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ON GENERAL MATRICES HAVING THE PERRON-FROBENIUS PROPERTY*

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Dedicated to Hans Schneider on the occasion of his 80^{th} *birthday*

Abstract. A matrix is said to have the Perron-Frobenius property if its spectral radius is an eigenvalue with a corresponding nonnegative eigenvector. Matrices having this and similar properties are studied in this paper as generalizations of nonnegative matrices. Sets consisting of such generalized nonnegative matrices are studied and certain topological aspects such as connectedness and closure are proved. Similarity transformations leaving such sets invariant are completely described, and it is shown that a nonnilpotent matrix eventually capturing the Perron-Frobenius property is in fact a matrix that already has it.

Key words. Perron-Frobenius property, Generalization of nonnegative matrices, Eventually nonnegative matrices, Eventually positive matrices.

AMS subject classifications. 15A48.

1. Introduction. A real matrix A is called nonnegative (respectively, positive) if it is entry-wise nonnegative (respectively, positive) and we write $A \geq 0$ (respectively, $A > 0$). This notation and nomenclature is also used for vectors. A column or a row vector v is called *semipositive* if v is nonzero and nonnegative. Likewise, if v is nonzero and entry-wise nonpositive then v is called *seminegative*. We denote the spectral radius of a matrix A by $\rho(A)$.

Following [29], we say that a real square matrix A has the *Perron-Frobenius property* if $Av = \rho(A)v$ for some semipositive vector v. In the latter case, we call v a *right Perron-Frobenius eigenvector* for A. Similarly, if $u^T A = \rho(A)u^T$ for some semipositive vector u then we call u a *left Perron-Frobenius eigenvector* for A.

The Perron-Frobenius property is naturally associated with nonnegative matrices. Perron [21] proved that a square positive matrix has the following properties:

1. Its spectral radius is a simple positive eigenvalue.

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2. The eigenvector corresponding to the spectral radius can be chosen to be positive (called a Perron vector).
3. No other eigenvalue has the same modulus.

This result was extended by Frobenius [9] to nonnegative irreducible matrices and consequently to nonnegative matrices, using a perturbation argument. In the latter case, the spectral radius is an eigenvalue with a nonnegative eigenvector.

These results and other extensions, all of which are known now as the Perron-Frobenius theory, have been widely applied to problems with nonnegative matrices, and also with M-matrices and H-matrices; see, e.g., the monographs [2], [14], [25], [30]. Applications include stochastic processes [25], Markov chains [28], population models [18], solution of partial differential equations [1], and asynchronous parallel iterative methods [10], among others.

In this paper, we are interested in matrices having the Perron-Frobenius property but which are not necessarily nonnegative.

We begin by mentioning some relevant earlier work. A real square matrix A is said to be *eventually nonnegative* (respectively, *eventually positive*) if $A^k \geq 0$ (respectively, $A^k > 0$) for all $k \geq k_0$ for some positive integer k_0. Eventually positive matrices do satisfy properties 1–3 and in general, nonnilpotent eventually nonnegative matrices possess the Perron-Frobenius property. We note here that an eventually nonnegative matrix A which is nilpotent may not have the Perron-Frobenius property. We illustrate this with the following example.

Example 1.1. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, then $A^2 = 0$. Hence, A is an eventually nonnegative matrix which is nilpotent and 0 is its only eigenvalue. Moreover, all the eigenvectors of A are of the form $[t, -t]^T$ for some $t \in \mathbb{R}$, $t \neq 0$. Therefore, A does not possess the Perron-Frobenius property. On the other hand, the zero matrix possesses the Perron-Frobenius property despite the fact that it is nilpotent. Hence, a nilpotent matrix may or may not have the Perron-Frobenius property.

Friedland [8] showed that for eventually nonnegative matrices the spectral radius is an eigenvalue. Such an eigenvalue and any other eigenvalue whose modulus is equal to the spectral radius is called a *dominant* eigenvalue. Furthermore, if a matrix has only one dominant eigenvalue (regardless of its multiplicity), then we call such an eigenvalue *strictly dominant*.

Zaslavsky, McDonald, and Tam [31], [32] studied the Jordan form of eventually nonnegative matrices. Carnochan Naqvi and McDonald [5] studied combinatorial properties of eventually nonnegative matrices whose index is 0 or 1 by considering their Frobenius normal forms. Eschenbach and Johnson [7] studied sign patterns of a

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matrix requiring the spectral radius to be an eigenvalue. They called such a property the *Perron property*. Other earlier papers looking at issues relating to the spectral radius being an eigenvalue, at positive or nonnegative corresponding eigenvector, or at matrices with these properties, include [11], [12], [13], [16], [17], [19], [26], [27], [29].

We mention in passing the work of Rump [23], [24], who generalized the concept of a positive dominant eigenvalue, but this is not related to the questions addressed in this paper.

Different nomenclature for different "Perron properties" appear in the literature. In [7], as we already mentioned, the *Perron property* stands for having the spectral radius as an eigenvalue, whereas, in [11] the *weak Perron property* stands for having the spectral radius as a simple positive and strictly dominant eigenvalue. In [19], the *Perron-Frobenius property* stands for having the spectral radius as a positive eigenvalue with a nonnegative eigenvector; this is also the definition used in [6]. On the other hand, in this paper we say that a real matrix A possesses the *Perron-Frobenius property* if $\rho(A)$ (whether it is zero or positive) is an eigenvalue of A having a nonnegative eigenvector (which is the same as the definition introduced in [29]). Moreover, we say that A possesses the *strong Perron-Frobenius property* if A has a simple, positive, and strictly dominant eigenvalue with a positive eigenvector (which is the same as the definition introduced in [19]).

Following [16], we let PFn denote the collection of $n \times n$ real matrices whose spectral radius is a simple positive and strictly dominant eigenvalue having positive left and right eigenvectors. Equivalently, we can say that PFn is the collection of matrices A such that both A and A^T possess the strong Perron-Frobenius property. Similarly, WPFn denotes the collection of $n \times n$ real matrices whose spectral radius is an eigenvalue having nonnegative left and right eigenvectors. Equivalently, WPFn is the collection of matrices A such that both A and A^T possess the Perron-Frobenius property.

In this paper, we address several issues relating to matrices having the Perron-Frobenius property and the sets consisting of these matrices. We first show how PFn, WPFn, and other sets of matrices having the Perron-Frobenius property relate to one another (section 2). In section 3, we consider an extension of the Nonnegative Basis theorem (due to Rothblum [22]) to an eventually nonnegative matrix with index at most 1.

Eventually nonnegative matrices generalize nonnegative matrices, and WPFn generalizes eventually nonnegative matrices. A natural question is then, what can be said about matrices that are "eventually in WPFn" or that "eventually possess" the Perron-Frobenius property? For nonnilpotent matrices, we show that "eventually possessing" the Perron-Frobenius property is actually the same as possessing

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the Perron-Frobenius property (section 4). We completely describe similarity transformations preserving the Perron-Frobenius property (section 5) and we study some topological aspects such as connectedness and closure of the collection of matrices having the Perron-Frobenius property and other sets of interest (section 6). In this latter section, we prove a new result which asserts that a given pair of semipositive vectors with a nonzero inner product can be mapped to a pair of positive vectors by real orthogonal matrices arbitrarily close to the identity. This new tool may be useful in other contexts as well. Applications to generalizations of M-matrices are presented in the forthcoming paper [6].

2. Relations between sets. We present in this short section inclusion relations between the different sets defined in the introduction.

We begin by mentioning that

$$\text{PFn} = \{\text{Eventually Positive Matrices in } \mathbb{R}^{n\times n}\}, \tag{2.1}$$

which follows from [32, Theorem 4.1 and Remark 4.2].

The proof of the following lemma can be found in [19]. Here, we have added the necessary hypothesis of having at least one nonzero eigenvalue or equivalently having a positive spectral radius.

LEMMA 2.1. *If A is a real $n\times n$ eventually nonnegative matrix that has a nonzero eigenvalue, then $\rho(A)$ is a positive eigenvalue of A with corresponding nonnegative right and left eigenvectors. Hence, $A\in$ WPFn.*

Observe that from Example 1.1 and Lemma 2.1 all eventually nonnegative matrices are inside WPFn with the exception of some nilpotent matrices. In fact, the set of nonnilpotent eventually nonnegative matrices is a proper subset of WPFn as we show in the following proposition.

PROPOSITION 2.2. *The collection of $n \times n$ nonnilpotent eventually nonnegative matrices ($n \geq 2$) is properly contained in WPFn.*

Proof. It suffices to find a matrix A in WPFn which is not eventually nonnegative. Consider the matrix $A = J \oplus [-1]$ where J is the matrix of dimension $(n-1)$ having all its entries equal to 1. Then, $A^k = [(n-1)^{(k-1)}J] \oplus [(-1)^k]$. Clearly, A is not eventually nonnegative because the (n,n)-entry of A keeps alternating signs. However, $A \in$ WPFn since $\rho(A) = n-1$ and there is a semipositive vector $v = [1,\ldots,1,0]^T \in \mathbb{R}^n$ satisfying $v^TA = \rho(A)v^T$ and $Av = \rho(A)v$. $\square$

Hence, Proposition 2.2 establishes that all the inclusions are proper in the following statement:

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$$\begin{aligned}\mathrm{PFn} &= \{\text{Eventually Positive Matrices in } \mathbb{R}^{n\times n}\}\\ &\subset \{\text{Nonnilpotent eventually nonnegative matrices in } \mathbb{R}^{n\times n}\}\\ &\subset \mathrm{WPFn}.\end{aligned}$$

Moreover, it turns out that an irreducible matrix in WPFn does not have to be eventually nonnegative as Examples 2.4 and 2.5 below show.

The following theorem is due to Zaslavsky and Tam [32, Theorem 3.6]. We present it here for completeness, since it is a useful tool to construct irreducible matrices in WPFn which are not eventually nonnegative.

In Theorem 2.3 below, we denote the multiset of all elementary Jordan blocks of an $n \times n$ complex matrix A by $\mathcal{U}(A)$, we say that an $n \times n$ complex matrix A is eventually real if A^k is real for all integers $k \geq k_0$ for some positive integer k_0, and we denote the 1×1 elementary Jordan block corresponding to 0 by $J_1(0)$.

Theorem 2.3. *Let A be an $n \times n$ complex matrix.*

(i) The matrix A can be expressed uniquely in the form $B + C$, where B is a matrix in whose Jordan form the singular elementary Jordan blocks, if any, are all 1×1 and C is a nilpotent matrix such that $BC = CB = 0$.
(ii) With B, C as given in (i), the collection of non-singular elementary Jordan blocks in $\mathcal{U}(A)$ is the same as the collection of non-singular elementary Jordan blocks in $\mathcal{U}(B)$, and the collection of singular elementary Jordan blocks in $\mathcal{U}(A)$ can be obtained from $\mathcal{U}(C)$ by removing from it r $J_1(0)$'s, where $r = \text{rank } B$.
(iii) With B as given in (i), A is eventually real (eventually non-negative, eventually positive) if and only if B is real (eventually non-negative, eventually positive).

The following example was inspired by [5, Example 3.1], and its construction follows from Theorem 2.3.

Example 2.4. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix},$$

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$$\text{and } C = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that A is an irreducible matrix. Also, note that $\rho(A) = 2$ and that for $v = [2, 2, 1, 1]^T$ and $w = [1, 1, 0, 0]^T$, we have that $Av = \rho(A)v$ and $w^T A = \rho(A)w^T$. Thus, A is an irreducible matrix in WPF4. Furthermore, it is easy to see that $A = B + C$ and that $BC = CB = C^2 = 0$. Hence, $A^j = B^j$ for all $j \geq 2$. But B is not eventually nonnegative since the lower right 2×2 diagonal block of B^j keeps alternating signs. Thus, A is not eventually nonnegative.

In the following example, we present a nonsingular irreducible matrix A in WPF3 which is not eventually nonnegative. Unlike Example 2.4, we do not make use of Theorem 2.3 to show that A is not eventually nonnegative because in this case the decomposition $A = B + C$ given by Theorem 2.3 is simply $A = A + 0$. More precisely, we show in Example 2.5 below that there is a matrix A in WPF3 that satisfies the following: A is not eventually nonnegative although the spectral radius of A is a simple positive (but not strictly dominant) eigenvalue of A having corresponding positive right and left Perron-Frobenius eigenvectors.

Example 2.5. Consider the symmetric matrix

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

By direct calculation, one can find that the eigenvalues of A are 2, -2, and -1. Hence, $\rho(A) = 2$ is a simple positive eigenvalue which is not strictly dominant. Moreover, $v = [1, 1, 2]^T$ is a positive right and left eigenvector of A that corresponds to $\rho(A)$. For every odd positive integer k, the matrix A^k is not nonnegative since trace $A^k = 2^k + (-2)^k + (-1)^k = -1 < 0$. Hence, A is not eventually nonnegative.

We end this section by noting that the set of $n \times n$ matrices having the spectral radius as an eigenvalue with a corresponding positive eigenvector properly includes the set PFn and has elements outside WPFn. This can be seen from the following example which is [19, Example 2.1].

Example 2.6. Consider the following matrix:

$$A = \frac{1}{5} \begin{bmatrix} 5 & 10 & 5 \\ -2 & 5 & 5 \\ -2 & 25 & 40 \end{bmatrix}.$$

The matrix A possesses the strong Perron-Frobenius property, whereas its transpose A^T does not even possess a nonnegative eigenvector corresponding to $\rho(A)$. All the

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eigenvectors of A^T corresponding to $\rho(A)$ have both positive and negative entries. Hence $A \notin$ WPF3.

3. The algebraic eigenspace and the classes of an eventually nonnegative matrix. Let $\sigma(A)$ denote the spectrum of a matrix A. The index of a matrix A is the smallest nonnegative integer k such that rank A^k = rank A^{k+1}.

Our main result in this section is Theorem 3.8 which generalizes Rothblum's result [22] on the algebraic eigenspace of a nonnegative matrix and its basic classes. The tools we use in its proof are taken from the work of Carnochan Naqvi and McDonald [5], who showed that if A is an eventually nonnegative matrix whose index is 0 or 1 then the matrices A and A^g share some combinatorial properties for large prime numbers g. We review those results here in a slightly more general form, namely by expanding the set of powers g for which the results hold.

Following the notation of [5], for any real matrix A, we define a set of integers D_A (the *denominator set* of the matrix A) as follows

$$\begin{aligned} D_A \quad := \quad & \{d \mid \theta - \alpha = c/d, \text{ where } re^{2\pi i\theta},\ re^{2\pi i\alpha} \in \sigma(A),\ r > 0,\ c \in \mathbb{Z}^+, \\ & d \in \mathbb{Z}\backslash\{0\},\ \gcd(c,d) = 1, \text{ and } |\theta - \alpha| \notin \{0,1,2,\ldots\}\}. \end{aligned}$$

The set D_A captures the denominators of the reduced fractions that represent the argument differences (normalized by a factor of $1/2\pi$) of two distinct eigenvalues of A lying on the same circle in the complex plane with the origin as its center. In other words, if two distinct eigenvalues of A lie on the same circle in the complex plane with the origin as its center and their argument difference is a rational multiple of 2π, then the denominator of this rational multiple in the lowest terms belongs to D_A. Note that the set D_A defined above is empty if and only if *one* of the following statements is true:

1. A has no distinct eigenvalues lying on the same circle in the complex plane with the origin as its center.
2. The argument differences of the distinct eigenvalues of A that lie on the same circle in the complex plane with the origin as its center are irrational multiples of 2π.

Note also that D_A is always a finite set, and that 1 is never an element of D_A. Moreover, $d \in D_A$ if and only if $-d \in D_A$.

We define now the following sets of integers:

$$\begin{aligned} P_A \quad &:= \quad \{kd \mid k \in \mathbb{Z},\ d > 0 \ \text{ and } \ d \in D_A\} \quad && \text{(Problematic Powers of } A). \\ N_A \quad &:= \quad \{1,2,3,\ldots\}\backslash P_A && \text{(Nice Powers of } A). \end{aligned}$$

Since D_A is finite and 1 is never an element of D_A, N_A is always an infinite set. In

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particular, N_A contains all the prime numbers that are larger than the maximum of D_A.

The eigenspace of A for the eigenvalue λ is denoted by $E_\lambda(A)$ and the (complex) Jordan canonical form of A is denoted by $J(A)$. By $G_\lambda(A)$ we denote the generalized eigenspace of matrix A for the eigenvalue λ. The following result follows from [15, Thereom 6.2.25].

Lemma 3.1. *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)$, $\lambda \neq 0$. Then for all $k \in N_A$ we have $E_\lambda(A) = E_{\lambda^k}(A^k)$ and the elementary Jordan blocks of λ^k in $J(A^k)$ are obtained from the elementary Jordan blocks of λ in $J(A)$ by replacing λ with λ^k.*

We present now some further notation. Let $\Gamma(A)$ denote the usual directed graph of a real matrix A; see, e.g., [2, 14]. A vertex j has access to a vertex m if there is a path from j to m in $\Gamma(A)$. If j has access to m and m has access to j, then we say j and m communicate. If n is the order of the matrix A, then the communication relation is an equivalence relation on $\langle n\rangle := \{1, 2, \ldots, n\}$ and an equivalence class is called a *class* of the matrix A. If in the graph $\Gamma(A)$ a vertex m has access to a vertex j which happens to be in a class α of matrix A, then we say m has access to α. If $A \in \mathbb{C}^{n\times n}$ and α, β are ordered subsets of $\langle n\rangle$, then $A[\alpha, \beta]$ denotes the submatrix of A whose rows are indexed by α and whose columns are indexed by β according to the prescribed orderings of the ordered sets α and β. For simplicity, we write $A[\alpha]$ for $A[\alpha, \alpha]$. If A is in $\mathbb{C}^{n\times n}$ and $\kappa = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is an ordered partition of $\langle n\rangle$ (see, e.g., [4, 5]), then A_κ denotes the block matrix whose $(i, j)^{th}$ block is $A[\alpha_i, \alpha_j]$. The reduced graph of A, denoted by $R(A)$, is the graph whose vertices are the classes of A and there is an edge (α, β) in $R(A)$ if and only if $A[\alpha, \beta]$ is nonzero. Moreover, we say that a class α has access to a class β if in the reduced graph, $R(A)$, α has access to β. We define the transitive closure of $R(A)$ as the graph with the same vertices as $R(A)$ but in which there is an edge (α, β) if and only if α has access to β in $R(A)$. A class of A is called an initial class if it is not accessed by any other class of A, and it is called a final class if it does not access any other class of A. If α is a class of A for which $\rho(A[\alpha]) = \rho(A)$, then we call α a basic class. We note here that the concept of a basic class is normally used for a square nonnegative matrix [2], and that it is justifiable to extend this definition to an arbitrary square real matrix because, in view of the Frobenius normal form of A, for every class α, we have $\rho(A[\alpha]) \leq \rho(A)$. (In other words, the possibility that $\rho(A[\alpha]) > \rho(A)$ is ruled out.)

The following three results follow from Lemma 3.1 with the same proofs as in [5].

Lemma 3.2. *Suppose that $A \in \mathbb{R}^{n\times n}$ with index equal to 0 or 1, and that $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, if for some ordered partition $\kappa = (\alpha_1, \alpha_2)$ of $\langle n\rangle$ we have $(A^g)[\alpha_1, \alpha_2] = 0$ and $(A^g)[\alpha_2]$ is irreducible or a 1×1 zero block, then $A[\alpha_1, \alpha_2] = 0$.*

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Lemma 3.3. *Suppose that $A \in \mathbb{R}^{n\times n}$ with index equal to 0 or 1, and that $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, if $(A^g)_\kappa$ is in the Frobenius normal form for some ordered partition κ, then A_κ is also in the Frobenius normal form for the same partition κ.*

Lemma 3.4. *Suppose that $A \in \mathbb{R}^{n\times n}$ with index equal to 0 or 1, and that $A^s \geq 0$ for all $s \geq m$. Then, for all $g \in N_A \cap \{m, m+1, m+2, \ldots\}$, the transitive closures of the reduced graphs of A and A^g are the same.*

We will use the next result to prove the main theorem of this section.

Lemma 3.5. *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)$, then $G_\lambda(A) = G_{\lambda^k}(A^k)$ for all $k \in N_A$.*

Proof. There exists a nonsingular matrix P such that $P^{-1}AP = J_1 \oplus J_2$, where J_1 is the direct sum of all elementary Jordan blocks of A corresponding to λ and J_2 is the direct sum of all elementary Jordan blocks corresponding to other eigenvalues of A. Then, we have $P^{-1}A^kP = J_1^k \oplus J_2^k$. Since $k \in N_A$, it is clear that $\lambda^k \notin \sigma(J_2^k)$. So, λ^k is an eigenvalue of A^k of multiplicity m where m is the size of J_1. Note that the subspace spanned by the first m columns of P is equal to $G_\lambda(A)$. On the other hand, from $A^kP = P(J_1^k \oplus J_2^k)$, we also see that the subspace spanned by the first m columns of P is invariant under A^k and the restriction of A^k to the latter subspace is represented by the upper triangular matrix J_1^k. It follows that the latter subspace is included in $G_{\lambda^k}(A^k)$ and hence it is $G_{\lambda^k}(A^k)$ since they have the same dimension. ∎

Remark 3.6. Lemma 3.5 does not hold in general if the power k to which the matrix A is raised is not in the set of "nice powers" N_A. Take, e.g., the diagonal matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. If $\lambda = 1$, then $G_\lambda(A) = Span\ \{e_1\}$ whereas (for $k = 2 \in D_A$) $G_{\lambda^2}(A^2) = \mathbb{R}^2$. Hence, $G_\lambda(A)$ is a proper subspace of $G_{\lambda^2}(A^2)$.

We note that since for any eventually nonnegative matrix A having index 0 or 1, the transitive closures of the reduced graphs of A and A^p are the same for $p \in N_A$ large enough (by Lemma 3.4) and the generalized eigenspaces $G_\lambda(A)$ and $G_{\lambda^p}(A^p)$, $\lambda \neq 0$, are the same for $p \in N_A$ (by Lemma 3.5), it is natural to expect that any result one has on the generalized eigenspaces of nonnegative matrices and their classes to carry over to eventually nonnegative matrices having index 0 or 1. In particular, this situation applies to the following result due to Rothblum [22, Theorem 3.1]:

Theorem 3.7. *Let A be a square nonnegative matrix having m basic classes $\alpha_1, \ldots, \alpha_m$. Then, $G_{\rho(A)}(A)$ contains a basis consisting of m semipositive vectors $v^{(1)}, \ldots, v^{(m)}$ associated with the basic classes of A such that $v_i^{(j)} > 0$ if and only if vertex i has access to α_j in $\Gamma(A)$, $j = 1, \ldots, m$, and any such collection is a basis for $G_{\rho(A)}(A)$.*

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We now show that this theorem holds as well for eventually nonnegative matrices A with index 0 or 1.

THEOREM 3.8. *Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix whose index is 0 or 1 and assume that A has m basic classes $\alpha_1, \ldots, \alpha_m$. Then, $G_{\rho(A)}(A)$ contains a basis consisting of m semipositive vectors $v^{(1)}, \ldots, v^{(m)}$ associated with the basic classes of A such that $v_i^{(j)} > 0$ if and only if vertex i has access to α_j in $\Gamma(A)$, $j = 1, \ldots, m$, and any such collection is a basis for $G_{\rho(A)}(A)$.*

Proof. Since A is eventually nonnegative, it follows that there is $p \in N_A$ such that $A^s \geq 0$ for all $s \geq p$. Let κ be an ordered partition of $\langle n\rangle$ that gives the Frobenius normal form of A^p and assume that A^p has m' basic classes. By Theorem 3.7, $G_{\rho(A^p)}(A^p)$ contains a basis consisting of m' semipositive vectors $v^{(1)}, \ldots, v^{(m')}$ associated with the basic classes of A^p such that $v_i^{(j)} > 0$ if and only if i has access to α_j in $\Gamma(A^p)$, $j = 1, \ldots, m'$, and any such collection is a basis for $G_{\rho(A^p)}(A^p)$. By Lemma 3.3, we know that the ordered partition κ also gives the Frobenius normal form of A and that $m' = m$. Moreover, by Lemma 3.5 we have $G_{\rho(A^p)}(A^p) = G_{\rho(A)}(A)$. Furthermore, we claim that i has access to α_j in $\Gamma(A^p)$ if and only if i has access to α_j in $\Gamma(A)$. To prove this claim, let β denote the class to which the index i belongs and consider the reduced graphs of A and A^p. By Lemma 3.4, the transitive closures of the reduced graphs of A and A^p are the same. Hence, the reduced graphs of A and A^p have the same access relations. Thus, β has access to α_j in the reduced graph of A if and only if β has access to α_j in the reduced graph of A^p. Since i communicates with any vertex in β, it follows that i has access to α_j in $\Gamma(A^p)$ if and only if i has access to α_j in $\Gamma(A)$, and thus, the theorem holds. ∎

REMARK 3.9. We note here that if A is any eventually nonnegative matrix in $\mathbb{R}^{n\times n}$ then by Theorem 2.3 the matrix A has the decomposition $A = B + C$ in which B is a real $n \times n$ eventually nonnegative matrix whose index is 0 or 1, $\rho(A) = \rho(B)$, the matrix C is a real nilpotent matrix, and $BC = CB = 0$. Moreover, $G_{\rho(A)}(A) = G_{\rho(B)}(B)$. Hence, if B has m basic classes $\alpha_1, \ldots, \alpha_m$. Then, $G_{\rho(A)}(A)$ contains a basis consisting of m semipositive vectors $v^{(1)}, \ldots, v^{(m)}$ associated with the basic classes of B such that $v_i^{(j)} > 0$ if and only if vertex i has access to α_j in $\Gamma(B)$, $j = 1, \ldots, m$, and any such collection is a basis for $G_{\rho(A)}(A)$. However, we point out here that the observation that $G_{\rho(A)}(A) = G_{\rho(B)}(B)$, though of interest, is not that useful as it seems difficult to determine B.

The following results are immediate from Theorem 3.8 and they are generalizations of theorems known for nonnegative matrices; see, e.g., [2].

PROPOSITION 3.10. *Suppose that $A \in \mathbb{R}^{n\times n}$ is an eventually nonnegative matrix with index equal to 0 or 1. Then, there is a positive eigenvector corresponding to $\rho(A)$ if and only if the final classes of A are exactly its basic classes.*

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PROPOSITION 3.11. *Suppose that $A \in \mathbb{R}^{n\times n}$ is an eventually nonnegative matrix with index equal to 0 or 1. Then, there are positive right and left eigenvectors corresponding to $\rho(A)$ if and only if all the classes of A are basic and final, i.e., A is permutationally similar to a direct sum of irreducible matrices having the same spectral radius.*

REMARK 3.12. Propositions 3.10 and 3.11 do not hold for eventually nonnegative matrices whose index is greater than 1. Consider, for example, the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The matrix A is an irreducible eventually nonnegative matrix whose index is 2. Hence, A has one class which is basic and final yet one can easily check that $\rho(A) = 2$ and that $E_{\rho(A)}(A) = Span\ \{[0,0,1,1]^T\}$, and therefore, A does not have a positive eigenvector corresponding to $\rho(A)$.

4. Matrices that eventually possess the Perron-Frobenius property. As we have seen, nonnilpotent eventually nonnegative matrices have the Perron-Frobenius property. It is then natural to ask, what can we say about matrices whose powers eventually possess the Perron-Frobenius property. We show in this short section that if such matrices are nonnilpotent then they must possess the Perron-Frobenius property.

THEOREM 4.1. *Let A be a nonnilpotent matrix in $\mathbb{R}^{n\times n}$. Then, A has the Perron-Frobenius property (respectively, the strong Perron-Frobenius property) if and only if for some positive integer k_0, A^k has the Perron-Frobenius property (respectively, the strong Perron-Frobenius property) for all $k \geq k_0$.*

Proof. We show the case of the matrix having Perron-Frobenius property, the other case being analogous. Suppose that $A \in \mathbb{R}^{n\times n}$ has the Perron-Frobenius property. For any $\lambda \in \sigma(A)$ and all $k \geq 1$, we have $E_\lambda(A) \subseteq E_{\lambda^k}(A^k)$. In particular, this is true for $\lambda = \rho(A)$. Using the fact that $\rho(A^k) = (\rho(A))^k$, we see that $E_{\rho(A)}(A) \subseteq E_{\rho(A)^k}(A^k)$. Therefore, A^k has the Perron-Frobenius property for all $k \geq 1$. Conversely, suppose that there is a positive integer k_0 such that A^k has the Perron-Frobenius property for all $k \geq k_0$. By picking distinct prime numbers k_1 and k_2 larger than k_0 and using the fact that the eigenvalues of A are k_ith roots of the eigenvalues of A^{k_i} for $i = 1, 2$, we can see that $\rho(A)$ must be an eigenvalue of A. Furthermore, by picking $k \in N_A \cap \{k_0, k_0+1, k_0+2, \ldots\}$ ("nice powers" k of A that are larger than k_0), we have $E_{\rho(A)}(A) = E_{\rho(A)^k}(A^k)$ (this follows from Lemma 3.1). Hence, we can choose a nonnegative eigenvector of A corresponding to $\rho(A)$. Thus, A has the Perron-Frobenius property. ∎

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Corollary 4.2. *Let A be a nonnilpotent matrix in $\mathbb{R}^{n\times n}$. Then, $A \in WPFn$ (respectively, $A \in PFn$) if and only if for some positive integer k_0, $A^k \in WPFn$ (respectively, $A^k \in PFn$), for all $k \geq k_0$.*

5. Similarity matrices preserving the Perron-Frobenius property. If S is a positive diagonal matrix or a permutation matrix then clearly $S^{-1}AS$ possesses the Perron-Frobenius property whenever A does. This observation leads to the following question: which similarity matrices S preserve the Perron-Frobenius property, the strong Perron-Frobenius property, eventual nonnegativity, or being in WPFn or in PFn? In other words, for which similarity transformations are these sets of matrices invariant? We first prove a preliminary lemma that leads to answering this question.

Lemma 5.1. *Let S be an $n \times n$ real matrix which has a positive entry and a negative entry ($n \geq 2$). If S is of rank one but not expressible as xy^T, where x is a nonnegative vector, or is of rank two or more, then there is a positive vector $v \in \mathbb{R}^n$ such that Sv has a positive entry and a negative entry.*

Proof. If S is a rank-one matrix with the given property, then S is expressible as xy^T , where x is a vector which has a positive entry and a negative entry. Choose any positive vector v such that $y^Tv \neq 0$. Then Sv, being a nonzero multiple of x, clearly has a positive entry and a negative entry.

Suppose that S is of rank two or more and assume to the contrary that for each positive vector $v \in \mathbb{R}^n$, the vector Sv is either nonnegative or nonpositive. Choose positive vectors v_1, v_2 such that Sv_1 is semipositive, Sv_2 is seminegative, and Sv_1, Sv_2 are not multiples of each other. It is clear that there exists a scalar λ, $0 < \lambda < 1$, such that $S((1-\lambda)v_1 + \lambda v_2)$ has both a positive entry and a negative entry. But $(1-\lambda)v_1 + \lambda v_2$ is a positive vector. Thus, we arrive at a contradiction. □

By $GL(n, \mathbb{R})$ we denote the set of nonsingular matrices in $\mathbb{R}^{n\times n}$. We call a matrix S *monotone* if $S \in GL(n, \mathbb{R})$ and S^{-1} is nonnegative. We call a matrix $S \in \mathbb{R}^{n\times n}$ a monomial matrix if S has exactly one nonzero entry in each row and each column.

Theorem 5.2. *For any $S \in GL(n, \mathbb{R})$, the following statements are equivalent:*

(i) *Either S or $-S$ is monotone.*
(ii) *$S^{-1}AS$ has the Perron-Frobenius property (respectively, the strong Perron-Frobenius property) for all matrices A having the Perron-Frobenius property (respectively, the strong Perron-Frobenius property).*
(iii) *$S^{-1}AS$ has a positive right Perron-Frobenius eigenvector whenever $A \in \mathbb{R}^{n\times n}$ has a positive right Perron-Frobenius eigenvector.*

Proof. First, we show that (i) is equivalent to (ii). Suppose (i) is true. We prove only the case when S is monotone because $(-S)^{-1}A(-S) = S^{-1}AS$. If A is a ma-

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trix with the Perron-Frobenius property (respectively, the strong Perron-Frobenius property) and v is a right Perron-Frobenius eigenvector of A, then $S^{-1}v$ is an eigenvector of $S^{-1}AS$ corresponding to $\rho(A)$. The nonsingularity of S implies that none of the rows of S^{-1} is 0. Therefore, $S^{-1}v$ is a semipositive (respectively, positive) vector. Also, $\rho(A)$ is an eigenvalue (respectively, a simple positive and strictly dominant eigenvalue) of $S^{-1}AS$ since $S^{-1}AS$ and A have the same characteristic polynomial. This shows that $(i) \Rightarrow (ii)$. Conversely, suppose (i) is not true, i.e., S and $-S$ are both not monotone. Then, in such a case, S^{-1} must have a positive entry and a negative entry. By Lemma 5.1, there is a positive vector v such that $S^{-1}v$ has a positive entry and a negative entry. Consider the matrix $A = vv^T \in$ PFn and note that $\rho(A) = v^T v > 0$ and $Av = \rho(A)v$. For such a matrix, we have $E_{\rho(A)}(A) = Span\{v\}$. Since the eigenvectors in $E_{\rho(A)}(S^{-1}AS)$ are of the form $S^{-1}w$ for some eigenvector $w \in E_{\rho(A)}(A)$, it follows that $E_{\rho(A)}(S^{-1}AS)$ does not have a nonnegative eigenvector. Thus, $S^{-1}AS$ does not have the Perron-Frobenius property (respectively, the strong Perron-Frobenius property). Hence, (ii) is not true, which shows that $(ii) \Rightarrow (i)$. Hence, (i) is equivalent to (ii). The proof of the equivalence of (i) and (iii) is analogous to the proof of the equivalence of (i) and (ii), and thus, we omit it. □

We note that we can apply Theorem 5.2 to A^T, thus obtaining a similar result for left eigenvectors.

THEOREM 5.3. *For any $S \in GL(n, \mathbb{R})$, the following statements are equivalent:*

(i) *Either S or $-S$ is a nonnegative monomial matrix.*
(ii) *$S^{-1}AS$ is nonnegative for all nonnegative $A \in \mathbb{R}^{n\times n}$.*
(iii) *$S^{-1}AS \in$ PFn for all $A \in$ PFn.*
(iv) *$S^{-1}AS$ is a nonnilpotent eventually nonnegative matrix for any nonnilpotent eventually nonnegative matrix $A \in \mathbb{R}^{n\times n}$.*
(v) *$S^{-1}AS$ is a nonnilpotent matrix in WPFn whenever A is a nonnilpotent matrix in WPFn.*
(vi) *$S^{-1}AS \in$ WPFn for all $A \in$ WPFn.*
(vii) *$S^{-1}AS$ has positive right and left Perron-Frobenius eigenvectors whenever $A \in \mathbb{R}^{n\times n}$ has positive right and left Perron-Frobenius eigenvectors.*
(viii) *$S^{-1}AS$ is positive for all positive $A \in \mathbb{R}^{n\times n}$.*

Proof. Note that (i) is equivalent to saying that S and S^{-1} are both nonnegative or both nonpositive. We show that (i) is equivalent to (vi), the other cases being analogous. Suppose that (i) is true. Then, S and S^{-1} are both nonnegative or both nonpositive. We consider only the case when S and S^{-1} are both nonnegative because $(-S)^{-1}A(-S) = S^{-1}AS$. Let S^{-T} denote the transpose of the inverse of matrix S. Note that S^T and S^{-T} are both nonnegative. By Theorem 5.2, the matrix $S^{-1}AS$ has the Perron-Frobenius property whenever A does and $(S^{-1}AS)^T = S^T A^T S^{-T}$ has the Perron-Frobenius property whenever A^T does. Thus, $S^{-1}AS \in$ WPFn whenever

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$A \in$ WPFn, which shows $(i) \Rightarrow (vi)$. Conversely, suppose that (i) is not true. Then, we have four cases: 1. S^{-1} has a positive entry and a negative entry, 2. S^{-1} has a positive entry and S has a negative entry, 3. S^{-1} has a negative entry and S has a positive entry, and 4. S has a positive and a negative entry. We only consider cases 1 and 2 since the other two cases are analogous. Suppose that case 1 holds, i.e., S^{-1} has a positive entry and a negative entry. Then, by the argument given in the proof of Theorem 5.2, there is a positive matrix A such that $S^{-1}AS$ does not have a Perron-Frobenius eigenvector. If case 2 holds, i.e., S^{-1} has a positive entry and S has a negative entry, then we consider two subcases: I. S^{-1} has a positive entry and a negative entry and II. S^{-1} is a nonnegative matrix. If subcase I holds then we are back to case 1, so we are done. If subcase II holds then either S has a positive entry (and a negative entry), and this is analogous to case 1, or S is nonpositive. But S cannot be nonpositive because this would imply that $I = S^{-1}S \leq 0$, a contradiction. Hence, in all cases, there is a positive matrix A such that $S^{-1}AS$ does not have a Perron-Frobenius eigenvector. □

6. Topological properties. In this section, we prove some topological properties of sets consisting of matrices with the Perron-Frobenius property. In subsection 6.1, we show that many of the sets we study are simply connected. This includes PFn, WPFn, the set of matrices with the Perron-Frobenius property, the set of matrices with the strong Perron-Frobenius property, etc. However, other related sets, such as the set of nonnilpotent eventually nonnegative matrices and the set of nonnilpotent matrices with the Perron-Frobenius property, while they might be simply connected, we can only prove that they are path-connected. At the end of Subsection 6.1, we present a counter-example showing that none of the sets of matrices that we considered is convex. In Subsection 6.2, we study the closure of some of the sets considered in Subsection 6.1 and we prove some additional results on path-connectedness.

6.1. Simply connected and path-connected sets. The following theorem is due to Brauer; see [3, Theorem 27].

THEOREM 6.1. *Let A be an $n \times n$ real matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let v be a real eigenvector of A associated with some real eigenvalue λ_k and let w be any n-dimensional real vector. Then, the matrix $B = A + vw^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + w^T v, \lambda_{k+1}, \ldots, \lambda_n$.*

COROLLARY 6.2. *Let A be a matrix in $\mathbb{R}^{n\times n}$ with the Perron-Frobenius property, and let v be its right Perron-Frobenius eigenvector. If $w \in \mathbb{R}^n$, is such that $v^T w > 0$ then for all scalars $\epsilon > 0$ the following holds:*

(i) The matrix $B = A + \epsilon v w^T$ has the Perron-Frobenius property.
(ii) $\rho(A) < \rho(B)$.

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(iii) If $v > 0$ then B has the strong Perron-Frobenius property.

The following lemma is a well-known lemma about eigenvalue and eigenvector continuity; see, e.g., [20, p. 45].

LEMMA 6.3. *The eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ are continuous functions of the elements of the matrix A. Moreover, if λ is a simple eigenvalue of A and x is a corresponding eigenvector then for any $E \in \mathbb{C}^{n\times n}$ the matrix $A+E$ has an eigenvalue λ_E and an eigenvector x_E such that $\lambda_E \to \lambda$ and $x_E \to x$ as $E \to 0$.*

THEOREM 6.4. *PFn is contractible, and thus, simply connected.*

Proof. Johnson and Tarazaga proved in [16, Theorem 2] that PFn is path-connected. Thus, it is enough to show that any loop in PFn can be shrunk to a point. Let $\{A_t \ : \ t \in [0, 1]\}$ be a loop of matrices in PFn. For all $t \in [0, 1]$, let v_t and w_t be respectively the unit right and left Perron-Frobenius eigenvectors of A_t. Note that by Lemma 6.3 the vectors v_t and w_t depend continuously on t. Moreover, for all scalars $\epsilon \in [0,\infty)$ and all scalars $t \in [0, 1]$, define the matrix $B_{(t,\epsilon)} := A_t + \epsilon v_t w_t^T$. Note that the matrix $B_{(t,\epsilon)}$ depends continuously on both t and ϵ. Consider the loop $\{B_{(t,\epsilon)} \ : \ t \in [0, 1]\}$ defined for each $\epsilon \in [0,\infty)$. By Corollary 6.2, the loop $\{B_{(t,\epsilon)} \ : \ t \in [0, 1]\}$ is in PFn for all $\epsilon \in [0,\infty)$. We claim that for large values of $\epsilon \in [0,\infty)$, the loop $\{B_{(t,\epsilon)} \ : \ t \in [0, 1]\}$ is a loop of positive matrices. To see this, pick $t_0 \in [0, 1]$. There is some large value of $\epsilon \in [0,\infty)$ which depends on t_0, call it ϵ_0, such that $B_{(t_0,\epsilon_0)}$ is a positive matrix. Note that $B_{(t_0,\epsilon)}$ must also be a positive matrix for all scalars $\epsilon \geq \epsilon_0$. Since the entries of the matrix $B_{(t,\epsilon)}$ depend continuously on both t and ϵ, it follows that there is an open neighborhood $U_0 \subset \mathbb{R}^2$ around the point $(t_0, \epsilon_0) \in [0, 1] \times [0,\infty)$ such that whenever $(t, \epsilon) \in U_0 \cap ([0, 1] \times [0,\infty))$ then $B_{(t,\epsilon)}$ is a positive matrix. Thus, there is an open interval L_0 around t_0 such that $B_{(t,\epsilon)}$ is a positive matrix whenever $t \in L_0 \cap [0, 1]$ and $\epsilon \geq \epsilon_0$. Hence, we have shown that for every $t_0 \in [0, 1]$ there is an open interval L_0 around t_0 and a nonnegative scalar ϵ_0 such that $B_{(t,\epsilon)}$ is a positive matrix whenever $t \in L_0 \cap [0, 1]$ and $\epsilon \geq \epsilon_0$. By compactness of the interval [0,1], we can select finitely many open intervals, say k open intervals, $L_1, L_2, \ldots, L_k$ with k nonnegative scalars $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ such that $[0, 1] \subset \bigcup_{j=1}^{k} L_j$ and for each $j \in \{1, 2, \ldots, k\}$, the matrix $B_{(t,\epsilon)}$ is a positive matrix whenever $t \in L_j \cap [0, 1]$ and $\epsilon \geq \epsilon_j$. Let $\mu = max\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$. Hence, the loop $\{B_{(t,\epsilon)} \ : \ t \in [0, 1]\}$ is a loop of positive matrices whenever $\epsilon \geq \mu$. And thus, it can be continuously shrunk to a point. As a consequence, the loop $\{A_t \ : \ t \in [0, 1]\}$ can be continuously shrunk to a positive matrix. ▯

Similarly, we have the following theorem.

THEOREM 6.5. *The following sets of matrices in $\mathbb{R}^{n\times n}$ are contractible, and thus, simply connected:*

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(i) The set of matrices whose spectral radius is a simple positive eigenvalue with corresponding positive right and left eigenvectors.
(ii) The set of matrices whose spectral radius is a simple positive eigenvalue with a corresponding positive right eigenvector.
(iii) The set of matrices with the strong Perron-Frobenius property.

We point out here that a result similar to that of part (ii) in Theorem 6.5 can be obtained by replacing "right eigenvector" with "left eigenvector". Likewise, a result similar to that of part (iii) in Theorem 6.5 can be obtained for the set of matrices $\{A \in \mathbb{R}^{n\times n} \ : \ A^T$ has the strong Perron-Frobenius property$\}$.

THEOREM 6.6. *The following sets of matrices in $\mathbb{R}^{n\times n}$ are contractible, and thus, simply connected:*

(i) The set of matrices having the Perron-Frobenius property.
(ii) The set of eventually nonnegative matrices.
(iii) WPFn.

Proof. The proof is the same for (i), (ii), and (iii). Every matrix A in the set under consideration is connected by the path $\{A_t \ : \ t \in [0,1]\}$ to the zero matrix where A_t is defined as $(1-t)A$ for all $t \in [0,1]$. Note that the matrix A_t is in the set under consideration for all $t \in [0,1]$. Hence, the set under consideration is path-connected. Similarly, every loop of matrices in the set under consideration shrinks continuously to the zero matrix. ∎

THEOREM 6.7. *The set of nonnilpotent matrices in $\mathbb{R}^{n\times n}$ having the Perron-Frobenius property is path-connected.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be a nonnilpotent matrix having the Perron-Frobenius property. Since the collection of positive matrices is a convex subset of the set of nonnilpotent matrices with the Perron-Frobenius property, it is enough to show that there is a path of nonnilpotent matrices having the Perron-Frobenius property that connects matrix A to some positive matrix B. The proof goes as follows: connect matrix A to a matrix $\tilde{A}$ having a positive right Perron-Frobenius eigenvector by a path $\{A_t \ : \ t \in [0,1]\}$ consisting of nonnilpotent matrices having the Perron-Frobenius property and then connect the matrix $\tilde{A}$ to a positive matrix B by a path $\{B_\epsilon \ : \ \epsilon \in [0,1]\}$ of nonnilpotent matrices having the Perron-Frobenius property.

If A has a positive right Perron-Frobenius eigenvector then define $\tilde{A} := A$ and define $A_t := A$ for all t in $[0,1]$. The path $\{A_t \ : \ t \in [0,1]\}$ is the first of the two desired paths. Otherwise, consider the real Jordan canonical form of A. We denote the real Jordan canonical form of A by $J_{\mathbb{R}}(A)$. Hence, we may write $A = VJ_{\mathbb{R}}(A)V^{-1}$ where $V = [v\ w_2\ w_3 \cdots w_n]$ and v is a right Perron-Frobenius eigenvector of A. For every scalar $t \geq 0$, we construct the vector v_t by replacing the zero entries of v by

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t, and we construct a new matrix $V_t = [v_t\ w_2\ w_3 \cdots w_n]$. Since $V_0 = V \in GL(n, \mathbb{R})$ and since $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, there is a positive scalar δ such that whenever $0 \le t \le \delta$ we have $V_t \in GL(n, \mathbb{R})$. Define $A_t := V_{\delta t} J_{\mathbb{R}}(A) V_{\delta t}^{-1}$ for $0 \le t \le 1$. Then, for all t in $[0, 1]$, the matrix A_t has $\rho(A)$ as a positive dominant eigenvalue with a corresponding nonnegative eigenvector $v_{\delta t}$. Furthermore, the vector $v_{\delta t}$ is positive for all $0 < t \le 1$. Let $\tilde{A} = A_1$ and let v_δ be its corresponding positive eigenvector. The path $\{A_t\ :\ t \in [0, 1]\}$ is the first of the two desired paths.

Let w be any positive vector. For all scalars $\epsilon \ge 0$, define the matrix $K_\epsilon := \tilde{A} + \epsilon v_\delta w^T$. By Corollary 6.2, the matrix K_ϵ is a nonnilpotent matrix that possesses the Perron-Frobenius property for all $\epsilon \ge 0$. Since $v_\delta w^T$ is a positive matrix, K_ϵ is positive for some large value of ϵ, say μ. Define $B_\epsilon := K_{\mu\epsilon}$ for all ϵ in $[0, 1]$. Hence, the path $\{B_\epsilon\ :\ \epsilon \in [0, 1]\}$ is the second of the two desired paths. □

THEOREM 6.8. *The following sets of matrices in $\mathbb{R}^{n\times n}$ are path-connected:*

(i) The set consisting of nonnilpotent eventually nonnegative matrices.
(ii) The set of eventually nonnegative matrices whose spectral radius is a simple positive eigenvalue.

Proof. We note first that the set of nonnilpotent nonnegative matrices in $\mathbb{R}^{n\times n}$ is a path-connected subset of the set of nonnilpotent eventually nonnegative matrices. This statement is true because any nonnilpotent nonnegative matrix $A \in \mathbb{R}^{n\times n}$ is connected by the path $\{(1 - t)A + tI\ \ :\ \ t \in [0, 1]\}$ of nonnilpotent nonnegative matrices to the identity matrix $I \in \mathbb{R}^{n\times n}$. Hence, to prove that the set given in (i) is path-connected, it is enough to show that every matrix in it connects to a nonnilpotent nonnegative matrix by a path lying completely in the set given in (i).

Suppose that $A \in \mathbb{R}^{n\times n}$ is a nonnilpotent matrix that satisfies $A^k \ge 0$ for all $k \ge m$. Define the matrix $A_t := (1 - t)A + tA^m$ for t in $[0, 1]$ and consider the path of matrices $\{A_t\ :\ t \in [0, 1]\}$. Note that $(A_t)^k \ge 0$ for all $k \ge m$ and all $t \in [0, 1]$ and that $A_1 = A^m \ge 0$. Furthermore, the eigenvalues of A_t are of the form $(1-t)\lambda + t\lambda^m$ where $\lambda \in \sigma(A)$. For every $\lambda \in \sigma(A)$, we have $|(1 - t)\lambda + t\lambda^m| \le (1 - t)|\lambda| + t|\lambda|^m \le (1 - t)\rho(A) + t\,(\rho(A))^m$. But, $(1 - t)\rho(A) + t\,(\rho(A))^m$ is a positive eigenvalue of A_t because $\rho(A) > 0$. Hence, $\rho(A_t) = (1 - t)\rho(A) + t\rho(A)$ is a positive eigenvalue of A_t for all $t \in [0, 1]$, and thus, A_t is a nonnilpotent eventually nonnegative matrix for all $t \in$ [0,1]. This proves that the set given in (i) is path-connected. The proof of the path-connectedness of the set given in (ii) is very similar, and thus, we omit it. □

We mention here that another result on path-connectedness can be found in Theorem 6.14 below.

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The following example (which is taken from [16, page 329]) shows that none of the sets of "generalized nonnegative matrices" which we mentioned in Sections 1 and 2 is a convex set.

EXAMPLE 6.9. Let $A = \begin{bmatrix} 20 & 1 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & -10 \end{bmatrix}$ and let $B = \begin{bmatrix} -10 & 1 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & 20 \end{bmatrix}$. Observe that A and B are matrices in PFn. However, the matrix

$$C = \frac{1}{2}A + \frac{1}{2}B = \begin{bmatrix} 5 & 1 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

is a nonnilpotent matrix which does not possess the Perron-Frobenius property. Hence by Lemma 2.1, the matrix C is not even eventually nonnegative. Therefore, PFn is not necessarily convex. Furthermore, in view of the relationship among the different sets given in Section 2, we conclude from this example that none of the following sets of matrices in $\mathbb{R}^{n\times n}$ is necessarily convex: WPFn, the set of eventually nonnegative matrices, the set of nonnilpotent eventually nonnegative matrices, the set of matrices with the Perron-Frobenius property, the set of nonnilpotent matrices with the Perron-Frobenius property, and the set of matrices with the strong Perron-Frobenius property. In particular, for $n = 3$, none of them is convex.

6.2. Closure of sets. We begin by a series of lemmas leading to a new result that asserts that a given pair of semipositive vectors with a nonzero inner product can be mapped to a pair of positive vectors by real orthogonal matrices arbitrarily close to the identity. We point out here that the results that appear in this subsection and that involve a matrix norm $||\cdot||$ are true for any choice of matrix norm on $\mathbb{R}^{n\times n}$. If k is a positive integer, then we denote the $k \times k$ identity matrix by I_k.

LEMMA 6.10. *For any semipositive vector v_1 and for any positive scalar ϵ, there is a path $\{Q_t \ : \ t \in [0, 1]\}$ of real orthogonal matrices such that $Q_0 = I_n$, $Q_t v_1$ is a positive vector for all $t \in (0, 1]$, and $||Q_t - I_n|| < \epsilon$ for all $t \in [0, 1]$. Moreover, if the ith entry of v_1, call it γ, is positive then the ith entry of $Q_t v_1$ is $\gamma \cos(\delta t)$ for all t in $[0, 1]$ for a sufficiently small positive scalar δ which depends on ϵ.*

Proof. Assume without loss of generality that v_1 is a unit vector. If v_1 is a positive vector then define $Q_t := I_n$ for all $t \in [0, 1]$, otherwise take a semipositive unit vector v_2 orthogonal to v_1 such that $v_1 + v_2$ is positive. Extend $\{v_1, v_2\}$ to an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$. For every $t \in [0, \frac{\pi}{2}]$, define an $n \times n$ real orthogonal matrix P_t by $P_t v_1 = (\cos t)v_1 + (\sin t)v_2$, $P_t v_2 = (-\sin t)v_1 + (\cos t)v_2$ and $P_t v_i = v_i$ for $i = 3, \ldots, n$. Note that P_t depends continuously on t. Moreover, it is clear that $\lim_{t\to 0^+} P_t = I_n$. Hence, there is a positive scalar δ such that whenever $t \in [0, \delta]$ we have $||P_t - I_n|| < \epsilon$. Define the matrix $Q_t := P_{\delta t}$ for $t \in [0, 1]$. The path

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$\{Q_t \ : \ t \in [0,1]\}$ is the desired path of real orthogonal matrices. □

LEMMA 6.11. *Let $u = [u_1, \ldots, u_n]^T$ be a positive vector in $\mathbb{R}^n$ and let $v = \gamma e_1 = [\gamma, 0, \ldots, 0]^T$ be a vector in $\mathbb{R}^n$ where γ is a positive scalar. Then, for any positive scalar ϵ there is a path $\{Q_t \ : \ t \in [0,1]\}$ of real orthogonal matrices such that $Q_0 = I_n$, $Q_t u$ is a positive vector for all $t \in (0,1]$, $Q_t v$ is a positive vector for all $t \in (0,1]$, and $||Q_t - I_n|| < \epsilon$ for all $t \in [0,1]$. Moreover, the first entry of $Q_t v$ is $\gamma \cos(\delta t)$ for all t in $[0,1]$ for a sufficiently small positive scalar δ which depends on ϵ.*

Proof. Let $\alpha = (\sum_{j=2}^n (u_j)^2)^{1/2}$ and let $\tilde{u} = [0, \alpha^{-1}u_2, \ldots, \alpha^{-1}u_n]^T$. Then, $\tilde{u}$ is a semipositive unit vector orthogonal to e_1. For all scalars θ such that $0 < \theta < \frac{\pi}{2}$, define the $n \times n$ real orthogonal matrix P_θ by $P_\theta e_1 = (\cos\theta)e_1 + (\sin\theta)\tilde{u}$, $P_\theta \tilde{u} = (-\sin\theta)e_1 + (\cos\theta)\tilde{u}$ and $P_\theta w = w$ for all vectors w orthogonal to $\mathrm{Span}\{e_1, \tilde{u}\}$. Note that P_θ depends continuously on θ and that $\lim_{\theta \to 0^+} P_\theta = I_n$. Moreover, $P_\theta v = \gamma P_\theta e_1$ is a positive vector for all $0 < \theta < \frac{\pi}{2}$. Furthermore, since $u = u_1 e_1 + \alpha \tilde{u}$, it follows that $P_\theta u = u_1 P_\theta e_1 + \alpha P_\theta \tilde{u} = (u_1 \cos\theta - \alpha \sin\theta)e_1 + (u_1 \sin\theta + \alpha\cos\theta)\tilde{u}$. Hence, if we choose $\theta > 0$ sufficiently small, say less than or equal to a positive scalar δ, then $P_\theta u$ is a positive vector for all $0 < \theta \leq \delta$. Define $Q_0 := I_n$ and $Q_t := P_{\delta t}$ for all $t \in (0,1]$. Then, the path $\{Q_t \ : \ t \in [0,1]\}$ is the desired path. □

THEOREM 6.12. *Let $u = [u_1, \ldots, u_n]^T$ and $v = [v_1, \ldots, v_n]^T$ be semipositive vectors in $\mathbb{R}^n$. Then, the following statements are equivalent:*

(i) $u^T v > 0$.

(ii) For any positive scalar ϵ there is a path $\{Q_t \ : \ t \in [0,1]\}$ of $n \times n$ real orthogonal matrices such that $Q_0 = I_n$, $Q_t u$ is a positive vector for all $t \in (0,1]$, $Q_t v$ is a positive vector for all $t \in (0,1]$, and $||Q_t - I_n|| < \epsilon$ for all $t \in [0,1]$.

Proof. Suppose that (i) is true and let a positive scalar ϵ be given. We will show that (ii) is true by constructing the desired path $\{Q_t \ : \ t \in [0,1]\}$ of real orthogonal matrices in three steps: Step 1, Step 2, and Step 3. In Step j (where $j \in \{1,2,3\}$), we will define an $n \times n$ real orthogonal matrix $Q_{(j,t)}$ for all $t \in [0,1]$. The matrix $Q_{(j,t)}$ is meant to replace the zero entries by positive entries in specific locations of the semipositive vectors u and v. Furthermore, the matrices $Q_{(1,t)}$, $Q_{(2,t)}$, and $Q_{(3,t)}$ will be such that they commute with each other for all t in $[0,1]$, and such that the vectors $Q_{(3,t)}Q_{(2,t)}Q_{(1,t)}u$ and $Q_{(3,t)}Q_{(2,t)}Q_{(1,t)}v$ are positive for all t in $(0,1]$. Moreover, each matrix $Q_{(j,t)}$ will depend continuously on $t \in [0,1]$ and will satisfy $Q_{(j,0)} = I_n$ and $||Q_{(j,t)} - I_n|| < \epsilon$ for all $t \in [0,1]$. Hence, the matrices $Q_{(1,t)}$, $Q_{(2,t)}$, and $Q_{(3,t)}$ will satisfy $\lim_{t \to 0^+} Q_{(3,t)}Q_{(2,t)}Q_{(1,t)} = I_n$, and therefore, for a sufficiently small value of t, say γ, they will satisfy $||Q_{(3,t)}Q_{(2,t)}Q_{(1,t)} - I_n|| < \epsilon$ for all t in $[0,\gamma]$. After that, we define the matrix Q_t as the product $Q_{(3,\gamma t)}Q_{(2,\gamma t)}Q_{(1,\gamma t)}$ for all t in [0,1]. The path $\{Q_t \ : \ t \in [0,1]\}$ will be our desired path.

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To that end, partition the set $\langle n\rangle$ by writing $\langle n\rangle = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ where $\alpha_1 = \{j \mid u_j = v_j = 0\}$, $\alpha_2 = \{j \mid u_j > 0 \text{ and } v_j = 0\}$, $\alpha_3 = \{j \mid u_j > 0 \text{ and } v_j > 0\}$, and $\alpha_4 = \{j \mid u_j = 0 \text{ and } v_j > 0\}$. Let k_j denote the cardinality of α_j for $j = 1, 2, 3, 4$, and note that $k_3 \neq 0$ because $u^T v > 0$. We may assume that the elements of α_1 are the first k_1 integers in $\langle n\rangle$, the elements of α_2 are the following k_2 integers in $\langle n\rangle$, the elements of α_3 are the following k_3 integers in $\langle n\rangle$, and the elements of α_4 are the last k_4 integers in $\langle n\rangle$, i.e., $\alpha_1 = \{1, 2, \ldots, k_1\}$, $\alpha_2 = \{k_1 + 1, k_1 + 2, \ldots, k_1 + k_2\}$, $\alpha_3 = \{k_1+k_2+1, k_1+k_2+2, \ldots, k_1+k_2+k_3\}$, and $\alpha_4 = \{k_1+k_2+k_3+1, k_1+k_2+k_3+2, \ldots, n\}$.

Step 1: Let $\beta_1 = \alpha_2 \cup \{k_1 + k_2 + 1\}$. In this step, we will define an $n \times n$ real orthogonal matrix $Q_{(1,t)}$ for every t in $[0, 1]$ with the following properties:

- The matrix $Q_{(1,t)}$ depends continuously on t in $[0, 1]$.
- The matrix $Q_{(1,0)} = I_n$.
- For all t in $[0, 1]$, we have $||Q_{(1,t)} - I_n|| < \epsilon$.
- For all t in $(0, 1]$, the vectors $(Q_{(1,t)}u)[\beta_1]$ and $(Q_{(1,t)}v)[\beta_1]$ are positive.
- If α_2 is empty then the matrix $Q_{(1,t)} = I_n$ for all t in [0,1], otherwise, there is a sufficiently small positive scalar δ_1 which depends on ϵ such that for all t in $[0, 1]$ we have $(Q_{(1,t)}u)[\{k_1 + k_2 + 1\}] = u_{k_1+k_2+1} \cos(\delta_1 t)$ and $(Q_{(1,t)}v)[\{k_1 + k_2 + 1\}] = v_{k_1+k_2+1} \cos(\delta_1 t)$.
- The matrix $Q_{(1,t)}[\langle n\rangle \backslash \beta_1] = I_{n-k_2-1}$ for all t in $[0, 1]$.

As a result of these properties, the vectors $(Q_{(1,t)}u)[\alpha_2 \cup \alpha_3]$ and $(Q_{(1,t)}v)[\alpha_2 \cup \alpha_3]$ in $\mathbb{R}^{k_2+k_3}$ will be positive for all t in $(0, 1]$.

To construct the matrix $Q_{(1,t)}$, consider the set α_2. If α_2 is empty, then define $Q_{(1,t)} := I_n$ for all $t \in [0, 1]$, otherwise consider the vectors $u[\beta_1]$ and $v[\beta_1]$ in $\mathbb{R}^{k_2+1}$. The vector $u[\beta_1]$ is a positive vector while the vector $v[\beta_1]$ is a semipositive vector with only one positive entry. By Lemma 6.11, there is a path $\{P_t \ : \ t \in [0, 1]\}$ of $(k_2+1)\times(k_2+1)$ real orthogonal matrices such that $P_0 = I_{k_2+1}$, the vector $P_t\,(u[\beta_1])$ is a positive vector for all $t \in (0, 1]$, the inequality $||P_t - I_{k_2+1}|| < \epsilon$ holds for all $t \in [0, 1]$, and the last entries of the vectors $P_t\,(u[\beta_1])$ and $P_t\,(v[\beta_1])$ are respectively $u_{k_1+k_2+1} \cos(\delta_1 t)$ and $v_{k_1+k_2+1} \cos(\delta_1 t)$ for all t in $[0, 1]$ for a sufficiently small positive scalar δ_1 which depends on ϵ. Define $Q_{(1,t)} := I_{k_1} \oplus P_t \oplus I_{k_3+k_4-1}$ for all $t \in [0, 1]$. Then, clearly the matrix $Q_{(1,t)}$ satisfies the properties required in this step.

Step 2: Let $\beta_2 = \alpha_4 \cup \{k_1 + k_2 + 1\}$. In this step, we will define an $n \times n$ real orthogonal matrix $Q_{(2,t)}$ for every t in $[0, 1]$ with the following properties:

- The matrix $Q_{(2,t)}$ depends continuously on t in $[0, 1]$.
- The matrix $Q_{(2,0)} = I_n$.
- For all t in $[0, 1]$, we have $||Q_{(2,t)} - I_n|| < \epsilon$.
- For all t in $(0, 1]$, the vectors $(Q_{(2,t)}u)[\beta_2]$ and $(Q_{(2,t)}v)[\beta_2]$ are positive.

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- If α_4 is empty then the matrix $Q_{(2,t)} = I_n$ for all t in [0,1], otherwise, there is a sufficiently small positive scalar δ_2 which depends on ϵ such that for all t in $[0,1]$ we have $(Q_{(2,t)}u)[\{k_1+k_2+1\}] = u_{k_1+k_2+1}\cos(\delta_2 t)$ and $(Q_{(2,t)}v)[\{k_1+k_2+1\}] = v_{k_1+k_2+1}\cos(\delta_2 t)$.
- The matrix $Q_{(2,t)}[\langle n\rangle\backslash\beta_2] = I_{n-k_4-1}$ for all t in $[0,1]$.

As a result of these properties, the matrix $Q_{(2,t)}$ commutes with the matrix $Q_{(1,t)}$. Moreover, the vectors $(Q_{(2,t)}Q_{(1,t)}u)[\alpha_2\cup\alpha_3\cup\alpha_4]$ and $(Q_{(2,t)}Q_{(1,t)}v)[\alpha_2\cup\alpha_3\cup\alpha_4]$ in $\mathbb{R}^{k_2+k_3+k_4}$ will be positive for all t in $(0,1]$.

To construct the matrix $Q_{(2,t)}$, consider the set α_4. If α_4 is empty, then define $Q_{(2,t)} := I_n$ for all $t\in[0,1]$, otherwise consider the vectors $u[\beta_2]$ and $v[\beta_2]$ in $\mathbb{R}^{k_4+1}$. The vector $u[\beta_2]$ is a positive vector while the vector $v[\beta_1]$ is a semipositive vector with only one positive entry. By the argument given in Step 1, a matrix $Q_{(2,t)}$ satisfying the properties required in this step exists.

Step 3: Let $\beta_3 = \alpha_1\cup\{k_1+k_2+1\}$. In this step, we will define an $n\times n$ real orthogonal matrix $Q_{(3,t)}$ for every t in $[0,1]$ with the following properties:

- The matrix $Q_{(3,t)}$ depends continuously on t in $[0,1]$.
- The matrix $Q_{(3,0)} = I_n$.
- For all t in $[0,1]$, we have $||Q_{(3,t)} - I_n|| < \epsilon$.
- For all t in $(0,1]$, the vectors $(Q_{(3,t)}u)[\beta_3]$ and $(Q_{(3,t)}v)[\beta_3]$ are positive.
- If α_1 is empty then the matrix $Q_{(3,t)} = I_n$ for all t in [0,1], otherwise, there is a sufficiently small positive scalar δ_3 which depends on ϵ such that for all t in $[0,1]$ we have $(Q_{(3,t)}u)[\{k_1+k_2+1\}] = u_{k_1+k_2+1}\cos(\delta_3 t)$ and $(Q_{(3,t)}v)[\{k_1+k_2+1\}] = v_{k_1+k_2+1}\cos(\delta_3 t)$.
- The matrix $Q_{(3,t)}[\langle n\rangle\backslash\beta_3] = I_{n-k_1-1}$ for all t in $[0,1]$.

As a result of these properties, the matrix $Q_{(3,t)}$ commutes with both $Q_{(1,t)}$ and $Q_{(2,t)}$. Moreover, the vectors $Q_{(3,t)}Q_{(2,t)}Q_{(1,t)}u$ and $Q_{(3,t)}Q_{(2,t)}Q_{(1,t)}v$ in $\mathbb{R}^n$ will be positive for all t in $(0,1]$.

To construct the matrix $Q_{(3,t)}$, consider the set α_1. If α_1 is empty, then define $Q_{(3,t)} := I_n$ for all $t\in[0,1]$, otherwise consider the vectors $u[\beta_3]$ and $v[\beta_3]$ in $\mathbb{R}^{k_1+1}$ and let $\tilde{\delta} := v_{k_1+k_2+1}/u_{k_1+k_2+1} > 0$. Then, the vector $v[\beta_3] = \tilde{\delta}u[\beta_3]$ is a semipositive vector with only one positive entry. By Lemma 6.10, there is a path $\{P_t \,:\, t\in[0,1]\}$ of $(k_1+1)\times(k_1+1)$ real orthogonal matrices such that $P_0 = I_{k_1+1}$, the vectors $P_t\,(u[\beta_3])$ and $P_t\,(v[\beta_3])$ are positive vectors for all $t\in(0,1]$, the inequality $||P_t - I_{k_1+1}|| < \epsilon$ holds for all $t\in[0,1]$, and the last entries of the vectors $P_t\,(u[\beta_3])$ and $P_t\,(v[\beta_3])$ are respectively $u_{k_1+k_2+1}\cos(\delta_3 t)$ and $v_{k_1+k_2+1}\cos(\delta_3 t)$ for all t in $[0,1]$ for a sufficiently small positive scalar δ_3 which depends on ϵ. Define the matrix $Q_{(3,t)}$ as follows: $Q_{(3,t)}[\beta_3] = P_t$ and $Q_{(3,t)}[\langle n\rangle\backslash\beta_3] = I_{n-k_1-1}$. Then, the matrix $Q_{(3,t)}$ satisfies the

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properties required in this step.

Hence, $(i) \Rightarrow (ii)$. Conversely, if (ii) is true then let a positive scalar ϵ be given and let $\{Q_t : t \in [0,1]\}$ be the path of real orthogonal matrices guaranteed by (ii). Then, for any $t \in (0,1]$, we have $u^T v = (Q_t u)^T (Q_t v) > 0$. This shows that $(ii) \Rightarrow (i)$. ∎

The following lemma is [14, Lemma 6.3.10]. By y^* we denote the conjugate transpose of a vector $y \in \mathbb{C}^n$.

Lemma 6.13. *Let A be a matrix in $\mathbb{C}^{n\times n}$ and let λ be a simple eigenvalue of A. If x and y are, respectively, right and left eigenvectors of A corresponding to the eigenvalue λ then $y^* x \neq 0$.*

Theorem 6.14. *The set of matrices in $\mathbb{R}^{n\times n}$ whose spectral radius is a simple positive eigenvalue with corresponding nonnegative right and left eigenvectors is path-connected.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be a matrix whose spectral radius $\rho(A)$ is a simple positive eigenvalue with corresponding nonnegative right and left eigenvectors, u and v, respectively. Then, by Lemma 6.13 we have $u^T v > 0$. And thus, by Theorem 6.12, for any given positive scalar ϵ, there is a path $\{Q_t : t \in [0,1]\}$ of $n \times n$ real orthogonal matrices such that $Q_0 = I_n$, $Q_t u$ is a positive vector for all $t \in (0,1]$, $Q_t v$ is a positive vector for all $t \in (0,1]$, and $||Q_t - I_n|| < \epsilon$ for all $t \in [0,1]$. Define the matrix $A_t := Q_t A Q_t^T$ for all t in [0,1] and consider the path $\{A_t : t \in [0,1]\}$. Note that $A_0 = A$ and that for all $t \in (0,1]$ the spectral radius $\rho(A_t)$ is a simple positive eigenvalue of A_t with corresponding positive right and left eigenvectors. Set $B = A_1$ and let $x,\ y \in \mathbb{R}^n$ be positive right and left eigenvectors of B corresponding to $\rho(B)$. Define the matrix $B_t := B + txy^T$ for all $t \in [0,\infty)$. The matrix B_t is a continuous function of $t \in [0,\infty)$ and by Theorem 6.1 its spectral radius $\rho(B_t)$ is a simple positive eigenvalue of B_t with corresponding positive right and left eigenvectors. Moreover, for some large values of t, the matrix B_t is positive. ∎

Theorem 6.15. *The closure of the set of $n \times n$ real matrices with the strong Perron-Frobenius property is equal to the set of $n \times n$ real matrices with the Perron-Frobenius property.*

Proof. The fact that the closure of the set of $n \times n$ real matrices with the strong Perron-Frobenius property is included in the set of $n\times n$ real matrices with the Perron-Frobenius property easily follows by taking limits. To prove the reverse inclusion, let A be an $n \times n$ real matrix that has the Perron-Frobenius property. Choose is an $n \times n$ nonsingular real matrix P such that $P^{-1}AP$ is in real Jordan canonical form. We may assume that the first diagonal block of $P^{-1}AP$ is $J_k(\rho(A))$ and the first column of P is semipositive. For any $\epsilon > 0$, let P_ϵ denote the matrix obtained from P by

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replacing the zero entries (if any) in the first column of P by ϵ and also let J_ϵ denote the matrix obtained from $P^{-1}AP$ by replacing its (1,1)-entry by $\rho(A)+\epsilon$. Note that P_ϵ is nonsingular for $\epsilon > 0$ sufficiently small, and also that for any such ϵ, the matrix $P_\epsilon J_\epsilon P_\epsilon^{-1}$ has the strong Perron-Frobenius property. Since $\lim_{\epsilon\to 0^+} P_\epsilon J_\epsilon P_\epsilon^{-1} = A$, it follows that A belongs to the closure of the set of $n\times n$ real matrices with the strong Perron-Frobenius property. □

Corollary 6.16. *The following sets of matrices are closed subsets of* $\mathbb{R}^{n\times n}$*:*

(i) The set of matrices having the Perron-Frobenius property.
(ii) WPFn.

Remark 6.17. Theorem 6.15 does not imply that WPFn is the closure of PFn. It is not generally true that the closure of PFn is equal to WPFn. Take, for example PF2. The set PF2 is characterized as the set of all 2×2 matrices with positive off-diagonal entries and positive trace [16]. Hence, any convergent sequence of matrices in PF2 would converge to a 2×2 matrix whose off-diagonal entries and trace are nonnegative. If we look at the matrix $A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$, which is a matrix in WPF2, we can see that there is no sequence of matrices in PF2 converging to the matrix A since its (1,2)-entry, which is an off-diagonal entry, is negative. On the other hand, if we restrict our attention to the set $\{A \in \text{WPFn} \ : \ \rho(A) \text{ is a simple eigenvalue}\}$, we get the following result.

Theorem 6.18. *If A is a matrix in WPFn such that $\rho(A)$ is a simple eigenvalue then for every scalar $\epsilon > 0$ there is a matrix B in PFn such that $||B - A|| < \epsilon$.*

Proof. Consider the real Jordan canonical form of matrix A, which we denote by $J_{\mathbb{R}}(A)$. The matrix A can be written as $VJ_{\mathbb{R}}(A)V^{-1} = V\,[[\rho(A)]\oplus J_2]\,V^{-1}$ where J_2 is the direct sum of all diagonal blocks appearing in $J_{\mathbb{R}}(A)$ that correspond to eigenvalues other than $\rho(A)$. Let u and v be, respectively, the first column of V and the transpose of the first row of V^{-1}. Thus, u and v are, respectively, right and left eigenvectors of A corresponding to $\rho(A)$. Moreover, we may assume that u and v are semipositive. Then, $u^Tv = 1$. Let a scalar $\epsilon > 0$ be given. By Theorem 6.12, there is a path $\{Q_t \ : \ t \in [0,1]\}$ of real orthogonal matrices such that $Q_0 = I_n$, $Q_t u$ is a positive vector for all $t \in (0,1]$, $Q_t v$ is a positive vector for all $t \in (0,1]$, and $||Q_t - I_n|| < \epsilon$ for all $t \in [0,1]$. Define the matrix $B_t := Q_t V\,[[\rho(A)+t]\oplus J_2]\,V^{-1}Q_t^T$ for all t in $[0,1]$. The matrix B_t is in PFn for all t in $(0,1]$. Furthermore, for a sufficiently small value of t in $(0,1]$, say t_0, we have $||B_{t_0} - A|| < \epsilon$. The matrix $B := B_{t_0}$ is the desired matrix. □

In our next result, we show that every normal matrix in WPFn is the limit of normal matrices in PFn. Since WPFn is a closed set (by Corollary 6.16) and since the set of normal matrices in $\mathbb{R}^{n\times n}$ is also a closed set, it is easy to see that the next

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result indeed shows that the closure of the set of normal matrices in PFn is the set of normal matrices in WPFn.

THEOREM 6.19. *If A is a normal matrix in WPFn then for any positive scalar ϵ there is a normal matrix B in PFn such that $||B - A|| < \epsilon$.*

Proof. Let A be a normal matrix in WPFn. Then,

$$A = V\,[[\rho(A)] \oplus M_2 \oplus \cdots \oplus M_k]\,V^T,$$

where V is a real orthogonal matrix and each M_i for $i = 2, \ldots, k$ is a 1×1 real block or a positive scalar multiple of a 2×2 real orthogonal block; see, e.g., [14, Theorem 2.5.8]. Moreover, the first column of V, which we denote by v, is both a right and a left Perron-Frobenius eigenvector of A. Let ϵ be any given positive real number. Choose a scalar $\delta > 0$ sufficiently small so that the matrix $B_\delta = V\,[[\rho(A) + \delta] \oplus M_2 \oplus \cdots \oplus M_k]\,V^T$ satisfies $||B_\delta - A|| < \frac{\epsilon}{2}$, where $||\cdot||$ is any norm of the set of $n \times n$ real matrices. By Lemma 6.10, there is a path $\{Q_t \;:\; t \in [0, 1]\}$ of $n \times n$ real orthogonal matrices such that $Q_0 = I_n$, $Q_t v$ is a positive vector for all $t \in (0, 1]$, and $||Q_t - I_n|| < \frac{\epsilon}{2}$ for all $t \in [0, 1]$. Note that $Q_t B_\delta Q_t^T$ is a real normal matrix with a simple positive strictly dominant eigenvalue $\rho(A) + \delta$ and a corresponding positive right and left eigenvector (namely, $Q_t v$) for all $t \in (0, 1]$, i.e., $Q_t B_\delta Q_t^T$ is in PFn for all $t \in (0, 1]$. For a sufficiently small value of $t \in (0, 1]$, say t_0, we have $||Q_{t_0} B_\delta Q_{t_0}^T - A|| < \epsilon$. Set $B := Q_{t_0} B_\delta Q_{t_0}^T$. Then, B is the desired matrix. ∎

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