Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

ELA

NONNEGATIVITY OF SCHUR COMPLEMENTS OF NONNEGATIVE IDEMPOTENT MATRICES*

SHMUEL FRIEDLAND† AND ELENA VIRNIK‡

Abstract. Let A be a nonnegative idempotent matrix. It is shown that the Schur complement of a submatrix, using the Moore-Penrose inverse, is a nonnegative idempotent matrix if the submatrix has a positive diagonal. Similar results for the Schur complement of any submatrix of A are no longer true in general.

Key words. Nonnegative idempotent matrices, Schur complement, Moore-Penrose inverse, generalized inverse.

AMS subject classifications. 15A09, 15A15, 15A48.

1. Introduction. Let $\langle n\rangle := \{1,\ldots,n\}$ and assume that $\alpha \subset \langle n\rangle$, $\alpha^c := \langle n\rangle\backslash\alpha$, $\beta \subset \langle n\rangle$ are three nonempty sets. For $A \in \mathbb{R}^{n\times n}$, denote by $A[\alpha,\beta]$ the submatrix of A composed of the rows and columns indexed by the sets α and β, respectively. Assume that $A[\alpha,\alpha]$ is invertible. Then, the α Schur complement of A, which is equal to the Schur complement of $A[\alpha,\alpha]$, is given by

$$A(\alpha) := A[\alpha^c,\alpha^c] - A[\alpha^c,\alpha]A[\alpha,\alpha]^{-1}A[\alpha,\alpha^c]. \tag{1.1}$$

If $A[\alpha,\alpha]$ is not invertible we define

$$A_{\mathrm{ginv}}(\alpha) := A[\alpha^c,\alpha^c] - A[\alpha^c,\alpha]A[\alpha,\alpha]^{\mathrm{ginv}}A[\alpha,\alpha^c], \tag{1.2}$$

for some semi-inverse $A[\alpha,\alpha]^{\mathrm{ginv}}$ [1]. The α Moore-Penrose Schur complement of A is defined as

$$A_{\dagger}(\alpha) := A[\alpha^c,\alpha^c] - A[\alpha^c,\alpha]A[\alpha,\alpha]^{\dagger}A[\alpha,\alpha^c],$$

where $A[\alpha,\alpha]^{\dagger}$ is the Moore-Penrose inverse of $A[\alpha,\alpha]$ [3, 5, 6].

*Received by the editors April 22, 2008. Accepted for publication August 12, 2008. Handling Editor: Miroslav Fiedler. This research is partly supported by the DFG Research Center MATHEON in Berlin.

†Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 322 Science and Engineering Offices (M/C 249), 851 S. Morgan Street Chicago, IL 60607-7045 (friedlan@uic.edu), and Visiting Professor, Berlin Mathematical School, Institut für Mathematik, Technische Universität Berlin, Strasse des 17. Juni 136, D-10623 Berlin, FRG.

‡Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG (virnik@math.tu-berlin.de).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

ELA

Assume that A is a nonnegative idempotent matrix, i.e., $A^2 = A \in \mathbb{R}_+^{n\times n}$. In this note we show that if $A[\alpha, \alpha]$ has a positive diagonal then $A_\dagger(\alpha)$ is a nonnegative idempotent matrix. We give an example of A, where $A[\alpha, \alpha]$ has a nonpositive diagonal, and $A_\dagger(\alpha)$ has positive and negative entries. We show that for certain $A[\alpha, \alpha]$ with a nonpositive diagonal, which includes the above example, one can define a semi-inverse such that $A_{\mathrm{ginv}}(\alpha)$ is nonnegative and idempotent. We do not know if this result holds in general. Our results follow from Flor's theorem [4], using manipulations with block matrices. Our study was motivated by the analysis of positive differential-algebraic equations (DAEs) [2, 7].

2. Main result. First, we recall the following facts [1]. For $U \in \mathbb{R}^{m\times n}$, a matrix $U^{\mathrm{ginv}} \in \mathbb{R}^{n\times m}$ is called a semi-inverse of U if the following conditions hold

$$UU^{\mathrm{ginv}}U = U, \quad U^{\mathrm{ginv}}UU^{\mathrm{ginv}} = U^{\mathrm{ginv}}. \tag{2.1}$$

If $0 \neq U = \mathbf{x}\mathbf{y}^\top$ then

$$U^\dagger = \frac{1}{(\mathbf{x}^\top\mathbf{x})(\mathbf{y}^\top\mathbf{y})}\mathbf{y}\mathbf{x}^\top.$$

If we assume that U is a direct sum of matrices $U = \oplus_{i=1}^s U_i$, then $U^\dagger = \oplus_{i=1}^s U_i^\dagger$.

For our main result we need the following simplification of Flor's theorem [4].

LEMMA 2.1. *Any nonzero nonnegative idempotent matrix $B \in \mathbb{R}_+^{n\times n}$ is permutationally similar to the following 3×3 block matrix*

$$P := \begin{bmatrix} J & JG & 0 \\ 0 & 0 & 0 \\ FJ & FJG & 0 \end{bmatrix}, \ J \in \mathbb{R}_+^{n_1\times n_1}, G \in \mathbb{R}_+^{n_1\times n_2}, F \in \mathbb{R}_+^{n_3\times n_1}, \tag{2.2}$$

where $n = n_1 + n_2 + n_3, 1 \le n_1, 0 \le n_2, 0 \le n_3$. F, G are arbitrary nonnegative matrices, and J is a direct sum of $k \ge 1$ rank one positive idempotent matrices $J_i \in \mathbb{R}_+^{l_i\times l_i}$, i.e.,

$$J = \oplus_{i=1}^k J_i,\ J_i = \mathbf{u}_i\mathbf{v}_i^\top, \mathbf{0} < \mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}_+^{l_i},\ \mathbf{v}_i^\top\mathbf{u}_i = 1,\ i = 1, \ldots, k. \tag{2.3}$$

Proof. Flor's theorem states that B is permutationally similar to the following block matrix [4]

$$C := \begin{bmatrix} J & JG_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ F_1J & F_1JG_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

Here, $J \in \mathbb{R}_+^{n_1 \times n_1}$ is of the form (2.3), $G_1 \in \mathbb{R}_+^{n_1 \times m_2}, F_1 \in \mathbb{R}_+^{n_3 \times n_1}$ are arbitrary nonnegative matrices, and the last m_4 rows and columns of C are zero. Hence, $n_1 + m_2 + n_3 + m_4 = n$ and $0 \leq m_2, n_3, m_4$. If $m_4 = 0$ then C is of the form (2.2). It remains to show that C is permutationally similar to P if $m_4 > 0$.

Interchanging the last row and column of C with the $(n_1 + m_2 + 1)$-st row and column of C we obtain a matrix C_1. Then, we interchange the $(n-1)$-st row and column of C_1 with the $(n_1 + m_2 + 2)$-nd row and column of C_1. We continue this process until we obtain the idempotent matrix P with $n_2 = m_2 + m_4$ zero rows located at the rows $n_1 + 1, \ldots, n_1 + n_2$. It follows that P is of the form

$$P := \begin{bmatrix} J & G & 0 \\ 0 & 0 & 0 \\ F & H & 0 \end{bmatrix}, G \in \mathbb{R}_+^{n_1 \times n_2}, F \in \mathbb{R}_+^{n_3 \times n_1}, H \in \mathbb{R}_+^{n_3 \times n_3}.$$

Since $P^2 = P$ we have that

$$G = JG,\ F = FJ,\ H = FG = (FJ)(JG) = FJG.$$

Hence, P is of the form (2.2). $\square$

Theorem 2.2. *Let $A \in \mathbb{R}_+^{n \times n}$ be a nonnegative idempotent matrix. We assume that for $\emptyset \neq \alpha \subsetneq \langle n \rangle$, the submatrix $A[\alpha, \alpha]$ has a positive diagonal. Then $A_\dagger(\alpha)$ is a nonnegative idempotent matrix. Furthermore,*

$$\operatorname{rank} A_\dagger(\alpha) = \operatorname{rank} A - \operatorname{rank} A[\alpha, \alpha]. \tag{2.4}$$

Proof. Without loss of generality we may assume that A is of the form (2.2). Since $A[\alpha, \alpha]$ has a positive diagonal, we deduce that $A[\alpha, \alpha]$ is a submatrix of J. First we consider the special case $A[\alpha, \alpha] = J$. Using the identity $JJ^\dagger J = J$, we obtain that $A_\dagger(\alpha) = 0$. Since $\operatorname{rank} A = \operatorname{rank} J$, also the equality in (2.4) holds.

Let J, F, G be defined as in (2.2) and assume now that $A[\alpha, \alpha]$ is a strict submatrix of J. In the following, for an integer j we write $j + \langle m \rangle$ for the index set $\{j+1, \ldots, j+m\}$. Let $\alpha' := \langle n_1 \rangle \backslash \alpha$, $\beta := n_1 + \langle n_2 \rangle$ and $\gamma := n_1 + n_2 + \langle n_3 \rangle$. Then,

$$A[\alpha^c, \alpha] A[\alpha, \alpha]^\dagger A[\alpha, \alpha^c] = \begin{bmatrix} J[\alpha', \alpha] \\ 0 \\ (FJ)[\gamma, \alpha] \end{bmatrix} J[\alpha, \alpha]^\dagger \begin{bmatrix} J[\alpha, \alpha'] & (JG)[\alpha, \beta] & 0 \end{bmatrix}$$

$$= \begin{bmatrix} J[\alpha', \alpha] J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] & J[\alpha', \alpha] J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta] & 0 \\ 0 & 0 & 0 \\ (FJ)[\gamma, \alpha] J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] & (FJ)[\gamma, \alpha] J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta] & 0 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

On the other hand, we have

$$A[\alpha^c,\alpha^c] = \begin{bmatrix} J[\alpha',\alpha'] & (JG)[\alpha',\beta] & 0 \\ 0 & 0 & 0 \\ (FJ)[\gamma,\alpha'] & FJG & 0 \end{bmatrix}.$$

Thus, the nonnegativity of $A_\dagger(\alpha)$ is equivalent to the following, (entrywise), inequalities

$$\begin{aligned} &J[\alpha',\alpha'] \geq J[\alpha',\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'],\\ &(JG)[\alpha',\beta] \geq J[\alpha',\alpha]J[\alpha,\alpha]^\dagger (JG)[\alpha,\beta],\\ &(FJ)[\gamma,\alpha'] \geq (FJ)[\gamma,\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'],\\ &FJG \geq (FJ)[\gamma,\alpha]J[\alpha,\alpha]^\dagger (JG)[\alpha,\beta]. \end{aligned}$$

Without loss of generality, we may assume that J is permuted such that the indices of the first q blocks J_i are contained in α^c, the indices of the following blocks J_i for $i = q+1,\ldots,q+p$ are split between α and α^c and the indices of the blocks J_i for $i = q+p+1,\ldots,q+p+\ell = k$ are contained in α. Partitioning the vectors $\mathbf{u}_i, \mathbf{v}_i$ in (2.3) according to α and α^c as

$$\mathbf{u}_i^\top = (\mathbf{a}_i^\top, \mathbf{x}_i^\top), \quad \mathbf{v}_i^\top = (\mathbf{b}_i^\top, \mathbf{y}_i^\top) \ \text{ for } \ i = q+1,\ldots,q+p,$$

we obtain that

$$J[\alpha',\alpha'] = (\oplus_{i=1}^q J_i) \oplus_{i=q+1}^{q+p} \mathbf{a}_i\mathbf{b}_i^\top,\ J[\alpha,\alpha] = (\oplus_{j=q+1}^{q+p} \mathbf{x}_i\mathbf{y}_i^\top) \oplus_{i=q+p+1}^{q+p+\ell} J_i.$$

Note that

$$q = \operatorname{rank} J - \operatorname{rank} A[\alpha,\alpha] = \operatorname{rank} A - \operatorname{rank} A[\alpha,\alpha]. \tag{2.5}$$

We will only consider the case $q,p,\ell > 0$, as other cases follow similarly. We have

$$J[\alpha,\alpha]^\dagger = (\oplus_{i=q+1}^{q+p} \frac{1}{(\mathbf{x}_i^\top\mathbf{x}_i)(\mathbf{y}_i^\top\mathbf{y}_i)}\mathbf{y}_i\mathbf{x}_i^\top) \oplus_{i=q+p+1}^{q+p+\ell} \frac{1}{(\mathbf{u}_i^\top\mathbf{u}_i)(\mathbf{v}_i^\top\mathbf{v}_i)}\mathbf{v}_i\mathbf{u}_i^\top, \tag{2.6}$$

$$J[\alpha,\alpha'] = \begin{bmatrix} 0 & \oplus_{i=q+1}^{q+p}\mathbf{x}_i\mathbf{b}_i^\top \\ 0 & 0 \end{bmatrix},\ J[\alpha',\alpha] = \begin{bmatrix} 0 & 0 \\ \oplus_{i=q+1}^{q+p}\mathbf{a}_i\mathbf{y}_i^\top & 0 \end{bmatrix}, \tag{2.7}$$

and hence,

$$\begin{aligned} J[\alpha',\alpha]J[\alpha,\alpha]^\dagger &= \begin{bmatrix} 0 & 0 \\ \oplus_{i=q+1}^{q+p}\frac{1}{\mathbf{x}_i^\top\mathbf{x}_i}\mathbf{a}_i\mathbf{x}_i^\top & 0 \end{bmatrix},\\ J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] &= \begin{bmatrix} 0 & \oplus_{i=q+1}^{q+p}\frac{1}{\mathbf{y}_i^\top\mathbf{y}_i}\mathbf{y}_i\mathbf{b}_i^\top \\ 0 & 0 \end{bmatrix},\\ J[\alpha',\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] &= \begin{bmatrix} 0 & 0 \\ 0 & \oplus_{i=q+1}^{q+p}\mathbf{a}_i\mathbf{b}_i^\top \end{bmatrix}. \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

Therefore, we obtain

$$J[\alpha',\alpha'] - J[\alpha',\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] = \begin{bmatrix} \oplus_{i=1}^q J_i & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

which proves (2.5).

We now show the inequalities (2.5) and (2.5). First, we observe that JG and FJ have the following block form

$$JG = \begin{bmatrix} \mathbf{u}_1\mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_k\mathbf{g}_k^\top \end{bmatrix},\ FJ = \begin{bmatrix} \mathbf{f}_1\mathbf{v}_1^\top & \cdots & \mathbf{f}_k\mathbf{v}_k^\top \end{bmatrix},\ \mathbf{g}_i \in \mathbb{R}_+^{n_2}, \mathbf{f}_i \in \mathbb{R}_+^{n_3} \text{ for } i = 1,\ldots,k.$$

Hence, we obtain

$$(JG)[\alpha,\beta] = \begin{bmatrix} \mathbf{x}_{q+1}\mathbf{g}_{q+1}^\top \\ \vdots \\ \mathbf{x}_{q+p}\mathbf{g}_{q+p}^\top \\ \mathbf{u}_{q+p+1}\mathbf{g}_{q+p+1}^\top \\ \vdots \\ \mathbf{u}_k\mathbf{g}_k^\top \end{bmatrix}, \tag{2.8}$$

$$(JG)[\alpha',\beta] = \begin{bmatrix} \mathbf{u}_1\mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_q\mathbf{g}_q^\top \\ \mathbf{a}_{q+1}\mathbf{g}_{q+1}^\top \\ \vdots \\ \mathbf{u}_{q+p}\mathbf{g}_{q+p}^\top \end{bmatrix}, \tag{2.9}$$

$$(FJ)[\gamma,\alpha] = \begin{bmatrix} \mathbf{f}_{q+1}\mathbf{y}_{q+1}^\top & \cdots & \mathbf{f}_{q+p}\mathbf{y}_{q+p}^\top & \mathbf{f}_{q+p+1}\mathbf{v}_{q+p+1}^\top & \cdots & \mathbf{f}_k\mathbf{v}_k^\top \end{bmatrix}, \tag{2.10}$$

$$(FJ)[\gamma,\alpha'] = \begin{bmatrix} \mathbf{f}_1\mathbf{v}_1^\top & \cdots & \mathbf{f}_q\mathbf{v}_q^\top & \mathbf{f}_{q+1}\mathbf{b}_{q+1}^\top & \cdots & \mathbf{f}_{q+p}\mathbf{b}_{q+p}^\top \end{bmatrix}. \tag{2.11}$$

We use (2.8) to deduce that

$$(FJ)[\gamma,\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] = \begin{bmatrix} 0 & \cdots & 0 & \mathbf{f}_{q+1}\mathbf{b}_{q+1}^\top & \cdots & \mathbf{f}_{q+p}\mathbf{b}_{q+p}^\top \end{bmatrix}.$$

Therefore, we have

$$(FJ)[\gamma,\alpha'] - (FJ)[\gamma,\alpha]J[\alpha,\alpha]^\dagger J[\alpha,\alpha'] = \begin{bmatrix} \mathbf{f}_1\mathbf{v}_1^\top & \cdots & \mathbf{f}_q\mathbf{v}_q^\top & 0 & \cdots & 0 \end{bmatrix}. \tag{2.12}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

Similarly, using (2.8), we obtain

$$(JG)[\alpha',\beta] - J[\alpha',\alpha]J[\alpha,\alpha]^{\dagger}(JG)[\alpha,\beta] = \begin{bmatrix} \mathbf{u}_1\mathbf{g}_1^{\top} \\ \vdots \\ \mathbf{u}_q\mathbf{g}_q^{\top} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence, the inequalities (2.5) and (2.5) hold.

We now show the last inequality (2.5). To this end, we observe that

$$FJG = (FJ)(JG) = \sum_{i=1}^{k} \mathbf{f}_i\mathbf{g}_i^{\top}. \tag{2.13}$$

Multiplying (2.6), (2.8) and (2.10) we obtain that

$$(FJ)[\gamma,\alpha]J[\alpha,\alpha]^{\dagger}(JG)[\alpha,\beta] = \sum_{i=q+1}^{k} \mathbf{f}_i\mathbf{g}_i^{\top}.$$

Hence,

$$FJG - (FJ)[\gamma,\alpha]J[\alpha,\alpha]^{\dagger}(JG)[\alpha,\beta] = \sum_{i=1}^{q} \mathbf{f}_i\mathbf{g}_i^{\top} \geq 0.$$

In particular, this proves that (2.5) holds.

It is left to show that $A_{\dagger}(\alpha)$ is an idempotent matrix. Clearly, if $q = 0$ then $A_{\dagger}(\alpha) = 0$. So $A_{\dagger}(\alpha)$ is a trivial idempotent matrix, and (2.5) yields (2.4).

Assuming finally that $q > 0$, it follows that $A_{\dagger}(\alpha)$ has the block form (2.2) with $J = \oplus_{i=1}^{q} J_i \oplus 0$. Hence $A_{\dagger}(\alpha)$ is an idempotent matrix whose rank is q, and (2.5) yields (2.4). ∎

Corollary 2.3. *Let $A \in \mathbb{R}_{+}^{n\times n}, A \neq 0$ be idempotent. If $\alpha \subsetneqq \langle n\rangle$ is chosen such that $A[\alpha,\alpha]$ is an invertible matrix, then $A[\alpha,\alpha]$ is diagonal.*

Proof. Note that the number ℓ in the proof of Theorem 2.2 is either zero or the corresponding blocks J_i are positive 1×1 matrices for $i = q+p+1,\ldots,q+p+\ell$. Furthermore, for the split blocks, we also have that $\mathbf{x}_i\mathbf{y}_i^T \in \mathbb{R}^{1\times 1}$, for $i = q+1,\ldots,q+p$, since $\mathbf{x}_i\mathbf{y}_i^T$ is of rank 1. Therefore, $A[\alpha,\alpha]$ is diagonal. ∎

Corollary 2.4. *Let $A \in \mathbb{R}_{+}^{n\times n}, A \neq 0$ be idempotent. If $\alpha \subsetneqq \langle n\rangle$ is chosen such that $A[\alpha,\alpha]$ is an invertible matrix, then the standard Schur complement (1.1) is nonnegative.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

Corollary 2.5. *Let $A \in \mathbb{R}_+^{n\times n}, A \neq 0$ be idempotent. Choose $\alpha \subsetneqq \langle n\rangle$, such that $I - A[\alpha,\alpha]$ is invertible. Then, $\tilde{A}(\alpha)$ defined by*

$$\tilde{A}(\alpha) := A[\alpha^c,\alpha^c] + A[\alpha^c,\alpha](I - A[\alpha,\alpha])^{-1}A[\alpha,\alpha^c]$$

is a nonnegative idempotent matrix.

To prove this Corollary 2.5 we need the following fact for idempotent matrices, which is probably known.

Lemma 2.6. *Let $A \in \mathbb{R}^{n\times n}, A \neq 0$ be idempotent given as a 2×2 block matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Assume that $I - A_{22} \in \mathbb{R}^{n-m}$ is invertible. Then $B := A_{11} + A_{12}(I - A_{22})^{-1}A_{21}$ is idempotent.*

Proof. Let

$$E = (I - A_{22})^{-1}A_{21},\ D = A_{21} + A_{22}E,\ z = \begin{bmatrix} x \\ Ex \end{bmatrix} \in \mathbb{R}^n,\ x \text{ any vector in } \mathbb{R}^m.$$

Note that $Az = \begin{bmatrix} Bx \\ Dx \end{bmatrix}$. As $A^2z = Az$ and x is an arbitrary vector, we obtain the equalities

$$A_{11}B + A_{12}D = B, \quad A_{21}B + A_{22}D = D. \tag{2.14}$$

From the second equality of (2.14) we obtain $D = EB$. Substituting this equality into the first equality of (2.14) we obtain that $B^2 = B$. □

Proof of Corollary 2.5. The assumption that $I - A[\alpha,\alpha]$ is invertible implies that $A[\alpha,\alpha]$ does not have an eigenvalue 1, i.e., $\rho(A[\alpha,\alpha]) < 1$. Hence, $I - A[\alpha,\alpha]$ is an M-matrix [1] and $(I - A[\alpha,\alpha])^{-1} \geq 0$. The assertion of Corollary 2.5 now follows using Lemma 2.6. □

3. Additional results.

3.1. An example. In this subsection we assume that the nonnegative idempotent matrix A is of the special form

$$A := \begin{bmatrix} J & JG \\ 0 & 0 \end{bmatrix}. \tag{3.1}$$

Furthermore, we assume that $A[\alpha,\alpha]$ has a zero on its main diagonal. We give an example where $A_\dagger(\alpha)$ may fail to be nonnegative. To this end, we first start with the following known result.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

Lemma 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a singular matrix of the following form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{bmatrix},\ A_{11} \in \mathbb{R}^{p\times p}, A_{12} \in \mathbb{R}^{p\times(n-p)},$$

for some $1 \le p < n$. Then $(A^\dagger)^\top$ has the same block form as A.

Proof. Let $r = \operatorname{rank} A$. So $r \le p$. Then the reduced singular value decomposition of A is of the form $U_r\Sigma_r V_r^\top$, where $U_r, V_r \in \mathbb{R}^{n\times r}, U_r^\top U_r = V_r V_r^\top = I_r$ and Σ_r is a diagonal matrix, whose diagonal entries are the positive singular values of A.

Clearly, $AA^\top = \begin{bmatrix} A_{11}A_{11}^\top + A_{12}A_{12}^\top & 0 \\ 0 & 0 \end{bmatrix}$. Hence all eigenvectors of $AA^\top$, corresponding to positive eigenvalues are of the form $(\mathbf{x}^\top, \mathbf{0}^\top)^\top, \mathbf{x} \in \mathbb{R}^p$. Thus $U_r^\top = [U_{r1}^\top\ 0_{r\times(n-p)}]$ where $U_{r1} \in \mathbb{R}^{p\times r}$. Recall that $A^\dagger = V_r\Sigma_r^{-1}U_r^\top$. The above form of U_r establishes the lemma. ☐

In the following example we permute some rows and columns of A, in order to find the Schur complement of the right lower block.

Example 3.2. Consider a nonnegative idempotent matrix in the block form

$$B = \left[\begin{array}{ccc|cc} \mathbf{u}_1\mathbf{v}_1^\top & 0 & \mathbf{u}_1\mathbf{s}_1^\top & \mathbf{u}_1\mathbf{t}_1^\top & 0 \\ 0 & \mathbf{a}_2\mathbf{b}_2^\top & \mathbf{a}_2\mathbf{s}_2^\top & \mathbf{a}_1\mathbf{t}_2^\top & \mathbf{a}_2\mathbf{y}_2^\top \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x}_2\mathbf{b}_2^\top & \mathbf{x}_2\mathbf{s}_2^\top & \mathbf{x}_1\mathbf{t}_2^\top & \mathbf{x}_2\mathbf{y}_2^\top \end{array}\right].$$

Then,

$$B[\alpha,\alpha] = \begin{bmatrix} 0 & 0 \\ \mathbf{x}_1\mathbf{t}_2^\top & \mathbf{x}_2\mathbf{y}_2^\top \end{bmatrix},\ B[\alpha,\alpha]^\dagger = \begin{bmatrix} 0 & \frac{\mathbf{t}_2\mathbf{x}_2^\top}{(\mathbf{x}_2^\top\mathbf{x}_2)(\mathbf{t}_2^\top\mathbf{t}_2 + y_2^T y_2)} \\ 0 & \frac{\mathbf{y}_2\mathbf{x}_2^\top}{(\mathbf{x}_2^\top\mathbf{x}_2)(\mathbf{t}_2^\top\mathbf{t}_2 + \mathbf{y}_2^\top y_2)} \end{bmatrix},$$

and

$$B[\alpha^c,\alpha]B[\alpha,\alpha]^\dagger B[\alpha,\alpha^c] = \begin{bmatrix} 0 & \frac{\mathbf{t}_1^\top\mathbf{t}_2\mathbf{u}_1\mathbf{b}_2^\top}{\mathbf{t}_2^\top\mathbf{t}_2 + \mathbf{y}_2^\top\mathbf{y}_2} & \frac{\mathbf{t}_1^\top\mathbf{t}_2\mathbf{u}_1\mathbf{s}_2^\top}{\mathbf{t}_2^\top\mathbf{t}_2 + \mathbf{y}_2^\top\mathbf{y}_2} \\ 0 & \mathbf{a}_2\mathbf{b}_2^\top & \mathbf{a}_2\mathbf{s}_2^\top \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $B_\dagger(\alpha)_{11} > 0,\ B_\dagger(\alpha)_{12} \le 0$ and the Moore-Penrose inverse Schur complement is neither nonnegative nor nonpositive if $t_1^T t_2 > 0$.

3.2. Nonnegativity of semi-inverse Schur complement. In this section we extend the results of Section 2 for idempotent matrices of the form (2.2) for some Schur complements with zero diagonal entries. We start with the following simple observation.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

ELA

PROPOSITION 3.3. *Let the assumptions of Lemma 3.1 hold. Suppose that*

$$A_{11}(A_{11})^{\dagger}A_{12} = A_{12}.$$

Then $A^{\text{ginv}} = \begin{bmatrix} (A_{11})^{\dagger} & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{bmatrix}$ *is a semi-inverse of* A*. In particular any principle submatrix of an idempotent matrix as in (3.1) with at least one zero diagonal element has a semi-inverse of this form.*

Proof. The proposition follows by checking the conditions in (2.1). □ Note that condition $A_{11}(A_{11})^{\dagger}A_{12} = A_{12}$ holds in general for idempotent matrices A of the form as in (3.1).

The following theorem states the general result of this subsection.

THEOREM 3.4. *Let* $A \in \mathbb{R}_+^{n\times n}$ *be of the form (2.2), where* $n_2 + n_3 \geq 1$ *and the condition in (2.3) holds. Furthermore, let* $\alpha_1 \subset \langle n\rangle$ *be of the following form*

$$\begin{array}{ll} \textit{either} & \alpha_1 = \alpha \cup \beta, \emptyset \neq \beta \subseteq n_1 + \langle n_2\rangle, \\ \textit{or} & \alpha_1 = \alpha \cup \gamma, \emptyset \neq \gamma \subseteq n_1 + n_2 + \langle n_3\rangle, \end{array} \tag{3.2}$$

where $\alpha \subseteq \langle n_1\rangle$*. Then, there exists a semi-inverse* $A^{\text{ginv}}[\alpha_1, \alpha_1]$ *of* $A[\alpha_1, \alpha_1]$ *such that* $A_{\text{ginv}}(\alpha_1)$ *as defined in (1.2) is a nonnegative idempotent matrix. The rank of* $A_{ginv}(\alpha_1)$ *is equal to the multiplicity of the eigenvalue* 1 *in* $A[\alpha', \alpha']$*, where* $\alpha' = \langle n_1\rangle\backslash\alpha$*. In particular, if* 1 *is not an eigenvalue of* $A[\alpha', \alpha']$*, then* $A_{\text{ginv}}(\alpha) = 0$*.*

Proof. First we consider the case that $\alpha_1 = \alpha \cup \beta$. If $\alpha = \emptyset$, then $A[\alpha_1, \alpha_1]$ and $A[\alpha_1, \alpha_1]^{\text{ginv}}$ are zero matrices for any semi-inverse and $A_{\text{ginv}}(\alpha_1) = A[\alpha_1^c, \alpha_1^c]$. Using the proof of Theorem 2.2 we obtain that $A_{\text{ginv}}(\alpha_1)$ is a nonnegative idempotent matrix of rank k.

Assuming now that $\alpha \neq \emptyset$, we observe that $A[\alpha_1, \alpha_1]$ satisfies the assumption of Proposition 3.3. Defining $A[\alpha_1, \alpha_1]^{\text{ginv}}$ as in Proposition 3.3 and following the arguments of the proof of Theorem 2.2 we deduce the theorem in this case.

We assume now that $\alpha_1 = \alpha\cup\gamma$. If $\alpha = \emptyset$ we obtain that $A_{\text{ginv}}(\alpha_1)$ is a nonnegative idempotent matrix of rank k as above. Assuming finally that $\alpha \neq \emptyset$, we have that $A[\alpha_1, \alpha_1]^{\top}$ satisfies the assumption of Proposition 3.3. Define $(A[\alpha_1, \alpha_1]^{\top})^{\text{ginv}}$ as in Proposition 3.3 and let $A[\alpha_1, \alpha_1]^{\text{ginv}} := ((A[\alpha_1, \alpha_1]^{\top})^{\text{ginv}})^{\top}$. Repeating the arguments of the proof of Theorem 2.2 we deduce the theorem in this case. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 426-435, September 2008

REFERENCES

[1] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*, Second edition. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1994.

[2] R. Bru, C. Coll, and E. Sánchez. Structural properties of positive linear time-invariant difference-algebraic equations. *Linear Algebra Appl.*, 349:1–10, 2002.

[3] S. L. Campbell and C. D. Meyer, Jr. *Generalized Inverses of Linear Transformations*. Dover Publications Inc., New York, 1991.

[4] P. Flor. On groups of non-negative matrices. *Compositio Mathematica*, 21(4):376–382, 1969.

[5] R. Penrose. A generalized inverse for matrices. *Proc. Cambridge Phil. Soc.*, 51:406–413, 1955.

[6] C. R. Rao and S. K. Mitra. *Generalized inverse of matrices and its applications*. John Wiley and Sons Inc., 1971.

[7] E. Virnik. Stability analysis of positive descriptor systems. *Linear Algebra Appl.*. To appear.

[8] F. Zhang, ed. *The Schur Complement and its Applications*, vol. 4 of Numerical Methods and Algorithms, Springer-Verlag, New York, 2005.