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REAL DIMENSION OF THE LIE ALGEBRA OF S-SKEW-HERMITIAN MATRICES*

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Abstract. Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices over the complex numbers. Let $S \in M_n(\mathbb{C})$. A matrix $A \in M_n(\mathbb{C})$ is said to be S-skew-Hermitian if $SA^* = -AS$ where A^* is the conjugate transpose of A. The set $\mathfrak{u}_S$ of all S-skew-Hermitian matrices is a Lie algebra. In this paper, we give a real dimension formula for $\mathfrak{u}_S$ using the Jordan block decomposition of the cosquare $S(S^*)^{-1}$ of S when S is nonsingular.

Key words. Generalized skew-Hermitian matrices, Matrix Lie algebras.

AMS subject classifications. 20H20, 17B05, 17B45.

1. Main result. The unitary group of degree n, denoted by $U(n)$, is the Lie group of all $n \times n$ unitary matrices under matrix multiplication. Associated with $U(n)$ is its Lie algebra $\mathfrak{u}(n)$ consisting of all $n \times n$ skew-Hermitian matrices where the Lie bracket is the commutator. The notion of unitary groups and the algebras of skew-Hermitian matrices can be generalized naturally in the following manner. Let $S \in M_n(\mathbb{C})$. We say that a matrix $A \in \mathrm{GL}_n(\mathbb{C})$ is S-unitary if $ASA^* = S$. Here, $\mathrm{GL}_n(\mathbb{C})$ is the set of all nonsingular $n \times n$ matrices over $\mathbb{C}$. Let U_S be the set of all S-unitary matrices. We see that U_S is a Lie group with a corresponding Lie algebra $\mathfrak{u}_S = \{A \in M_n(\mathbb{C}) \mid SA^* = -AS\}$. An element of $\mathfrak{u}_S$ is said to be an S-skew-Hermitian matrix. Observe that if S is the identity matrix I_n, we recover $U(n)$ and $\mathfrak{u}(n)$. In the past years, there has been a great amount of research work extending to S-unitary and S-skew-Hermitian matrices the linear algebraic properties of their usual counterparts (see e.g., [1, 5, 6, 10]).

In [3], the problem of determining whether two Lie groups U_S and U_T are isomorphic is investigated. The strategy employed in the paper is to look at the associated Lie algebras $\mathfrak{u}_S$. We know that if two Lie algebras have different real dimensions, then their parent Lie groups are necessarily non-isomorphic. Thus, a significant part of the problem can be addressed by providing a dimension formula for $\mathfrak{u}_S$.

Such a dimension formula is given in [7, Theorem 3.3] by De Terán and Dopico (see also [8]). The derivation of their formula employs the idea of reducing the matrix S into its so-called canonical form for *congruence introduced by Horn and Sergeichuk in [9, Theorem 1.1]. Denote by $J_l(\beta)$ the Jordan block of size $l \in \mathbb{N}$ corresponding to $\beta \in \mathbb{C}$. In the *congruence decomposition, the matrix S is written as a direct sum of matrix blocks:

1. (Type 0) $J_k(0)$.
2. (Type I) $k \times k$ matrix $\alpha\Gamma_k$ where $|\alpha| = 1$ and

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$$\Gamma_k = \begin{bmatrix} 0 & & & & & (-1)^{k+1} \\ & & & & \iddots & (-1)^k \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix}.$$

3. (Type II) $H_{2k}(\mu)$, which is the skew sum of $J_k(\mu)$ and the size k identity matrix I_k where $|\mu| > 1$ i.e.,

$$H_{2k}(\mu) = \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix}.$$

Their formula is composed of 8 summands that record the contribution to the real dimension of $\mathfrak{u}_S$ of the three blocks listed above and their interaction to one another. When S is nonsingular, the number of summands is 4. In particular, the summand corresponding to two Type I blocks involves cases according to the parity of the size of the blocks.

The motivation of the paper of De Terán and Dopico is to solve the matrix equation $SA^* + AS = 0$ (with S fixed). In contrast, our motivation is to provide an insight on the structure of $\mathfrak{u}_S$ and find a convenient Lie algebra basis. This paper continues the authors' earlier works ([3] and [4, Corollary 2.1]) on $\mathfrak{u}_S$ where S is normal and nonsingular. In particular, we generalize the dimension formula given in [3, Theorem 3.19] by eliminating the normality condition on S.

To this end, we develop an approach that depends on the Jordan block decomposition of the cosquare $S(S^*)^{-1}$ of S when $S \in \mathrm{GL}_n(\mathbb{C})$. For simplicity, we write S^{-*} for $(S^*)^{-1}$.

Define $\mathrm{mult}(SS^{-*}; \beta, l)$ as the number of Jordan blocks $J_l(\beta)$ appearing in the Jordan normal form of SS^{-*}. We have the following main result.

Theorem 1.1. *Let $S \in \mathrm{GL}_n(\mathbb{C})$. Let Λ be the set of distinct eigenvalues of SS^{-*}. Then*

$$\dim_{\mathbb{R}}(\mathfrak{u}_S) = \sum_{m,l=1}^{n} \sum_{\beta \in \Lambda} \min(l, m)\, \mathrm{mult}(SS^{-*}; \beta, l)\, \mathrm{mult}(SS^{-*}; \overline{\beta}^{-1}, m).$$

If, in addition, S is normal, then SS^{-*} is unitary, and hence, diagonalizable. The distinct eigenvalues of SS^{-*} are $\beta_1, \ldots, \beta_k$ where $\beta_i = \alpha_i/|\alpha_i|$ for some eigenvalue α_i of S. We have $\beta_i = \overline{\beta_j}^{-1}$ if and only if $i = j$. Then the following corollary simplifies the formula given in [3, Theorem 3.19].

Corollary 1.2. *Let $S \in \mathrm{GL}_n(\mathbb{C})$ be normal. Let $\beta_1, \ldots, \beta_k$ be the distinct eigenvalues of SS^{-*} and let $\mathrm{mult}(SS^{-*}, \beta_i)$ be the algebraic multiplicity of β_i as an eigenvalue of SS^{-*}. Then*

$$\dim_{\mathbb{R}}(\mathfrak{u}_S) = \sum_{i=1}^{k} \mathrm{mult}(SS^{-*}, \beta_i)^2.$$

The formula in Theorem 1.1, in contrast with the formula given by De Terán and Dopico, provides a uniform treatment for Type I and Type II matrices into one case and avoids discussion on the parity. Hence, the formula allows for a simpler computation of $\dim_{\mathbb{R}} \mathfrak{u}_S$ when S is nonsingular.

To end this section, we provide some examples illustrating Theorem 1.1.

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Example 1.3. For $\mu \neq 0$, consider

$$H_{2k}(\mu) = \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix}.$$

The cosquare of $H_{2k}(\mu)$ is similar to $J_k(\mu) \oplus J_k(\bar{\mu}^{-1})$. By Theorem 1.1, if $|\mu| = 1$, then $\dim_{\mathbb{R}}(\mathfrak{u}_{H_{2k}(\mu)}) = 4k$. Otherwise, $\dim_{\mathbb{R}}(\mathfrak{u}_{H_{2k}(\mu)}) = 2k$.

Example 1.4. Here and throughout the article, we use the symbol $\mathfrak{i}$ to denote the square root of -1. Let $S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \mathfrak{i} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that S is not normal and as such, [3, Theorem 3.19] cannot be used. The cosquare of S is similar to $J_1(1) \oplus J_2(1)$. By Theorem 1.1, $\dim_{\mathbb{R}} \mathfrak{u}_S = 5$. The following may be taken as a basis for $\mathfrak{u}_S$:

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathfrak{i} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathfrak{i} \\ \mathfrak{i} & 0 & 0 \end{bmatrix}, \begin{bmatrix} \mathfrak{i} & 0 & 0 \\ 0 & \mathfrak{i} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ \mathfrak{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

2. Proof of Theorem 1.1. We say that the set $\{v_1, v_2, \ldots, v_l\} \subset \mathbb{C}^n = \mathbb{C}^{n\times 1}$ is a *Jordan chain* of a matrix A associated with an eigenvalue α, if it satisfies the following properties:

$$Av_1 = \alpha v_1$$

$$Av_{i+1} = \alpha v_{i+1} + v_i \ \text{ (for } 1 \leq i < l\text{)},$$

and, moreover, there is no vector $w \in \mathbb{C}^n$ such that $Aw = \alpha w + v_l$. The vectors $v_1, v_2, \ldots, v_l$ are *generalized eigenvectors* associated with α (see [2, Sec. 9.4 - 9.5]). For two Jordan chains $C = \{v_1, \ldots, v_l\}$ and $D = \{u_1, \ldots, u_m\}$ (which may be associated with the same eigenvalue), we define the vector space

$$\mathbb{V}(A; C, D) = \sum_{\substack{1\leq i\leq l \\ 1\leq j\leq m}} \mathbb{C} v_i (Au_j)^* + \sum_{\substack{1\leq i\leq l \\ 1\leq j\leq m}} \mathbb{C} u_j (Av_i)^*.$$

2.1. CASE 1: Distinct Jordan blocks. Let $S \in \mathrm{GL}_n(\mathbb{C})$. Consider Jordan chains $C = \{v_1, \ldots, v_l\}$ and $D = \{u_1, \ldots, u_m\}$ of SS^{-*} associated with the eigenvalues α and α' of SS^{-*}, respectively. Without loss of generality, we may assume $l \leq m$ (by interchanging the roles of the Jordan chains C and D, if needed). Suppose $C \cup D$ is linearly independent over $\mathbb{C}$. Note that if $\alpha \neq \alpha'$, then $C \cup D$ is always linearly independent. The condition is imposed, in particular, to the case where $\alpha = \alpha'$ and their corresponding Jordan chains are distinct.

For $2 \leq i \leq l$, we have

$$SS^{-*} v_i = \alpha v_i + v_{i-1},$$

and

$$SS^{-*} v_1 = \alpha v_1.$$

Likewise, for $2 \leq j \leq m$,

$$SS^{-*} u_j = \alpha' u_j + u_{j-1},$$

and

$$SS^{-*} u_1 = \alpha' u_1.$$

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Extend $C \cup D$ to a basis $\{w_1, \ldots, w_n\}$ of $\mathbb{C}^n$. Let $W = (w_1, \ldots, w_n) \in \mathrm{GL}_n(\mathbb{C})$, and let $\{e_i \mid 1 \leq i \leq n\}$ be the standard basis of $\mathbb{C}^n$. Then $We_i = w_i$. Let us consider the linear map

$$\Phi : X \longmapsto W^{-1}XSW^{-*},$$

where $X \in M_n(\mathbb{C})$. Then

$$\begin{aligned}\Phi(w_i(S^{-*}w_j)^*) &= W^{-1}w_iw_j^*S^{-1}SW^{-*}\\ &= W^{-1}w_i(W^{-1}w_j)^*\\ &= e_ie_j^*.\end{aligned}$$

Clearly, $\{e_ie_j^* \mid 1 \leq i, j \leq n\}$ is a basis of $M_n(\mathbb{C})$. Consequently, $\{w_i(S^{-*}w_j)^* \mid 1 \leq i, j \leq n\}$ is a linearly independent set. For each pair i, j satisfying $1 \leq i \leq l$ and $1 \leq j \leq m$, define

$$E_{ij} = v_i(S^{-*}u_j)^*,$$

and

$$E'_{ji} = u_j(S^{-*}v_i)^*.$$

Then $\{E_{ij}, E'_{ji} \mid 1 \leq i \leq l, 1 \leq j \leq m\}$ is a linearly independent set.

Denote by V_{ij} the subspace $\mathbb{C}E_{ij} + \mathbb{C}E'_{ji}$ of $M_n(\mathbb{C})$. Then, $\mathbb{V}(S^{-*}; C, D)$ is the direct sum

$$\mathbb{V}(S^{-*}; C, D) = \sum_{\substack{1\leq i\leq l\\ 1\leq j\leq m}} \mathbb{C}E_{ij} + \sum_{\substack{1\leq i\leq l\\ 1\leq j\leq m}} \mathbb{C}E'_{ji} = \sum_{\substack{1\leq i\leq l\\ 1\leq j\leq m}} V_{ij}.$$

For $1 \leq k \leq l + m - 1$, define the vector space V_k as

$$V_k = \sum_{i+j=k+1} V_{ij}.$$

From the definition of V_{ij}, it follows that $\dim_{\mathbb{R}}(V_{ij}) = 4$. Therefore,

$$\dim_{\mathbb{R}}(V_k) = \begin{cases} 4k & \text{if } 1 \leq k < l,\\ 4l & \text{if } l \leq k \leq m,\\ 4(l-k+m) & \text{if } m < k.\end{cases}$$

For simplicity, we set $V_{\leq k} = \sum_{t\leq k} V_t$ and $V_0 = \{0\}$. Let $\pi_k : \mathbb{V}(S^{-*}; C, D) \longrightarrow V_k$ be the canonical projection from $\mathbb{V}(S^{-*}; C, D)$ onto V_k with respect to the basis $\{E_{ij}, E'_{ji} \mid 1 \leq i \leq l, 1 \leq j \leq m\}$. Also, we define the projections π_{ij}, $\pi_{<k}$, and $\pi_{\leq k}$ from $\mathbb{V}(S^{-*}; C, D)$ onto V_{ij}, $V_{\leq k-1}$, and $V_{\leq k}$, respectively.

Consider the $\mathbb{R}$-linear map $d : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ given by $d(A) = SA^*S^{-1} + A$, for $A \in M_n(\mathbb{C})$. Clearly, $\mathfrak{u}_S = \ker(d)$. Using the fact that $E_{ij}^* = S^{-*}u_jv_i^*$ and $(E'_{ji})^* = S^{-*}v_iu_j^*$, observe that, for all $1 \leq i \leq l$,

$$d(E_{ij}) = \begin{cases} \alpha' E'_{ji} + E_{ij} & \text{if } j = 1,\\ \alpha' E'_{ji} + E'_{j-1,i} + E_{ij} & \text{if } 1 < j \leq m\end{cases},$$

and, for all $1 \leq j \leq m$,

$$d(E'_{ji}) = \begin{cases} \alpha E_{ij} + E'_{ji} & \text{if } i = 1,\\ \alpha E_{ij} + E_{i-1,j} + E'_{ji} & \text{if } 1 < i \leq l.\end{cases}$$

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Let $E_{ij} = E'_{ji} = 0$ when $i = 0$ or $j = 0$. Then the above equations are reduced to the following equations:

$$d(E_{ij}) = \alpha' E'_{ji} + E'_{j-1,i} + E_{ij},$$

$$d(E'_{ji}) = \alpha E_{ij} + E_{i-1,j} + E'_{ji}.$$

Let $d_k = d|_{V_k}$ be the restriction of the map d on V_k. Similarly, define d_{ij}, $d_{<k}$, and $d_{\leq k}$ for V_{ij}, $V_{\leq k-1}$, and $V_{\leq k}$, respectively. Note that, if $i+j = k+1$, then the elements E_{ij}, E'_{ji}, $\mathfrak{i}E_{ij}$, and $\mathfrak{i}E'_{ji}$ are in V_k. Thus, d and d_k coincide for these particular elements. Let us consider the composition $\pi_k \circ d_k : V_k \longrightarrow V_k$. If $i+j = k+1$, then

$$\begin{aligned}
\pi_k \circ d_k(E_{ij}) &= E_{ij} + \alpha' E'_{ji},\\
\pi_k \circ d_k(E'_{ji}) &= E'_{ji} + \alpha E_{ij},\\
\pi_k \circ d_k(\mathfrak{i}E_{ij}) &= \mathfrak{i}E_{ij} - \mathfrak{i}\alpha' E'_{ji},\\
\pi_k \circ d_k(\mathfrak{i}E'_{ji}) &= \mathfrak{i}E'_{ji} - \mathfrak{i}\alpha E_{ij}.
\end{aligned}$$

Note that the last two identities use the fact that $d(\mathfrak{i}A) = -\mathfrak{i}d(A) + 2\mathfrak{i}A$, for $A \in M_n(\mathbb{C})$.

Lemma 2.1. *Let $1 \leq k \leq l+m-1$. Then the real dimension of the image* $\mathrm{Im}(\pi_k \circ d_k)$ *of the map* $\pi_k \circ d_k$ *is given by*

$$\begin{cases} \dim_{\mathbb{R}}(V_k) & \text{if } \alpha\overline{\alpha'} \neq 1,\\ \frac{1}{2}\dim_{\mathbb{R}}(V_k) & \text{otherwise.} \end{cases}$$

Moreover, if $i+j = k+1$ and $\alpha\overline{\alpha'} = 1$, then $\mathrm{Im}(\pi_{ij} \circ d_k) = \mathbb{R}(E_{ij} + \alpha' E'_{ji}) + \mathbb{R}\mathfrak{i}(E_{ij} - \alpha' E'_{ji})$.

Proof. Fix $1 \leq k \leq l+m-1$. Suppose $i+j = k+1$. Take $E = \{E_{ij}, \mathfrak{i}E_{ij}, E'_{ji}, \mathfrak{i}E'_{ji}\}$ as an ordered basis of V_{ij} over $\mathbb{R}$. Then we encode the coordinates of the matrices $\pi_k \circ d_k(E_{ij})$, $\pi_k \circ d_k(\mathfrak{i}E_{ij})$, $\pi_k \circ d_k(E'_{ji})$, and $\pi_k \circ d_k(\mathfrak{i}E'_{ji})$ with respect to E as the rows in the following matrix:

$$\begin{bmatrix} 1 & 0 & \Re(\alpha') & \Im(\alpha')\\ 0 & 1 & \Im(\alpha') & -\Re(\alpha')\\ \Re(\alpha) & \Im(\alpha) & 1 & 0\\ \Im(\alpha) & -\Re(\alpha) & 0 & 1 \end{bmatrix},$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of a complex number z, respectively. The determinant of this matrix is $(\alpha\overline{\alpha'} - 1)(\alpha'\overline{\alpha} - 1)$. If $\alpha\overline{\alpha'} \neq 1$, then $\mathrm{Im}(\pi_k \circ d_k)$ contains V_{ij} and, consequently, $V_k \subseteq \mathrm{Im}(\pi_k \circ d_k)$. So $\mathrm{Im}(\pi_k \circ d_k) = V_k$. On the other hand, if $\alpha\overline{\alpha'} = 1$, the rank of the above matrix is 2. So $\dim_{\mathbb{R}}(\mathrm{Im}(\pi_k \circ d_k)) = 2\dim_{\mathbb{R}}(V_k)/4$. Moreover, $\pi_k \circ d_k(E_{ij})$ and $\pi_k \circ d_k(\mathfrak{i}E_{ij})$ are linearly independent and span $\mathrm{Im}(\pi_{ij} \circ d_k)$. ∎

Note that if $\alpha\overline{\alpha'} \neq 1$, then $\ker(\pi_k \circ d_k) = \{0\}$ by Lemma 2.1. Suppose $\alpha\overline{\alpha'} = 1$. Recall that we assume $l \leq m$. Fix $1 \leq k \leq l+m-1$. Suppose $i+j = k+1$. Let

$$A_i = E_{ij} - \alpha' E'_{ji} \in V_k,$$

and

$$B_i = \mathfrak{i}(E_{ij} + \alpha' E'_{ji}) \in V_k.$$

Then

$$d_k(A_i) = E'_{j-1,i} - \overline{\alpha'} E_{i-1,j},$$

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and

$$d_k(B_i) = -\mathfrak{i}(E'_{j-1,i} + \overline{\alpha'} E_{i-1,j}).$$

Clearly, $\pi_k \circ d_k(A_i) = \pi_k \circ d_k(B_i) = 0$. In other words, $A_i, B_i \in \ker(\pi_k \circ d_k)$. Recall that $\pi_k \circ d_k(E_{ij}) = E_{ij} + \alpha' E'_{ji}$ and $\pi_k \circ d_k(\mathfrak{i}E_{ij}) = \mathfrak{i}E_{ij} - \mathfrak{i}\alpha' E'_{ji}$. For fixed i and j, note that the set $\{A_i, B_i, \pi_k \circ d_k(E_{ij}), \pi_k \circ d_k(\mathfrak{i}E_{ij})\}$ is a basis of V_{ij} over $\mathbb{R}$. Since $V(S^{-*}; C, D)$ is a direct sum of all V_{ij}, the set

$$\Omega_k = \{A_i, B_i \mid \max(1, k+1-m) \leq i \leq \min(k, l)\},$$

is linearly independent over $\mathbb{R}$. Moreover, since $V_{ij} \cap \Omega_k = \{A_i, B_i\}$ and $\dim_{\mathbb{R}}(V_{ij}) = 4$, the cardinality of $V_{ij} \cap \Omega_k$ is $\dim_{\mathbb{R}}(V_{ij})/2 = 2$. Summing over all i, we obtain the cardinality of Ω_k to be

$$\dim_{\mathbb{R}}(V_k)/2.$$

By Lemma 2.1,

$$\begin{aligned} \dim_{\mathbb{R}}(\ker(\pi_k \circ d_k)) &= \dim_{\mathbb{R}}(V_k) - \dim_{\mathbb{R}}(\operatorname{Im}(\pi_k \circ d_k)) \\ &= \dim_{\mathbb{R}}(V_k)/2. \end{aligned}$$

Hence, Ω_k is a basis of $\ker(\pi_k \circ d_k)$ over $\mathbb{R}$. From the form of the matrices $d(A_i) = d_k(A_i)$ and $d(B_i) = d_k(B_i)$ and the linear independence of $\{E_{ij}, E'_{ji} \mid i + j = k\} \subseteq V_{k-1}$ over $\mathbb{R}$, we see that

$$\Psi := d(\Omega_k)$$

is a linearly independent subset of V_{k-1} over $\mathbb{R}$. The set Ψ also spans $d(\ker(\pi_k \circ d_k))$ over $\mathbb{R}$.

We prove the following lemma.

Lemma 2.2. *Let $\alpha\overline{\alpha'} = 1$ and let q be the quotient map*

$$q : V_{k-1} \longrightarrow V_{k-1}/\operatorname{Im}(\pi_{k-1} \circ d_{k-1}).$$

If $k \leq m$, then $q(\Psi)$ spans $V_{k-1}/\operatorname{Im}(\pi_{k-1} \circ d_{k-1})$ over $\mathbb{R}$. If $k > m$, then $q(\Psi)$ is linearly independent over $\mathbb{R}$.

Proof. Fix k. By Lemma 2.1,

$$\operatorname{Im}(\pi_{k-1} \circ d_{k-1}) = \operatorname{Span}_{\mathbb{R}}\{E_{ij} + \alpha' E'_{ji}, \mathfrak{i}(E_{ij} - \alpha' E'_{ji}) \mid i + j = k\}.$$

This implies that $q(E_{ij}) = q(-\alpha' E'_{ji})$ and $q(\mathfrak{i}E_{ij}) = q(\mathfrak{i}\alpha' E'_{ji})$ for each pair i, j such that $i + j = k$. For $a \in \mathbb{C}$, it follows that $q(aE_{ij}) = q(-\bar{a}\alpha' E'_{ji})$. Let $A = \sum_{i+j=k+1} a_i E_{i,j-1} + a'_i E'_{j-1,i} \in V_{k-1}$ where $a_i, a'_i \in \mathbb{C}$. Then

$$q(A) = q\left(\sum_{i+j=k+1} (-\bar{a}_i\alpha' + a'_i) E'_{j-1,i}\right).$$

Thus, the quotients $q(\sum_{i+j=k+1} a_i E'_{j-1,i})$, with $a_i \in \mathbb{C}$, represent all the elements of $\operatorname{Im}(q)$.

Let $A_i = E_{ij} - \alpha' E'_{ji}, B_i = \mathfrak{i}(E_{ij} + \alpha' E'_{ji}) \in \Omega_k$ where $\max(1, k+1-m) \leq i \leq \min(k, l)$ and $j = j(i) = k + 1 - i$. Then

$$\begin{aligned} q \circ d(A_i) &= q(E'_{j-1,i} - \overline{\alpha'} E_{i-1,j}) \\ &= q(E'_{j-1,i} - \Re(\overline{\alpha'}) E_{i-1,j} - \Im(\overline{\alpha'}) \mathfrak{i} E_{i-1,j}) \\ &= q(E'_{j-1,i} + \Re(\overline{\alpha'}) \alpha' E'_{j,i-1} - \Im(\overline{\alpha'}) \mathfrak{i}\alpha' E'_{j,i-1}) \\ &= q(E'_{j-1,i} + (\alpha')^2 E'_{j,i-1}), \end{aligned}$$

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and likewise,

$$q \circ d(B_i) = q(-\mathfrak{i}(E'_{j-1,i} + (\alpha')^2 E'_{j,i-1})).$$

Suppose $k \leq m$. In this case, $1 \leq i \leq \min(k, l)$. We prove, by induction on i, that

$$\{q(E'_{k-i,i}), q(\mathfrak{i}E'_{k-i,i})\} \subset \operatorname{Span}_{\mathbb{R}} q(\Psi).$$

For the base case,

$$q \circ d(A_1) = q(E'_{k-1,1}),$$

and

$$q \circ d(B_1) = q(-\mathfrak{i}E'_{k-1,1}),$$

since $E'_{k,0} = 0$. Hence, $q(\Psi)$ contains $q(E'_{k-1,1})$ and $q(\mathfrak{i}E'_{k-1,1})$. The rest of the induction follows from the fact that $E'_{j,i-1}$ in the expressions for $q \circ d(A_i)$ and $q \circ d(B_i)$ above can be written as $E'_{k-(i-1),i-1}$. Hence, $q(\Psi)$ spans the quotients $q(\sum_{i+j=k+1} a_i E'_{j-1,i})$ over $\mathbb{R}$. Thus, $q(\Psi)$ spans $\operatorname{Im}(q)$ over $\mathbb{R}$.

Finally, we consider the case $m < k$. By the same argument above, we see that

$$q(\Psi) \cup \{q(E'_{j-1,1}), q(\mathfrak{i}E'_{j-1,1})\},$$

spans $q(V_{k-1})$ over $\mathbb{R}$. The cardinality of $q(\Psi) \cup \{q(E'_{j-1,1}), q(\mathfrak{i}E'_{j-1,1})\}$ is equal to $|\Omega_k| + 2 = 2(l-k+m+1)$. Moreover,

$$\begin{aligned} \dim_{\mathbb{R}} q(V_{k-1}) &= \dim_{\mathbb{R}} V_{k-1} - \dim_{\mathbb{R}} \operatorname{Im}(\pi_{k-1} \circ d_{k-1}) \\ &= \frac{1}{2} \dim_{\mathbb{R}} V_{k-1} \\ &= 2(l - k + m + 1). \end{aligned}$$

Therefore, $q(\Psi) \cup \{q(E'_{j-1,1}), q(\mathfrak{i}E'_{j-1,1})\}$ is a basis of $q(V_{k-1})$ over $\mathbb{R}$. Thus, $q(\Psi)$ is linearly independent over $\mathbb{R}$. ◻

Lemma 2.3. *Let $\alpha\overline{\alpha'} = 1$ and $k < l$. Then $V_{\leq k-1} \subset \operatorname{Im}(d_{\leq k})$.*

Proof. We prove this by induction on k. Clearly, the lemma holds when $k = 1$ as $V_{\leq 0} = 0$. Suppose that $V_{\leq k-2} \subset \operatorname{Im}(d_{\leq k-1})$ holds for $1 < k < l$. We show that $V_{\leq k-1} \subset \operatorname{Im}(d_{\leq k})$. By using the inductive hypothesis and the fact that $V_{\leq k-1} = V_{\leq k-2} \oplus V_{k-1}$, it is sufficient to show that $V_{k-1} \subset \operatorname{Im}(d_{\leq k})$. By Lemma 2.2, $\Psi \cup \operatorname{Im}(\pi_{k-1} \circ d_{k-1})$ spans V_{k-1} over $\mathbb{R}$ since $k < l \leq m$. From $\Psi \subset \operatorname{Im}(d_k)$ and the induction hypothesis, we have

$$\begin{aligned} \Psi \cup \operatorname{Im}(\pi_{k-1} \circ d_{k-1}) &\subset \operatorname{Im}(d_k) + \operatorname{Im}(\pi_{k-1} \circ d_{k-1}) \\ &\subset \operatorname{Im}(d_k) + \operatorname{Im}(d_{k-1}) + V_{\leq k-2} \\ &\subset \operatorname{Im}(d_k) + \operatorname{Im}(d_{\leq k-1}) \\ &\subset \operatorname{Im}(d_{\leq k}). \end{aligned}$$

In other words, the linear space $\operatorname{Im}(d_{\leq k})$ contains a basis of V_{k-1} over $\mathbb{R}$. Therefore, $V_{k-1} \subset \operatorname{Im}(d_{\leq k})$. ◻

Lemma 2.4. *The real dimension of $\ker(d_{\leq l+m-1})$ is given by*

$$\dim_{\mathbb{R}}(\ker(d_{\leq l+m-1})) = \begin{cases} 2\min(l, m) & \textit{if } \alpha\overline{\alpha'} = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

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Proof. Let $1 \leq k \leq l+m-1$. First, we assume $\alpha\overline{\alpha'} \neq 1$. By Lemma 2.1,

$$\operatorname{Im}(\pi_k \circ d_k) = V_k.$$

By induction on k, we show $\operatorname{Im}(d_{\leq k}) = V_{\leq k}$. For $k=1$, we have $\operatorname{Im}(d_{\leq 1}) = \operatorname{Im}(\pi_1 \circ d_1) = V_1 = V_{\leq 1}$. Suppose $k \geq 1$ is such that $\operatorname{Im}(d_{\leq k}) = V_{\leq k}$. Then

$$\begin{aligned}\operatorname{Im}(d_{\leq k+1}) &= \operatorname{Im}(d_{k+1}) \oplus \operatorname{Im}(d_{\leq k}) \\ &= \operatorname{Im}(\pi_{k+1} \circ d_{k+1}) \oplus \operatorname{Im}(d_{\leq k}) \\ &= V_{k+1} \oplus V_{\leq k} = V_{\leq k+1}.\end{aligned}$$

In particular, $\operatorname{Im}(d_{\leq l+m-1}) = V_{\leq l+m-1}$. Thus, $\dim_{\mathbb{R}}(\ker(d_{\leq l+m-1})) = 0$.

Now, we suppose that $\alpha\overline{\alpha'} = 1$. Recall that we assume $l \leq m$. Note that the subspaces $\ker(d_{\leq k})$ satisfy

$$\ker(d_{\leq 1}) \subseteq \ker(d_{\leq 2}) \subseteq \cdots \subseteq \ker(d_{\leq l+m-1}).$$

To prove the lemma, we claim that

$$\dim_{\mathbb{R}}(\ker(d_{\leq k})/\ker(d_{<k})) = \begin{cases} 2 & \text{if } k \leq l, \\ 0 & \text{otherwise.} \end{cases}$$

Using the above formula, we get

$$\dim_{\mathbb{R}}(\ker(d_{\leq l+m-1})) = 2l.$$

Recall that $\Psi = d(\Omega_k)$ spans $d(\ker(\pi_k \circ d_k))$. Moreover, $\dim_{\mathbb{R}}(\ker(\pi_k \circ d_k)) = \dim_{\mathbb{R}}(V_k)/2$. As such,

$$\dim_{\mathbb{R}}(\operatorname{Span}_{\mathbb{R}}(\Psi)) = \frac{1}{2}\dim_{\mathbb{R}}(V_k).$$

By Lemma 2.1,

$$\dim_{\mathbb{R}}(\operatorname{Im}(\pi_{k-1} \circ d_{k-1})) = \frac{1}{2}\dim_{\mathbb{R}}(V_{k-1}).$$

By Lemma 2.2, the sum $\operatorname{Span}_{\mathbb{R}}(\Psi) + \operatorname{Im}(\pi_{k-1} \circ d_{k-1})$ is the full space V_{k-1} if $k \leq m$. On the other hand, from the proof of the same lemma, we have $\dim_{\mathbb{R}}(\operatorname{Span}_{\mathbb{R}}(\Psi) + \operatorname{Im}(\pi_{k-1} \circ d_{k-1})) = \dim_{\mathbb{R}} V_{k-1} - 2$ if $k > m$. Since

$$\dim_{\mathbb{R}}(W_1 + W_2) = \dim_{\mathbb{R}}(W_1) + \dim_{\mathbb{R}}(W_2) - \dim_{\mathbb{R}}(W_1 \cap W_2),$$

where $W_1 = \operatorname{Span}_{\mathbb{R}}(\Psi)$ and $W_2 = \operatorname{Im}(\pi_{k-1} \circ d_{k-1})$, we obtain

$$\dim_{\mathbb{R}}(W_1 \cap W_2) = \begin{cases} \frac{1}{2}\left(\dim_{\mathbb{R}}(V_k) - \dim_{\mathbb{R}}(V_{k-1})\right) & \text{if } k \leq m, \\ \frac{1}{2}\left(\dim_{\mathbb{R}}(V_k) - \dim_{\mathbb{R}}(V_{k-1})\right) + 2 & \text{if } k > m. \end{cases}$$

Using the explicit formula for $\dim_{\mathbb{R}}(V_k)$, we obtain

$$\dim_{\mathbb{R}}(\operatorname{Span}_{\mathbb{R}}(\Psi) \cap \operatorname{Im}(\pi_{k-1} \circ d_{k-1})) = \begin{cases} 2 & \text{if } k \leq l, \\ 0 & \text{if } l < k. \end{cases}$$

Next, we show that

$$\dim_{\mathbb{R}}(\ker(d_{\leq k})/\ker(d_{<k})) = \dim_{\mathbb{R}}(\operatorname{Span}_{\mathbb{R}}(\Psi) \cap \operatorname{Im}(\pi_{k-1} \circ d_{k-1})).$$

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First, we exhibit that $\mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})$ contains $d \circ \pi_k(\ker(d_{\leq k}))$. Suppose $A \in \ker(d_{\leq k})$. Let $A = X + Y$ where $X \in V_{\leq k-1}$ and $Y \in V_k$. Then,

$$\begin{aligned} d \circ \pi_k(A) &= d(\pi_k(X)) + d(\pi_k(Y)) \\ &= d(\pi_k(Y)) \\ &= d(Y). \end{aligned}$$

Since $d(A) = 0$, then $d(Y) = -d(X) \in V_{\leq k-1}$. So $\pi_k \circ d_k(Y) = -\pi_k \circ d(X) = 0$, i.e. $Y \in \ker(\pi_k \circ d_k)$. This implies that $d \circ \pi_k(A) = d(Y) \in \mathrm{Span}_{\mathbb{R}}(\Psi)$. Moreover, $d \circ \pi_k(A) \in V_{k-1} \oplus V_k$ since $Y \in V_k$. Likewise, $d \circ \pi_k(A) = -d(X)$ implies that $d \circ \pi_k(A) \in V_{\leq k-1}$. Since $(V_{k-1} \oplus V_k) \cap V_{\leq k-1} = V_{k-1}$, then $d \circ \pi_k(A) \in V_{k-1}$. Consequently, $d \circ \pi_k(A) = -\pi_{k-1} \circ d(X)$. Now, we can write $X = \pi_{k-1}(X) + X'$ where X' is in $V_{\leq k-2}$ because $X \in V_{\leq k-1}$. Since the map d is closed on $V_{\leq k-2}$, i.e., the image of $d|_{V_{\leq k-2}}$ is inside $V_{\leq k-2}$, we have

$$\begin{aligned} \pi_{k-1} \circ d(X) &= \pi_{k-1} \circ d(\pi_{k-1}(X)) + \pi_{k-1} \circ d(X') \\ &= \pi_{k-1} \circ d(\pi_{k-1}(X)) \\ &= \pi_{k-1} \circ d_{k-1}(\pi_{k-1}(X)). \end{aligned}$$

This implies that $d \circ \pi_k(A) \in \mathrm{Im}(\pi_{k-1} \circ d_{k-1})$. Therefore, we get

$$d \circ \pi_k(A) \in \mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1}).$$

Define $\Delta : \ker(d \leq k) \longrightarrow \mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})$ to be the restriction of the map $d \circ \pi_k$ on $\ker(d \leq k)$. Note that there is an embedding

$$\ker(d \leq k)/\ker(\Delta) \hookrightarrow \mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1}).$$

We claim that $\ker(\Delta) = \ker(d_{<k})$ to conclude that

$$\dim_{\mathbb{R}}(\ker(d_{\leq k})/\ker(d_{<k})) \leq \dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})).$$

Indeed, since Ω_k and $\Psi = d(\Omega_k)$ are both linearly independent over $\mathbb{R}$ and $|\Omega_k| = |\Psi|$, we have

$$\dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(\Omega_k)) = \dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(d(\Omega_k))).$$

This implies that $d : \mathrm{Span}_{\mathbb{R}}(\Omega_k) \longrightarrow \mathrm{Span}_{\mathbb{R}}(d(\Omega_k))$ is an injective linear map. Moreover,

$$\ker(d_k) \subseteq \ker(\pi_k \circ d_k) = \mathrm{Span}_{\mathbb{R}}(\Omega_k).$$

Thus, $\ker(d_k) = 0$. Consequently,

$$\begin{aligned} \ker(\Delta) &= \ker(d_{<k}) \oplus \ker(d_k) \\ &= \ker(d_{<k}). \end{aligned}$$

Recall that, if $l < k$, then $\dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})) = 0$. Hence, if $l < k$, then

$$\dim_{\mathbb{R}}(\ker(d_{\leq k})/\ker(d_{<k})) = 0.$$

Suppose $k \leq l$. We show that the map Δ is surjective. Let $A \in \mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})$. Then $A = d(\sum_i a_i A_i + b_i B_i)$ where $a_i, b_i \in \mathbb{R}$ and $A_i, B_i \in \Omega_k$.

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Moreover, there exists $A' \in V_{k-1}$ such that $A = \pi_{k-1} \circ d(A')$. Let $\tilde{A} = \sum_i a_i A_i + b_i B_i - A' \in V_{\leq k}$. Then $d(\tilde{A}) = A - d(A') \in V_{\leq k-1}$. Since the projection of $d(A')$ onto V_{k-1} is A, then necessarily $d(\tilde{A}) \in V_{\leq k-2}$.

By Lemma 2.3, we have $V_{\leq k-2} \subset \mathrm{Im}(d_{\leq k-1})$ for all $k \leq l$. So, there exists $\hat{A} \in V_{\leq k-1}$ such that $d(\tilde{A}) = d(\hat{A})$. Thus, $\tilde{A} - \hat{A} \in \ker(d_{\leq k})$ but $\tilde{A} - \hat{A} \notin \ker(d_{<k})$ because $\pi_k(\tilde{A} - \hat{A}) = \sum_i a_i A_i + b_i B_i$. In other words, we obtain an element of $\ker(d_{\leq k})$ whose image via the map $d \circ \pi_k$ is $A = d(\sum_i a_i A_i + b_i B_i)$. This means that Δ is surjective. Thus,

$$\dim_{\mathbb{R}}(\ker(d_{\leq k})/\ker(d_{<k})) \geq \dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})).$$

Therefore, if $k \leq l$, we have

$$\dim_{\mathbb{R}}(\ker(d_{\leq k})/\ker(d_{<k})) = \dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(\Psi) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})) = 2. \qquad \square$$

LEMMA 2.5. *Let $S \in \mathrm{GL}_n(\mathbb{C})$. Let $C = \{v_1, \ldots, v_l\}$ and $D = \{u_1, \ldots, u_m\}$ be Jordan chains of SS^{-*} associated with the eigenvalues α and α', respectively. Assume that $C \cup D$ is linearly independent. Then*

$$\dim_{\mathbb{R}}(\mathfrak{u}_S \cap \mathbb{V}(S^{-*}; C, D)) = \begin{cases} 2\min(l, m) & \text{if } \alpha\overline{\alpha'} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that

$$\mathbb{V}(S^{-*}; C, D) = \sum_{k=1}^{l+m-1} V_k = V_{\leq l+m-1}.$$

Thus,

$$\ker(d_{\leq l+m-1}) = \mathfrak{u}_S \cap \mathbb{V}(S^{-*}; C, D).$$

Therefore,

$$\dim_{\mathbb{R}}(\ker(d_{\leq l+m-1})) = \dim_{\mathbb{R}}(\mathfrak{u}_S \cap \mathbb{V}(S^{-*}; C, D)).$$

The conclusion follows from Lemma 2.4. $\square$

2.2. CASE 2: Same Jordan block. Let $S \in \mathrm{GL}_n(\mathbb{C})$. Let $C = \{v_1, \ldots, v_l\}$ be a Jordan chain of SS^{-*} associated with the eigenvalue α of SS^{-*}. For $1 \leq i < l$, we have

$$SS^{-*}v_{i+1} = \alpha v_{i+1} + v_i,$$

and

$$SS^{-*}v_1 = \alpha v_1.$$

For $1 \leq i, j \leq l$, define

$$E_{ij} = v_i(S^{-*}v_j)^*$$

and set

$$\mathbb{V}(S^{-*}; C, C) = \sum_{1 \leq i,j \leq l} \mathbb{C}E_{ij}.$$

For $1 \leq k \leq 2l - 1$, define $V_k = \sum_{i+j=k+1} \mathbb{C}E_{ij}$. Note that

$$\dim_{\mathbb{R}}(V_k) = \begin{cases} 2k & \text{if } k \leq l, \\ 4l - 2k & \text{if } l < k. \end{cases}$$

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Define the maps d, d_k, and π_k as in Section 2.1. If $i+j=k+1$, then

$$\begin{aligned} \pi_k \circ d_k(E_{ij}) &= E_{ij} + \alpha E_{ji}, \\ \pi_k \circ d_k(\mathfrak{i} E_{ij}) &= \mathfrak{i} E_{ij} - \mathfrak{i}\alpha E_{ji}. \end{aligned}$$

Analogous to Lemma 2.1, if $\alpha\overline{\alpha} \neq 1$, then $\operatorname{Im}(\pi_k \circ d_k) = V_k$. If $\alpha\overline{\alpha} = 1$, observe that

$$\begin{bmatrix} E_{ji} + \alpha E_{ij} \\ \mathfrak{i}(E_{ji} - \alpha E_{ij}) \end{bmatrix} = \begin{bmatrix} \Re(\alpha) & \Im(\alpha) \\ \Im(\alpha) & -\Re(\alpha) \end{bmatrix} \begin{bmatrix} E_{ij} + \alpha E_{ji} \\ \mathfrak{i}(E_{ij} - \alpha E_{ji}) \end{bmatrix}.$$

Therefore, $\operatorname{Im}(\pi_k \circ d_k)$ is spanned over $\mathbb{R}$ by the set

$$\{E_{ij} + \alpha E_{ji}, \mathfrak{i}(E_{ij} - \alpha E_{ji}) \mid i \leq j, i+j = k+1\}.$$

In particular,

$$\dim_{\mathbb{R}}(\operatorname{Im}(\pi_k \circ d_k)) = \begin{cases} \dim_{\mathbb{R}}(V_k) & \text{if } \alpha\overline{\alpha} \neq 1, \\ \dfrac{1}{2}\dim_{\mathbb{R}}(V_k) & \text{if } \alpha\overline{\alpha} = 1. \end{cases}$$

Assume $\alpha\overline{\alpha} = 1$. Fix $1 \leq k \leq 2l-1$ and suppose $i+j=k+1$, where $1 \leq i, j \leq l$. Let

$$A_i = E_{ij} - \alpha E_{ji},$$

and

$$B_i = \mathfrak{i}(E_{ij} + \alpha E_{ji}).$$

Then

$$d(A_i) = E_{j-1,i} - \overline{\alpha} E_{i-1,j},$$

and

$$d(B_i) = -\mathfrak{i}(E_{j-1,i} + \overline{\alpha} E_{i-1,j}).$$

Hence, $A_i, B_i \in \ker(\pi_k \circ d_k)$. If k is odd, then $\{A_{\frac{k+1}{2}}, B_{\frac{k+1}{2}}\}$ is linearly dependent over $\mathbb{R}$. The set $\widetilde{\Omega_k}$ defined below is a basis of $\ker(\pi_k \circ d_k)$ over $\mathbb{R}$:

$$\begin{aligned} \widetilde{\Omega_k} &= \{A_i, B_i \mid \max(1, k+1-l) \leq i < (k+1)/2\} \\ &\cup \begin{cases} \emptyset & \text{if } k \text{ is even}, \\ \{B_{\frac{k+1}{2}}\} & \text{if } \alpha = 1 \text{ and } k \text{ is odd}, \\ \{A_{\frac{k+1}{2}}\} & \text{if } \alpha \neq 1 \text{ and } k \text{ is odd}. \end{cases} \end{aligned}$$

The set $\widetilde{\Psi} = d(\widetilde{\Omega_k})$ is contained in V_{k-1} and it spans $d(\ker(\pi_k \circ d_k))$ over $\mathbb{R}$.

Lemma 2.6. *Let $\alpha\overline{\alpha} = 1$ and let q be the quotient map*

$$q : V_{k-1} \longrightarrow V_{k-1}/\operatorname{Im}(\pi_{k-1} \circ d_{k-1}).$$

If $k \leq l$, then $q(\widetilde{\Psi})$ spans $V_{k-1}/\operatorname{Im}(\pi_{k-1} \circ d_{k-1})$ over $\mathbb{R}$. If $k > l$, then $q(\widetilde{\Psi})$ is linearly independent over $\mathbb{R}$.

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Proof. We consider the quotient map $q : V_{k-1} \to V_{k-1}/\operatorname{Im}(\pi_{k-1} \circ d_{k-1})$. The image set $\operatorname{Im}(q)$ is represented by the quotients $q(\sum_{i\leq \frac{k}{2}, i+j=k+1} b_i E_{j-1,i})$, where $b_i \in \mathbb{C}$. Indeed, recall that $\operatorname{Im}(\pi_{k-1} \circ d_{k-1})$ is spanned over $\mathbb{R}$ by the set

$$\{E_{ij} + \alpha E_{ji}, \mathfrak{i}(E_{ij} - \alpha E_{ji}) \mid i \leq j, i+j=k\}.$$

It follows that

$$q(E_{ij}) = q(-\alpha E_{ji}),$$

and

$$q(\mathfrak{i}E_{ij}) = q(\mathfrak{i}\alpha E_{ji}),$$

for each pair i, j such that $i + j = k$ and $i \leq j$. Moreover, for such a pair, $q(aE_{ij}) = q(-\overline{a}\alpha E_{ji})$ if $a \in \mathbb{C}$. Set $E_{\frac{k}{2},\frac{k}{2}} = 0$ if k is odd. Let $A = \sum_{i+j=k} a_i E_{ij} = \sum_{1\leq i\leq k-1} a_i E_{ij} \in V_{k-1}$ where $a_i \in \mathbb{C}$. Then

$$q(A) \quad = \quad \sum_{i<\frac{k}{2}} q(a_i E_{ij}) + q(a_{\frac{k}{2}} E_{\frac{k}{2},\frac{k}{2}}) + \sum_{i>\frac{k}{2}} q(a_i E_{ij}).$$

Since $i + j = k$, then $i < \frac{k}{2}$ implies $j > \frac{k}{2}$. We have

$$\begin{aligned} q(A) &= \sum_{j>\frac{k}{2}} (-\overline{a_i}\alpha E_{ji}) + q(a_{\frac{k}{2}} E_{\frac{k}{2},\frac{k}{2}}) + \sum_{i>\frac{k}{2}} q(a_i E_{ij}) \\ &= \sum_{j>\frac{k}{2}} q\left((-\bar{a}_i\alpha + a_j)E_{ji}\right) + q(a_{\frac{k}{2}} E_{\frac{k}{2},\frac{k}{2}}). \end{aligned}$$

By re-indexing,

$$q(A) = \sum_{i\leq\frac{k}{2}, i+j=k+1} q\left(b_i E_{j-1,i}\right),$$

for some $b_i \in \mathbb{C}$ where the sum runs over $\max(1, k+1-l) \leq i \leq \min(k, l)$.

Now, for $A_i, B_i \in \widetilde{\Omega_k}$, we have

$$q \circ d(A_i) = q(E_{j-1,i} + \alpha^2 E_{j,i-1}),$$

$$q \circ d(B_i) = q(-\mathfrak{i}(E_{j-1,i} + \alpha^2 E_{j,i-1})).$$

The rest of the proof follows as in Lemma 2.2. ∎

Let $\alpha\overline{\alpha} = 1$. We apply the same argument used in Lemma 2.4. Note that the cardinality of $\widetilde{\Omega_k}$ is given by

$$|\widetilde{\Omega_k}| = \begin{cases} k & \text{if } k \leq l, \\ 2l - k & \text{if } k > l. \end{cases}$$

Using this, we can show that

$$\dim_{\mathbb{R}}(\operatorname{Span}_{\mathbb{R}} \widetilde{\Psi}) = \frac{1}{2} \dim_{\mathbb{R}} V_k.$$

By Lemma 2.6, if $k \leq l$, then

$$\dim_{\mathbb{R}}(\operatorname{Span}_{\mathbb{R}}(\widetilde{\Psi}) \cap \operatorname{Im}(\pi_{k-1} \circ d_{k-1})) = \frac{1}{2} \dim_{\mathbb{R}}(V_k) - \frac{1}{2} \dim_{\mathbb{R}}(V_{k-1}) = 1.$$

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Moreover, if $k > l$, then

$$\mathrm{Span}_{\mathbb{R}}(\widetilde{\Psi}) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1}) = \{0\}.$$

So

$$\dim_{\mathbb{R}}(d(\ker(\pi_k \circ d_k)) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})) = 0.$$

To summarize, we have

$$\dim_{\mathbb{R}}(d(\ker(\pi_k \circ d_k)) \cap \mathrm{Im}(\pi_{k-1} \circ d_{k-1})) = \begin{cases} 1 & \text{if } k \leq l, \\ 0 & \text{if } l < k, \end{cases}$$

and consequently,

$$\dim_{\mathbb{R}}(\ker(d_{\leq k}) / \ker(d_{<k})) = \begin{cases} 1 & \text{if } k \leq l, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we obtain the following as an analogue of Lemma 2.4.

Lemma 2.7. *The real dimension of* $\ker(d_{\leq 2l-1})$ *is given by*

$$\dim_{\mathbb{R}}(\ker(d_{\leq 2l-1})) = \begin{cases} l & \text{if } \alpha\overline{\alpha} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have the following result parallel to Lemma 2.5.

Lemma 2.8. *Let* $S \in \mathrm{GL}_n(\mathbb{C})$. *Let* $C = \{v_1, \ldots, v_l\}$ *be a Jordan chain of* SS^{-*} *associated with the eigenvalue* α *of* SS^{-*}. *Then*

$$\dim_{\mathbb{R}}(\mathfrak{u}_S \cap \mathbb{V}(S^{-*}; C, C)) = \begin{cases} l & \text{if } \alpha\overline{\alpha} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Combining Lemmas 2.5 and 2.8 yields Theorem 1.1. Indeed, let $\mathscr{C}$ be a set of Jordan chains of SS^{-*} such that

i. for each eigenvalue β of SS^{-*} and $m \in \mathbb{N}$, there are $\mathrm{mult}(SS^{-*}; \beta, m)$ Jordan chains in $\mathscr{C}$ of length m and associated with β; and
ii. the union of the Jordan chains in $\mathscr{C}$ is linearly independent over $\mathbb{C}$.

Then $\mathscr{C}$ yields a basis of $\mathbb{C}^n$ over $\mathbb{C}$. For $C, D \in \mathscr{C}$, we have

$$\begin{aligned} \mathbb{V}(S^{-*}; C, D) &= \mathbb{V}(S^{-*}; D, C) \\ &= \mathrm{Span}_{\mathbb{R}}(\{u(S^{-*}v)^* \mid (u, v) \in (C \times D) \cup (D \times C)\}. \end{aligned}$$

Let $\mathscr{C}^{(2)}$ be the set of all subsets of $\mathscr{C}$ with size 2. Thus,

$$\begin{aligned} M_n(\mathbb{C}) &= \bigoplus_{C, D \in \mathscr{C}} \mathrm{Span}_{\mathbb{R}}(\{u(S^{-*}v)^* \mid u \in C, v \in D\}) \\ &= \bigoplus_{\{C, D\} \in \mathscr{C}^{(2)}} \mathbb{V}(S^{-*}; C, D) \oplus \bigoplus_{C \in \mathscr{C}} \mathbb{V}(S^{-*}; C, C). \end{aligned}$$

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Recall that $\ker(d) = \mathfrak{u}_S$ and that $\mathbb{V}(S^{-*}; C, D)$ is closed under the linear map d for $C, D \in \mathscr{C}$. Thus,

$$\begin{aligned}
\mathfrak{u}_S &= \mathfrak{u}_S \cap \left(\bigoplus_{\{C,D\}\in\mathscr{C}^{(2)}} \mathbb{V}(S^{-*}; C, D) \oplus \bigoplus_{C\in\mathscr{C}} \mathbb{V}(S^{-*}; C, C) \right) \\
&= \bigoplus_{\{C,D\}\in\mathscr{C}^{(2)}} (\mathfrak{u}_S \cap \mathbb{V}(S^{-*}; C, D)) \oplus \bigoplus_{C\in\mathscr{C}} (\mathfrak{u}_S \cap \mathbb{V}(S^{-*}; C, C)).
\end{aligned}$$

Hence,

$$\dim_{\mathbb{R}}(\mathfrak{u}_S) = \sum_{\{C,D\}\in\mathscr{C}^{(2)}} \dim_{\mathbb{R}}(\mathfrak{u}_S \cap \mathbb{V}(S^{-*}; C, D)) + \sum_{C\in\mathscr{C}} \dim_{\mathbb{R}}(\mathfrak{u}_S \cap \mathbb{V}(S^{-*}; C, C)).$$

For a chain $C \in \mathscr{C}$, let $l(C)$ be its length and let $\alpha(C)$ be the eigenvalue associated with it. We have

$$\begin{aligned}
\dim_{\mathbb{R}}(\mathfrak{u}_S) &= \sum_{\substack{\{C,D\}\in\mathscr{C}^{(2)} \\ \alpha(C)\overline{\alpha(D)}=1}} 2\min(l(C), l(D)) + \sum_{\substack{C\in\mathscr{C} \\ \alpha(C)\overline{\alpha(C)}=1}} l(C) \\
&= \sum_{\substack{C,D\in\mathscr{C} \\ \alpha(C)\overline{\alpha(D)}=1}} \min(l(C), l(D)).
\end{aligned}$$

Finally, counting the Jordan chains in the sum above, we obtain

$$\dim_{\mathbb{R}}(\mathfrak{u}_S) = \sum_{m,l=1}^{n} \sum_{\beta\in\Lambda} \min(l, m)\,\mathrm{mult}(SS^{-*}; \beta, l)\,\mathrm{mult}(SS^{-*}; \overline{\beta}^{-1}, m).$$

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REFERENCES

[1] M.N.M. Abara, D.I. Merino, and A.T. Paras. ϕ_S-Orthogonal matrices. *Linear Algebra Appl.*, 423(11):2834–2846, 2010.
[2] R. Bronson. *Matrix Methods: An Introduction, 2nd edition.* Academic Press, Inc., New York, 1991.
[3] J. Caalim, C. Canlubo, and Y. Tanaka. On the Lie algebra of associated to S-unitary matrices. *Linear Algebra Appl.*, 553:167–181, 2018.
[4] J. Caalim, V. Futorny, V.V. Sergeichuk, and Y. Tanaka. Isometric and selfadjoint operators on a vector space with nondegenerate diagonalizable form. *Linear Algebra Appl.*, 587:92–110, 2020.
[5] R.J. De la Cruz, D.I. Merino, and A.T. Paras. The ϕ_S polar decomposition of matrices. *Linear Algebra Appl.*, 434(1):4–13, 2011.
[6] R.J. De la Cruz and H. Faßbender. On the diagonalizability of a matrix by a symplectic equivalence, similarity or congruence transformation. *Linear Algebra Appl.*, 496:288–306, 2016.
[7] F. De Terán and F.M. Dopico. The equation $XA + AX^* = 0$ and the dimension of *congruence orbits. *Electron. J. Linear Algebra*, 22:448–465, 2011.
[8] F. De Terán and F.M. Dopico. The equation $XA + AX^T = 0$ and its application to the theory of orbits. *Linear Algebra Appl.*, 434:44–67, 2011.
[9] R. Horn and V.V. Sergeichuk. Canonical forms for complex matrix congruence and *congruence. *Linear Algebra Appl.*, 416:1010–1032, 2006.
[10] P. Saltenberger. Structure-preserving diagonalization of matrices in indefinite product spaces. *Electronic Journal of Linear Algebra*, 36: 21–37, 2020.