreover, by Lemma 2.7, $|m_i| = |m_j| \neq 0$ for every i, j. Therefore, $M = m(\mathrm{Id}_r \oplus -\mathrm{Id}_{n-r})$, and hence $M^{\#} = M_0^{\#}$. It remains to show that A is an immediate predecessor of M_0. To this end, assume $A^{\#} \subseteq B^{\#} \subseteq M_0^{\#}$ for some B. As above we see that $B = \alpha \mathrm{Id}_r \oplus \beta \mathrm{Id}_{n-r}$. Clearly, $\alpha \neq \beta$. Now, if $\alpha = -\beta$ then B is equivalent to M_0. In all other possibilities, B is equivalent to A. $\square$

2.2. Proof of the Theorem. We divide the proof in several steps.

Step 1. Clearly, the bijection Φ preserves the set of minimal matrices. So, if D is minimal diagonal then by Lemma 2.4, $\Phi(D) = UD_1U^*$ for some unitary U and some minimal diagonal D_1, recall that $D^{\#} = D_1^{\#}$. We may assume $U = \mathrm{Id}$, otherwise we replace Φ with a bijection $X \mapsto U^*\Phi(X)U$.

Step 2. Let us continue by proving that several sets are preserved by Φ. Since $D = \mathrm{diag}(d_1, \ldots, d_n)$ is minimal, Lemma 2.4 implies $|d_i| \neq |d_j|$ for $i \neq j$. Therefore $D^{\#} = \mathrm{Diag}$ by Example 1.3 and since $\Phi(D^{\#}) = \Phi(D)^{\#} = D_1^{\#} = D^{\#}$, the bijection Φ satisfies $\Phi(\mathrm{Diag}) = \mathrm{Diag}$.

It is easy to see, by Lemma 2.1 and Lemma 2.2, that A is a scalar matrix precisely when $A^{\#} = H_n$. So, $\Phi(\mathbb{R}\,\mathrm{Id}_n) = \mathbb{R}\,\mathrm{Id}_n$.

Since Φ preserves maximal diagonal matrices as well as non-maximal ones, it also

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preserves the set of equivalence classes of diagonal projections by Lemma 2.8.

The map Φ preserves the set of all immediate successors of a minimal diagonal matrix. Hence, it also permutes their quasi-commutants. By Corollary 2.6 and since $\Phi(\mathrm{Diag}) = \mathrm{Diag}$, the bijective map Φ maps the set $\mathfrak{D}_{ij} = \{\lambda E_{ij} + \overline{\lambda} E_{ji} : \lambda \in \mathbb{C}\}$ onto $\mathfrak{D}_{uv}$ for some indices u, v.

Let a matrix P be equivalent to a diagonal projection of rank-k. It is easy to see that $\mathfrak{D}_{ij} \subseteq P^{\#}$ for exactly $\frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2}$ different sets $\mathfrak{D}_{ij}$. Therefore P is equivalent to a diagonal projection of rank-one if and only if $\mathfrak{D}_{ij} \subseteq P^{\#}$ for exactly $\frac{1}{2}(n-1)(n-2)$ different sets $\mathfrak{D}_{ij}$ (note that a projection E_{ii} of rank-one is equivalent to a projection $\mathrm{Id}_n - E_{ii}$ of rank $(n-1)$). Since Φ bijectively permutes the sets $\mathfrak{D}_{ij}$ among themselves, this shows that P is equivalent to a diagonal projection of rank-one if and only if $\Phi(P)$ is equivalent to a diagonal projection of rank-one.

Let P be an arbitrary matrix equivalent to a rank-one projection. Since Φ is determined only up to equivalence we can assume that P is already a projection of rank-one, that is $P = VE_{11}V^*$ for some unitary V. We temporarily replace Φ by $\Psi : X \mapsto U_P \Phi(VXV^*) U_P^*$, where a unitary U_P is such that Ψ fixes the equivalence class of a minimal diagonal matrix. Applying the above arguments to Ψ we see that $\Psi(E_{11}) = U_P \Phi(P) U_P^*$ is equivalent to a diagonal rank-one projection. Therefore Φ preserves the set of rank one projections up to equivalence.

Step 3. By Corollary 2.3, each equivalence class contains at most one projection of rank-one. Hence Φ induces the well defined bijection on the set of rank-one projections, which we denote by ϕ. We claim that ϕ preserves orthogonality among rank-one projections. Namely, if P, Q are orthogonal rank-one projections, then they are simultaneously unitarily diagonalizable, that is $P = V_2 E_{11} V_2^*$ and $Q = V_2 E_{22} V_2^*$ for some unitary V_2. Consider temporarily a bijection $\Psi : X \mapsto U_2 \Phi(V_2 X V_2^*) U_2^*$, where unitary U_2 is such that Ψ fixes the equivalence class of a minimal diagonal matrix. By the above, $\Psi(E_{11}) \in [E_{ii}]$ and $\Psi(E_{22}) \in [E_{jj}]$ for some i, j, where $i \neq j$ because E_{11} and E_{22} are not equivalent. By Corollary 2.3 there exists precisely one rank-one idempotent inside $[U_2^* E_{ii} U_2]$ and precisely one inside $[U_2^* E_{jj} U_2]$, so $\phi(P) = U_2^* E_{ii} U_2$ and $\phi(Q) = U_2^* E_{jj} U_2$ are indeed orthogonal.

Step 4. It can be deduced from Wigner's unitary-antiunitary Theorem, see [4, Theorem 4.1], that there exists a unitary matrix U_3 such that $\phi(P) = U_3 P U_3^*$ for every rank-one projection P or $\phi(P) = U_3 \bar{P} U_3^* = U_3 P^t U_3^*$ for every rank-one projection P, where $\bar{P}$ denotes complex conjugation applied entry-wise (see also [5] for more details). So, $\Phi(E_{ii}) \in [U_3 E_{ii} U_3^*]$ for every i. Let us show that $U_3 \,\mathrm{Diag}\, U_3^* \subseteq \mathrm{Diag}$. Since $\Phi(\mathrm{Diag}) = \mathrm{Diag}$, it follows that $\Phi(E_{ii})$ is diagonal, hence the equivalence class $[U_3 E_{ii} U_3^*]$ contains at least one diagonal matrix. Since $[U_3 E_{ii} U_3^*] = U_3 [E_{ii}] U_3^* = \{\lambda U_3 E_{ii} U_3^* + \mu(\mathrm{Id} - U_3 E_{ii} U_3^*) : \lambda, \mu \in \mathbb{R},\ |\lambda| \neq |\mu|\}$ by Corollary 2.3, it follows that

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$[U_3 E_{ii} U_3^*] \subseteq \mathrm{Diag}$, so $U_3 E_{ii} U_3^* \in \mathrm{Diag}$ for every i. Every diagonal matrix is a linear combination of matrices E_{ii} and therefore $U_3 \,\mathrm{Diag}\, U_3^* \subseteq \mathrm{Diag}$.

Step 5. If necessary, we replace Φ by the map $X \mapsto U_3^* \Phi(X) U_3$ or by the map $X \mapsto U_3^* \Phi(X^t) U_3$, which we again denote by Φ, so that $\Phi(P) \in [P]$ for every rank-one projection P. Observe that Φ still satisfies the property $\Phi(\mathrm{Diag}) = \mathrm{Diag}$ and therefore also all its properties proven up to now remain valid.

Step 6. Let $\mathbf{x} \in \mathbb{C}^n$ be a column vector, i.e., a matrix of dimension $n \times 1$, with Euclidean norm 1. It easily follows that a rank-one projection $\mathbf{x}\mathbf{x}^*$ quasi-commutes with A precisely when $\mathbf{x}$ is an eigenvector of A. Since quasi-commutativity is preserved in both directions, and $\Phi(\mathbf{x}\mathbf{x}^*)^\# = (\mathbf{x}\mathbf{x}^*)^\#$, we see that A and $\Phi(A)$ have exactly the same eigenvectors. In particular, if $A = U_A(\lambda_1 \mathrm{Id}_{n_1} \oplus \ldots \oplus \lambda_k \mathrm{Id}_{n_k})U_A^*$ with λ_i pairwise distinct and U_A unitary, then also $\Phi(A) = U_A(\mu_1 \mathrm{Id}_{n_1} \oplus \ldots \oplus \mu_k \mathrm{Id}_{n_k})U_A^*$, with μ_i pairwise distinct. In particular, $\Phi(U_A \,\mathrm{Diag}\, U_A^*) = U_A \,\mathrm{Diag}\, U_A^*$.

Step 7. Next let us show that $\Phi(V \mathfrak{D}_{ij} V^*) = V \mathfrak{D}_{ij} V^*$, where V is an arbitrary unitary matrix. Recall $\mathfrak{D}_{ij} = \{\lambda E_{ij} + \overline{\lambda} E_{ji} : \lambda \in \mathbb{C}\}$, and introduce temporarily $\Psi : X \mapsto V^* \Phi(V X V^*) V$. By the above, $\Psi(\mathrm{Diag}) = \mathrm{Diag}$, and Ψ fixes all rank-one projections up to equivalence. Hence, as we have already proved, $\Psi(\mathfrak{D}_{ij}) = \mathfrak{D}_{uv}$ for some (u, v). Observe that a rank-one projection $E_{kk} \in \mathfrak{D}_{ij}^\#$ if and only if $k \notin \{i, j\}$. So, $\mathfrak{D}_{uv} = \Psi(\mathfrak{D}_{ij}) \subset \Psi(E_{kk})^\# = E_{kk}^\#$ whenever $k \notin \{i, j\}$, which is possible only when $\mathfrak{D}_{uv} = \mathfrak{D}_{ij}$. Clearly then $\Phi(V \mathfrak{D}_{ij} V^*) = V \Psi(\mathfrak{D}_{ij}) V^* = V \mathfrak{D}_{ij} V^*$ as claimed.

Step 8. Consider a general hermitian matrix $A = U_A(\lambda_1 \mathrm{Id}_{n_1} \oplus \ldots \oplus \lambda_k \mathrm{Id}_{n_k})U_A^*$, where U_A is unitary and $\lambda_i \in \mathbb{R}$ are pairwise distinct. We already know that $\Phi(A) = U_A(\mu_1 \mathrm{Id}_{n_1} \oplus \ldots \oplus \mu_k \mathrm{Id}_{n_k})U_A^*$ for pairwise distinct $\mu_i \in \mathbb{R}$. Moreover, $\lambda_i = -\lambda_j \neq 0$ if and only if $\mu_i = -\mu_j \neq 0$, since $U_A \mathfrak{D}_{(n_1+\ldots+n_{i-1}+1)\,(n_1+\ldots+n_{j-1}+1)} U_A^* \subseteq A^\#$ precisely when $\lambda_i = -\lambda_j$. Consequently, if $|\lambda_i|$ are pairwise distinct then $\Phi(A) \in [A]$. In particular, Φ fixes the equivalence class of a matrix $1 \,\mathrm{Id}_{n_1} \oplus 2 \,\mathrm{Id}_{n_2} \oplus \ldots \oplus k \,\mathrm{Id}_{n_k}$ for any choice of a positive integer k and any choice of positive integers $n_1, \ldots, n_k$ with $n_1 + \ldots + n_k = n$. Hence Φ also bijectively maps its quasi-commutant, which equals $H_{n_1} \oplus \ldots \oplus H_{n_k}$, onto itself. Since Φ is injective no other matrix can be mapped into this set.

Step 9. We next show that Φ fixes equivalence classes of each matrix of the form $F = V(\mathrm{diag}(\lambda, -\lambda, \mu) \oplus 0_{n-3})V^*$, where $\lambda, \mu \neq 0$, $|\lambda| \neq |\mu|$, and V unitary. Assume with no loss of generality that $V = \mathrm{Id}$. We already know that $\Phi(F) = \mathrm{diag}(\nu, -\nu, \eta) \oplus \zeta \,\mathrm{Id}_{n-3}$, where $|\nu|, |\eta|, |\zeta|$ are pairwise distinct, and $\nu \neq 0$. When $n \geq 4$ we have to consider three options either (i) $\eta, \zeta \neq 0$, or (ii) $\eta = 0$, $\zeta \neq 0$, or (iii) $\eta \neq 0$, $\zeta = 0$. If $\Phi(F)$ has the property (i) then $\Phi(F)^\# = (H_1 \oplus H_1 \oplus H_1 \oplus H_{n-3}) \cup \mathfrak{D}_{12}$. By Step 7 and Step 8, the set $\Omega = \Phi(F)^\#$ is bijectively mapped onto itself by Φ. Note that $(E_{12} + E_{21} + E_{nn}) \in F^\# \setminus \Omega$. Since Φ is injective, $\Phi(E_{12} + E_{21} + E_{nn}) \notin \Phi(\Omega) =$

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$\Omega = \Phi(F)^{\#} = \Phi(F^{\#})$, a contradiction.

Suppose $\Phi(F)$ has the property (ii). Then there exists $X = (E_{12} + E_{21} + E_{nn}) \in F^{\#} \setminus (H_1 \oplus H_1 \oplus H_1 \oplus H_{n-3})$ with the property that $X^{\#}$ contains the set $\{\lambda(E_{1n} + E_{2n}) + \bar{\lambda}(E_{n1} + E_{n2}) : \lambda \in \mathbb{C}\} = V_3 \mathfrak{D}_{12} V_3^*$ for unitary $V_3 = \frac{1}{\sqrt{2}}(E_{11} + E_{13} + E_{21} - E_{23}) + E_{n2} + \sum_{i=3}^{n-1} E_{i\,(i+1)}$. Observe that

$$\begin{aligned}\Phi(X) &\in \Phi(F)^{\#} \setminus \Phi(H_1 \oplus H_1 \oplus H_1 \oplus H_{n-3})\\ &= (\operatorname{diag}(\nu, -\nu, 0) \oplus \zeta \operatorname{Id}_{n-3})^{\#} \setminus (H_1 \oplus H_1 \oplus H_1 \oplus H_{n-3})\\ &= \{\lambda E_{12} + \overline{\lambda} E_{21} + \beta E_{33} : \lambda \in \mathbb{C} \setminus \{0\},\ \beta \in \mathbb{R}\} = \Xi.\end{aligned}$$

It is now easy to see that, for an arbitrary $Y \in \Xi$, the set $Y^{\#}$ does not contain the set $\Phi(V_3 \mathfrak{D}_{12} V_3^*) = V_3 \mathfrak{D}_{12} V_3^*$, a contradiction.

Hence $\Phi(F)$ has the property (iii) for $n \geq 4$. When $n = 3$ we have only two options (i) $\eta \neq 0$ or (ii) $\eta = 0$. Note that $F^{\#} = (H_1 \oplus H_1 \oplus H_1) \cup \mathfrak{D}_{12}$ is a union of two sets invariant for Φ. Hence, $\Phi(F^{\#}) = F^{\#}$. Since $\operatorname{diag}(\nu, -\nu, 0)^{\#} \supsetneq F^{\#}$, the second case is contradictory, as anticipated.

Step 10. Finally, we can show that Φ fixes equivalence class of an arbitrary hermitian matrix A. Decompose A as in Step 8. We already know that $\Phi(A) = U_A(\mu_1 \operatorname{Id}_{n_1} \oplus \ldots \oplus \mu_k \operatorname{Id}_{n_k}) U_A^*$, with μ_i pairwise distinct and $\mu_i = -\mu_j$ precisely when $\lambda_i = -\lambda_j$, $i \neq j$. By Lemma 2.2 it remains to show that in the case when $\lambda_{i_0} = -\lambda_{j_0} \neq 0$ we have $\lambda_k = 0$ if and only if $\mu_k = 0$. So assume $\lambda_k = 0$. Note that $X = E_{i_0 j_0} + E_{j_0 i_0} + 2E_{kk} \in A^{\#}$. Since X is unitarily equivalent to $\operatorname{diag}(1, -1, 2, 0, \ldots, 0)$, Step 9 implies that its equivalence class is fixed by Φ, hence $[X] = [\Phi(X)] \subseteq \Phi(A)^{\#}$. So also $\mu_k = 0$. In the same way we obtain that $\lambda_k \neq 0$ implies $\mu_k \neq 0$. ∎

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