Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 193-210, February 2021.

NONSPARSE COMPANION HESSENBERG MATRICES*

ALBERTO BOROBIA† AND ROBERTO CANOGAR†

Abstract. In recent years, there has been a growing interest in companion matrices. Sparse companion matrices are well known: every sparse companion matrix is equivalent to a Hessenberg matrix of a particular simple type. Recently, Deaett et al. [*Electron. J. Linear Algebra,* 35:223–247, 2019] started the systematic study of nonsparse companion matrices. They proved that every nonsparse companion matrix is nonderogatory, although not necessarily equivalent to a Hessenberg matrix. In this paper, the nonsparse companion matrices which are unit Hessenberg are described. In a companion matrix, the variables are the coordinates of the characteristic polynomial with respect to the monomial basis. A PB-companion matrix is a generalization, in the sense that the variables are the coordinates of the characteristic polynomial with respect to a general polynomial basis. The literature provides examples with Newton basis, Chebyshev basis, and other general orthogonal bases. Here, the PB-companion matrices which are unit Hessenberg are also described.

Key words. Companion matrix, Characteristic polynomial, Hessenberg matrix, Polynomial basis, Nilpotent matrix.

AMS subject classifications. 15A54, 15B99, 15A21.

1. Introduction. For a given field $\mathbb{F}$, a matrix A of order n is said to be *companion* provided: (i) A has n^2-n entries that are constants of $\mathbb{F}$; (ii) the n remaining entries of A are the variables $x_1,\ldots,x_n$; and (iii) the characteristic polynomial of A is

$$\det(\lambda I_n - A) = \lambda^n - x_1\lambda^{n-1} - \cdots - x_{n-1}\lambda - x_n. \tag{1.1}$$

The classical example is the Frobenius companion matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ x_n & x_{n-1} & \cdots & x_2 & x_1 \end{bmatrix}.$$

In 2003, Fiedler [8] introduced new companion matrices, which in turn produced an increased interest in companion matrices. In 2013, Ma and Zhan [10] studied those matrices of order n whose entries are in the field $\mathbb{F}(x_1,\ldots,x_n)$ of rational functions in the variables $x_1,\ldots,x_n$ and for which the characteristic polynomial is (1.1). They showed that such matrices are necessarily irreducible and have at least $2n-1$ nonzero entries. This allows us to define a *sparse companion matrix* as a companion matrix with $2n-1$ nonzero entries. In 2016, Garnett et al. [9] characterized the Ma-Zhan matrices.

A square matrix is a *unit lower Hessenberg matrix*, *ULH matrix* for short, if all its superdiagonal entries are equal to one and all its entries above the superdiagonal are equal to zero. In 2014, Eastman et al. [5, 6]

*Received by the editors on May 7, 2020. Accepted for publication on February 16, 2021. Handling Editor: Heike Fassbender. Corresponding Author: Roberto Canogar. Proyecto financiado por la Agencia Estatal de Investigación de España (PID2019-106362GB-I00/AEI/10.13039/501100011033) y (MTM2017-90682-REDT).

†Dpto. Matemáticas, Universidad Nacional de Educación a Distancia (UNED), Madrid, Spain (aborobia@mat.uned.es, rcanogar@mat.uned.es).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 193-210, February 2021.

showed that every sparse companion matrix can be transformed into a ULH matrix by some combination of transposition, permutation similarity, and diagonal similarity (we will be more precise in Theorems 3.1 and 3.2 below). In 2019, Deaett et al. [3] gave one example that shows that this is not necessarily the case for nonsparse companion matrices. On the other hand, they proved that any realization of a companion matrix is nonderogatory.

If A is a companion matrix, then $(1, -x_1, \dots, -x_n)$ are the coordinates of $\det(\lambda I_n - A)$ with respect to the monomial basis $\{\lambda^n, \lambda^{n-1}, \dots, \lambda, 1\}$ of the $(n+1)$-dimensional vector space $\mathbb{F}_n[\lambda]$ of polynomials of degree at most n on the variable λ. The concept of companion matrix can be generalized from the monomial basis to an arbitrary polynomial basis. For example, consider the Newton basis

$$\left\{\lambda \prod_{h=1}^{n-1}(\lambda-\gamma_h),\ \prod_{h=1}^{n-1}(\lambda-\gamma_h),\ \dots,\ (\lambda-\gamma_1), 1\right\},$$

that occurs on interpolation at the points $\lambda = \gamma_1, \dots, \lambda = \gamma_{n-1}, \lambda = 0$. Farahat and Ledermann [7] noted that a Frobenius companion matrix in which we modify the main diagonal,

$$A = \begin{bmatrix} \gamma_1 & 1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_{n-1} & 1 \\ x_n & \cdots & \cdots & x_2 & x_1 \end{bmatrix},$$

has characteristic polynomial

$$\det(\lambda I_n - A) = \lambda \prod_{h=1}^{n-1}(\lambda-\gamma_h) - x_1 \prod_{h=1}^{n-1}(\lambda-\gamma_h) - \cdots - x_{n-1}(\lambda-\gamma_1) - x_n.$$

Therefore, $(1, -x_1, \dots, -x_n)$ are the coordinates of $\det(\lambda I_n - A)$ with respect to the Newton basis.

By results of [1, 2, 11], to name a few references in this direction, we can go even further and regard

$$\begin{bmatrix} d_{11} & 1 & 0 & \cdots & 0 \\ d_{21} & d_{22} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ d_{n-1,1} & \cdots & d_{n-1,n-2} & d_{n-1,n-1} & 1 \\ x_n & \cdots & x_3 & x_2 & x_1 \end{bmatrix}, \tag{1.2}$$

as a companion matrix with respect to some polynomial basis of $\mathbb{F}_n[\lambda]$. We will not give the specific polynomial basis for this matrix since in Theorems 4.3 and 4.8 we will consider broader families of matrices (that includes (1.2) as a particular case) for which such a polynomial basis exists. And in the proof of Lemma 4.2, we will see how the polynomial basis is constructed. These kind of results have motivated us to generalize the concept of companion matrix[1].

[1]Garnett et al. in [9] employ the term generalized companion matrix for those matrices of order n with $2n-1$ nonzero entries, $n-1$ are ones and n are elements of the ring $\mathbb{F}[x_1, \dots, x_n]$, with characteristic polynomial equal to (1.1). In the literature there exist other generalizations of companion matrices (see for instance, the introduction of the recent work of De Terán and Hernando [4]).

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A matrix A of order n is *companion with respect to a polynomial basis*, or *PB-companion* by short, provided: (i) A has $n^2 - n$ entries that are in the field $\mathbb{F}$; (ii) the n remaining entries of A are the variables $x_1, \ldots, x_n$; and (iii) the characteristic polynomial of A is

$$\det(\lambda I_n - A) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_{n-1} p_1(\lambda) - x_n p_0(\lambda),$$

where $\{p_n(\lambda), \ldots, p_1(\lambda), p_0(\lambda)\}$ is a polynomial basis of $\mathbb{F}_n[\lambda]$. A companion matrix is PB-companion with respect to the monomial basis $\{\lambda^n, \ldots, \lambda, 1\}$ of $\mathbb{F}_n[\lambda]$.

One of the major applications of companion matrices is the numerical approximation of the roots of polynomials. It has been known since the early 19th century that for polynomials of degree greater than 4, an algebraic (closed-form) solution for the roots is usually not possible. Consequently, numerical methods in this context have become a major topic of interest. Among the most successful approaches to finding roots of polynomials is the construction of a matrix or matrix pencil whose eigenvalues coincide with those roots, followed by the numerical approximation of the said eigenvalues. Of course, companion matrices are the matrices that are needed.

Although much is known about the stability of numerical methods for companion matrices with respect to the monomial basis, little is known about the stability of these methods for nonmonomial bases such as Chebyshev bases, which are becoming increasingly more important. For example, [12] focuses on stability of linearization methods for polynomials expressed in certain nonmonomial bases. Also in [12], it is mentioned that the first step of the QR method involves the reduction to Hessenberg form, a motivation for starting with a Hessenberg matrix.

In this work, we will not be concerned by numerical questions. We will focus on a theoretical description of both the companion unit lower Hessenberg matrices and the PB-companion unit lower Hessenberg matrices. Understanding their structure might be useful.

2. Different sets of ULH matrices to be considered. In the ULH matrices that are candidates to be companion, the positions $(i_1, j_1), \ldots, (i_n, j_n)$ of the variables $x_1, \ldots, x_n$ below the subdiagonal play an essential role. In order to avoid redundancies, we define a total order in the set $\mathbb{Z} \times \mathbb{Z}$ by

$$(i_r, j_r) \prec (i_s, j_s) \text{ if and only if } \begin{cases} i_r - j_r < i_s - j_s \\ \text{or} \\ i_r - j_r = i_s - j_s \text{ and } i_r < i_s. \end{cases}$$

This total order establishes the convention that tracks the order in which variables $x_1, \ldots, x_n$ are placed.

For the proper development of this work, we will introduce several sets of ULH matrices:

- Let $\mathcal{H}$ be the set composed of ULH matrices of any order n that have $2n-1$ nonzero entries: $n-1$ entries in the superdiagonal equal to one, and n entries below the superdiagonal equal to $x_1, \ldots, x_n$ that are placed according to the order $\prec$. That is, $\mathcal{H}$ consists of all matrices $H_{(i_1,j_1),\ldots(i_n,j_n)}$ such that:

$$\begin{gathered} H_{(i_1,j_1),\ldots,(i_n,j_n)} \text{ is the ULH matrix of order } n \text{ with } 2n-1 \text{ nonzero entries} \\ \text{whose } (i_1,j_1),\ldots,(i_n,j_n) \text{ entries are } x_1,\ldots,x_n \\ \text{with } (i_1,j_1) \prec \cdots \prec (i_n,j_n) \text{ and } 1 \le j_k \le i_k \le n \text{ for each } k = 1,\ldots,n. \end{gathered}$$

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- Let $\mathcal{D}$ be the set composed of those matrices of $\mathcal{H}$ in which the variable x_1 is located in the diagonal, x_2 in the subdiagonal, x_3 in the second subdiagonal, and so forth. Let us describe the matrices of $\mathcal{D}$ more precisely:

$$\mathcal{D} = \{D_{i_1,i_2,\dots,i_n} | D_{i_1,i_2,\dots,i_n} = H_{(i_1,i_1),(i_2,i_2-1),\dots,(i_n,i_n-n+1)} \text{ with } i_k \geq k \text{ for each } k \in \{1,\dots,n\}\}.$$

- Let $\mathcal{C}$ be the set composed of those matrices of $\mathcal{D}$ in which the variables $x_1,\dots,x_n$ lay in the rectangular submatrix with top-right vertex the (i_1,i_1) entry and with bottom-left vertex the $(n,1)$ entry. Let us describe the matrices of $\mathcal{C}$ more precisely:

$$\mathcal{C} = \{C_{i_1,\dots,i_n} | C_{i_1,\dots,i_n} = D_{i_1,\dots,i_n} \text{ with } i_k - k + 1 \leq i_1 \leq i_k \text{ for each } k \in \{1,\dots,n\}\}. \tag{2.3}$$

- Let $\mathcal{G}$ be the set composed of those matrices of $\mathcal{C}$ which have at least one variable on each one of the rows $i_1,\dots,n$ and on each one of the columns $1,\dots,i_1$. Let us describe the matrices of $\mathcal{G}$ more precisely:

$$\begin{aligned}\mathcal{G} = \{G_{i_1,\dots,i_n} | G_{i_1,\dots,i_n} = C_{i_1,\dots,i_n} \text{ with } &\{i_1,\dots,i_n\} = \{i_1,\dots,n\} \text{ and}\\ &\{i_1, i_2-1,\dots,i_n-n+1\} = \{1,\dots,i_1\}\}.\end{aligned}$$

- And finally, let $\mathcal{F}$ be the set composed of those matrices of $\mathcal{G}$ such that for each $\in \{2,\dots,n\}$ the variable x_k is in the same row or in the same column than the variable x_{k-1}. So in a matrix of $\mathcal{F}$ the variables $x_1,\dots,x_n$ form a lattice path starting on the bottom-left corner with x_n and ending on the diagonal with x_1. Let us describe the matrices of $\mathcal{F}$ more precisely:

$$\mathcal{F} = \{F_{i_1,\dots,i_n} | F_{i_1,\dots,i_n} = G_{i_1,\dots,i_n} \text{ with } i_k \in \{i_{k-1}, i_{k-1}+1\} \text{ for each } k \in \{2,\dots,n\}\}.$$

As the following examples show, we have $\mathcal{H} \supsetneq \mathcal{D} \supsetneq \mathcal{C} \supsetneq \mathcal{G} \supsetneq \mathcal{F}$.

$H_{(1,1),(3,3),(2,1),(5,4),(5,2)} \in \mathcal{H} \setminus \mathcal{D}$	$D_{3,5,3,5,5} \in \mathcal{D} \setminus \mathcal{C}$	$C_{3,4,3,4,5} \in \mathcal{C} \setminus \mathcal{G}$
$\begin{bmatrix} x_1 & 1 & 0 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & x_5 & 0 & x_4 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ x_3 & 0 & x_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ x_5 & x_4 & 0 & x_2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ x_3 & 0 & x_1 & 1 & 0 \\ x_4 & 0 & x_2 & 0 & 1 \\ x_5 & 0 & 0 & 0 & 0 \end{bmatrix}$

$G_{3,4,3,5,5} \in \mathcal{G} \setminus \mathcal{F}$	$F_{3,4,4,4,5} \in \mathcal{F}$
$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ x_3 & 0 & x_1 & 1 & 0 \\ 0 & 0 & x_2 & 0 & 1 \\ x_5 & x_4 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_1 & 1 & 0 \\ x_4 & x_3 & x_2 & 0 & 1 \\ x_5 & 0 & 0 & 0 & 0 \end{bmatrix}$

Observations: 1. The choice of letters for the sets of matrices was made according to the following mnemonic rule: $\mathcal{H}$ for Hessenberg (this is the largest set to consider), $\mathcal{D}$ for diagonal (each variable is in a different diagonal), $\mathcal{C}$ for companion (these are the sparse companion ULH matrices), $\mathcal{G}$ for gather (rows i_1 to n and columns 1 to i_1 gather all variables with each row and each column having at least one), and $\mathcal{F}$ for Fiedler (these are the Fiedler companion matrices [8] as demonstrated in [5]).

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A publication of the International Linear Algebra Society
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2. Each matrix A of order n has an associated digraph of n vertices in which the vertex i is joined with the vertex j by an oriented edge if and only if the (i,j) entry of A is nonzero. If $A_1 \in \mathcal{H} \setminus \mathcal{D}$, $A_2 \in \mathcal{D} \setminus \mathcal{C}$, $A_3 \in \mathcal{C} \setminus \mathcal{G}$, $A_4 \in \mathcal{G} \setminus \mathcal{F}$, and $A_5 \in \mathcal{F}$, then it is not difficult to see that the digraph associated to A_i is not isomorphic to the digraph associated to A_j for $1 \leq i < j \leq 5$. In [5] the authors studied the digraphs associated to matrices of $\mathcal{C}$ and $\mathcal{F}$.

3. Companion ULH matrices. From now on we will use the following definitions and notations:

(a) Unless otherwise indicated, a matrix A is assumed to be a square matrix of order n.
(b) $P_A(\lambda)$ refers to the characteristic polynomial of A.
(c) The matrix A of order 0 is the *empty matrix*, with $\det(A) = 1$ and $P_A(\lambda) = 1$.
(d) The *constant part* of a matrix A whose entries are elements of the ring $\mathbb{F}[x_1, \dots, x_n]$ is the constant matrix obtained from A by setting $x_1 = \cdots = x_n = 0$.
(e) For each $k \in \{0, 1, \dots, n-1\}$ the *k-subdiagonal* of A is $(a_{k+1,1}, a_{k+2,2}, \dots, a_{n,n-k})$.
(f) $s_k(A)$ is the sum of the entries on the k-subdiagonal of A.
(g) $A[i_1, \dots, i_r; j_1, \dots, j_s]$ is the submatrix of A restricted to all entries in rows $i_1, \dots, i_r$ and columns $j_1, \dots, j_s$.
(h) The *k-block* of A is the submatrix $A[k, \dots, n; 1, \dots, k]$ of A with top-right corner at position (k,k) and bottom-left corner at position $(n,1)$.
(i) $A[k_1, \dots, k_r]$ is $A[k_1, \dots, k_r; k_1, \dots, k_r]$.
(j) The *k-leading principal submatrix of* A is $A[1, \dots, k]$.
(k) The *k-trailing principal submatrix of* A is $A[n-k+1, \dots, n]$.
(l) $A[i, \dots, j]$ is the empty matrix if $i, \dots, j$ is not a valid range for A.
(m) A *companion ULH matrix* is a companion matrix which is also ULH.
(n) Starting with a matrix A, if $B = A$ or B is obtained by changing some zero entries of A to nonzero constants, we will say that B is a *superpattern* of A. This terminology comes from [5].
(o) If starting with a ULH matrix A we obtain B by changing some zero entries of A below the superdiagonal to nonzero constants, we will say that B is a *ULH superpattern* of A.
(p) The set of all ULH superpatterns of all matrices of, respectively, $\mathcal{H}$, $\mathcal{D}$, $\mathcal{C}$, $\mathcal{G}$, and $\mathcal{F}$ is denoted by, respectively, $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{G}}$, and $\widetilde{\mathcal{F}}$.

3.1. From sparse to nonsparse. Eastman et al. characterized the sparse companion matrices.

THEOREM 3.1 ([6]). *A is a sparse companion matrix if and only if A can be obtained from some matrix of $\mathcal{C}$ by some combination of transposition, permutation similarity, and diagonal similarity.*

They also determined the sparse companion matrices that are in $\mathcal{H}$. Note that the set of sparse companion matrices of $\mathcal{H}$ is the same as the set of sparse companion ULH matrices. So we adapt their result to the notation we have introduced.

THEOREM 3.2 ([5, Theorem 4.1]). *The sparse companion ULH matrices are the matrices of $\mathcal{C}$.*

For nonsparse companion matrices we will no longer obtain a result like Theorem 3.1. Namely, Deaett et al. [3] gave one particular nonsparse companion matrix that cannot be transformed into a ULH matrix by any combination of transposition, permutation similarity, and diagonal similarity. So we will focus our efforts to obtain a result like Theorem 3.2 that is also valid for non-sparse companion ULH matrices. The following theorem tells us where we should look for the companion ULH matrices and how to characterize them.

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Theorem 3.3. *Any companion ULH matrix belongs to $\widetilde{\mathcal{C}}$. Moreover, any ULH superpattern A of $C_{i_1,\ldots,i_n}$ is a companion matrix if and only if the following conditions are met:*

(*i*) *The constant part of A is a nilpotent matrix;*
(*ii*) *$A[1,\ldots,i_k-k]$ is a constant nilpotent matrix or the empty matrix for each $k\in\{1,\ldots,n\}$;*
(*iii*) *$A[i_k+1,\ldots,n]$ is a constant nilpotent matrix or the empty matrix for each $k\in\{1,\ldots,n\}$.*

If the only nonzero entries of the i_1-block of A are $x_1,\ldots,x_n$ then condition (i) is redundant.

Proof. The set of companion ULH matrices is the same as the set of companion matrices of $\widetilde{\mathcal{H}}$, so in order to prove that any companion ULH matrix belongs to $\widetilde{\mathcal{C}}$ it is enough to demonstrate that any matrix of $\widetilde{\mathcal{H}}\setminus\widetilde{\mathcal{C}}$ is not companion.

If $B\in\widetilde{\mathcal{H}}\setminus\widetilde{\mathcal{D}}$, then some k-subdiagonal of B contains two variables, say x_r and x_s. Let us compute the characteristic polynomial of B using the Leibniz formula for the determinant of λI_n-B. Among all terms that contain x_r only one also contains $n-k-1$ instances of λ, which gives a unique term of type $\pm x_r\lambda^{n-k-1}$. The same happens for x_s, which gives a unique term of type $\pm x_s\lambda^{n-k-1}$. Then B is not companion since none of those two terms can be canceled.

If $B\in\widetilde{\mathcal{D}}\setminus\widetilde{\mathcal{C}}$ with the variable x_1 on the (i_1,i_1) entry of B, then some variable x_s is on the $(s-1)$-subdiagonal and out of the i_1-block. Again compute the characteristic polynomial of B using the Leibniz formula for the determinant of λI_n-B. Among all terms that include x_1 and x_s only one also contains $n-1-s$ instances of λ. This gives a unique term of type $\pm x_1x_s\lambda^{n-1-s}$ that cannot be canceled. Then, B is not companion.

Now we go to the second part of the theorem. Recall $P_A(\lambda)$ is the notation for the characteristic polynomial of a matrix that we introduced in (b) at the beginning of Section 3. Suppose $A\in\widetilde{\mathcal{C}}$. If the matrix A is a superpattern of $C_{i_1,\ldots,i_n}$, then all the terms of $P_A(\lambda)$ in which variables $x_1,\ldots,x_n$ do not appear conform to $P_{A_0}(\lambda)$ where A_0 is the constant part of A. On the other hand, as A is a ULH matrix and x_k is the (i_k,i_k-k+1) entry of A, then x_k appears in $P_A(\lambda)$ only in the term

$$-x_k\,P_{A[1,\ldots,i_k-k]}(\lambda)\,P_{A[i_k+1,\ldots,n]}(\lambda).$$

Note that $A[1,\ldots,i_k-k]$ is the empty matrix if $i_k=k$ and $A[i_k+1,\ldots,n]$ is the empty matrix if $i_k=n$. As the variables $x_1,\ldots,x_n$ are in the i_1-block of A then $A[1,\ldots,i_k-k]$ is a submatrix of the constant matrix $A[1,\ldots,i_1-1]$ and $A[i_k+1,\ldots,n]$ is a submatrix of the constant matrix $A[i_1+1,\ldots,n]$. Therefore,

$$P_A(\lambda)=P_{A_0}(\lambda)-\sum_{k=1}^{n}x_k\,P_{A[1,\ldots,i_k-k]}(\lambda)\,P_{A[i_k+1,\ldots,n]}(\lambda), \tag{3.4}$$

with $P_{A[1,\ldots,i_k-k]}(\lambda)P_{A[i_k+1,\ldots,n]}(\lambda)$ containing no variables in $\{x_1,\ldots,x_n\}$.

So A is a companion matrix if and only if equations (3.4) and (1.1) match, that is, if and only if

$$\begin{aligned}\lambda^n&=P_{A_0}(\lambda);\\ \lambda^{n-k}&=P_{A[1,\ldots,i_k-k]}(\lambda)\,P_{A[i_k+1,\ldots,n]}(\lambda)\quad\text{for each } k\in\{1,\ldots,n\};\end{aligned}$$

if and only if

$$\begin{aligned}\lambda^n&=P_{A_0}(\lambda);\\ \lambda^{i_k-k}&=P_{A[1,\ldots,i_k-k]}(\lambda)\quad\text{for each } k\in\{1,\ldots,n\};\\ \lambda^{n-i_k}&=P_{A[i_k+1,\ldots,n]}(\lambda)\quad\text{for each } k\in\{1,\ldots,n\};\end{aligned}$$

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if and only if

$$\begin{aligned} &A_0 \ \text{ is nilpotent;}\\ &A[1,\ldots,i_k-k] \ \text{ is a constant nilpotent matrix or the empty matrix for each } k\in\{1,\ldots,n\};\\ &A[i_k+1,\ldots,n] \ \text{ is a constant nilpotent matrix or the empty matrix for each } k\in\{1,\ldots,n\}.\end{aligned}$$

Let us prove the third part. If the only nonzero entries of the i_1-block of A are $x_1,\ldots,x_n$, then

$$A_0=\begin{bmatrix} A[1,\ldots,i_1-1] & e_{i_1-1} & 0\\ 0 & 0 & e_1^T\\ 0 & 0 & A[i_1+1,\ldots,n]\end{bmatrix}.$$

Since A_0 is block upper triangular, the nilpotence of A_0 follows from (ii) and (iii). ☐

3.2. Nested nilpotent ULH matrices. Let $n_1,\ldots,n_s$ be integers with $0\le n_1,\ldots,n_s\le n$. We will denote by $\mathfrak{N}_L^n(n_1,\ldots,n_s)$ the set composed of constant ULH matrices of order n such that for each $h\in\{1,\ldots,s\}$ with $n_h\neq 0$ the n_h-leading principal submatrix is nilpotent. And we will denote by $\mathfrak{N}_T^n(n_1,\ldots,n_s)$ the set composed of ULH matrices of order n such that for each $h\in\{1,\ldots,s\}$ with $n_h\neq 0$ the n_h-trailing principal submatrix is nilpotent. For convenience we have included the possibility that the sequence $n_1,\ldots,n_s$ contains repeated elements or elements equal to zero.

If $J_n=\begin{bmatrix} 0 & & 1\\ & \iddots & \\ 1 & & 0\end{bmatrix}$ is the *reverse unit matrix* then

$$\mathfrak{N}_T^n(n_1,\ldots,n_s)=\big(J_n\,\mathfrak{N}_L^n(n_1,\ldots,n_s)\,J_n\big)^T.$$

The transposition is necessary to transform ULH matrices into ULH matrices. An example:

$$A=\left[\begin{array}{rrr|r} 0 & 1 & 0 & 0\\ 3 & -5 & 1 & 0\\ 15 & -28 & 5 & 1\\ \hline 0 & 0 & 0 & 0\end{array}\right]\in\mathfrak{N}_L^4(1,3,4)\ \text{ and }\ \big(J_n\,A\,J_n\big)^T=\left[\begin{array}{r|rrr} 0 & 1 & 0 & 0\\ \hline 0 & 5 & 1 & 0\\ 0 & -28 & -5 & 1\\ 0 & 15 & 3 & 0\end{array}\right]\in\mathfrak{N}_T^4(1,3,4).$$

Now we will state Theorem 3.3 in terms of nested nilpotent matrices.

THEOREM 3.4. *Any companion ULH matrix belongs to $\widetilde{\mathcal{C}}$. Moreover, any ULH superpattern A of $C_{i_1,\ldots,i_n}$ is a companion matrix if and only if the following conditions are met:*

(i) *The constant part of A is a nilpotent matrix;*
(ii) $A[1,\ldots,i_1-1]\in\mathfrak{N}_L^{i_1-1}(i_1-1,i_2-2,\ldots,i_n-n)$*;*
(iii) $A[i_1+1,\ldots,n]\in\mathfrak{N}_T^{n-i_1}(n-i_1,n-i_2,\ldots,n-i_n)$*.*

If the only nonzero entries of the i_1-block of A are $x_1,\ldots,x_n$, then condition (i) can be omitted.

Proof. It is enough to prove that conditions (ii) and (iii) in Theorems 3.3 and 3.4 are equivalent. Taking into account (2.3) we conclude that $0\le i_k-k\le i_1-1$ for each $k\in\{1,\ldots,n\}$ and so conditions (ii) in both theorems are equivalent. Again, taking into account (2.3) we conclude that $i_1\le i_k\le n$ for each $k\in\{1,\ldots,n\}$ and so conditions (iii) in both theorems are equivalent. ☐

In Section 3.4, we will show how to parametrize the sets $\mathfrak{N}_L^n(n_1,\ldots,n_s)$ and $\mathfrak{N}_T^n(n_1,\ldots,n_s)$. First, we analyze a specially simple case.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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LEMMA 3.5. *$\mathfrak{N}_L^n(1,\dots,n)=\mathfrak{N}_T^n(1,\dots,n)=\{U_n\}$, where U_n is the upper shift matrix with ones on the superdiagonal and zeroes elsewhere.*

Proof. We will prove by induction that $\mathfrak{N}_L^n(1,\dots,n)=\{U_n\}$. For $n=1$, $\mathfrak{N}_L^1(1)=\{[\,0\,]\}=\{U_1\}$. Assume that it is true for k, that is, $\mathfrak{N}_L^k(1,\dots,k)=\{U_k\}$. Let $A\in\mathfrak{N}_L^{k+1}(1,\dots,k+1)$. As $A[1,\dots,k]\in\mathfrak{N}_L^k(1,\dots,k)$, then $A[1,\dots,k]=U_k$. As A is a ULH matrix then

$$A=\left[\begin{array}{ccc|c} & & & 0\\ & U_k & & \vdots\\ & & & 0\\ & & & 1\\ \hline a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1,k+1}\end{array}\right].$$

As A is nilpotent then $a_{k+1,1}=\cdots=a_{k+1,k+1}=0$. So $\mathfrak{N}_L^{k+1}(1,\dots,k+1)=\{U_{k+1}\}$.

On the other hand,

$$\mathfrak{N}_T^n(1,\dots,n)=(J_n\,\mathfrak{N}_L^n(n_1,\dots,n_r)\,J_n)^T=\{(J_n\,U_n\,J_n)^T\}=\{U_n\}.$$

3.3. Description of the companions of $\widetilde{\mathcal{G}}$. Even though Theorems 3.3 and 3.4 characterize the companion matrices of $\widetilde{\mathcal{C}}$, we can give a nice (and much easier to check) description of the companion matrices of the subset $\widetilde{\mathcal{G}}$ of $\widetilde{\mathcal{C}}$.

First. we define the sets $\widehat{\mathcal{C}}$, $\widehat{\mathcal{G}}$, and $\widehat{\mathcal{F}}$ composed by those matrices of, respectively, $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{G}}$, and $\widetilde{\mathcal{F}}$ such that all its nonzero entries are either in the superdiagonal (and are equal to 1) or in the i_1-block determined by the position of the variable x_1. Note that $\widehat{\mathcal{C}}\supsetneq\widehat{\mathcal{G}}\supsetneq\widehat{\mathcal{F}}$.

This structure implies that if A is a matrix of $\widehat{\mathcal{C}}$ then

$$P_A(\lambda)=\lambda^n-s_0(A)\lambda^{n-1}-\cdots-s_{n-2}(A)\lambda-s_{n-1}(A), \tag{3.5}$$

where $s_k(A)$ is the sum of the entries on the k-subdiagonal of A. Matching (3.5) with (1.1) we obtain the following characterization of the companion matrices of $\widehat{\mathcal{C}}$.

LEMMA 3.6. *$A\in\widehat{\mathcal{C}}$ is companion if and only if $s_{k-1}(A)=x_k$ for each $k\in\{1,\dots,n\}$.*

Let us see how to use this result to characterize the companion matrices of $\widetilde{\mathcal{G}}$.

THEOREM 3.7. *$A\in\widetilde{\mathcal{G}}$ is companion if and only if $A\in\widehat{\mathcal{G}}$ and $s_{k-1}(A)=x_k$ for each $k\in\{1,\dots,n\}$.*

Proof. The sufficiency follows from Lemma 3.6.

The necessity. By Lemma 3.6, it is enough to prove that $A\in\widehat{\mathcal{G}}$. Note that A is the superpattern of $G_{i_1,\dots,i_n}$ for some $i_1,\dots,i_n$. Then there exists at least one variable on each of the rows $i_1,i_1+1,\dots,n$ and one variable on each of the columns $1,\dots,i_1$. From Theorem 3.4 and Lemma 3.5 it follows that

$$\begin{aligned}&A[1,\dots,i_1-1]\in\mathfrak{N}_T^{i_1-1}(1,\dots,i_1-1)=\{U_{i_1-1}\};\ \text{and}\\ &A[i_1+1,\dots,n]\in\mathfrak{N}_T^{n-i_1}(1,\dots,n-i_1)=\{U_{n-i_1}\}.\end{aligned}$$

So A meets the conditions to be a matrix of $\widehat{\mathcal{G}}$. ∎

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A publication of the International Linear Algebra Society
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The interest of Theorem 3.7 is that for any $A \in \mathcal{G}$ we can parametrize the set of all ULH superpatterns of A which are companion matrices. We give an example of such a parametrization.

Example 3.8. Consider the matrix $G_{4,4,6,5,5,7,7} \in \mathcal{G}$. According to Theorem 3.7 the companion matrices which are superpatterns of $G_{4,4,6,5,5,7,7}$ are

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b & a & x_2 & x_1 & 1 & 0 & 0 \\ x_5 & x_4 & -a & 0 & 0 & 1 & 0 \\ 0 & d & c & x_3 & 0 & 0 & 1 \\ x_7 & x_6 & -d & -b-c & 0 & 0 & 0 \end{bmatrix} \quad \text{with } a, b, c, d \in \mathbb{F}.$$

A consequence of Theorem 3.7 is related with an open question stated by Deaett et al. [3]: "*We wonder if, in producing a companion matrix by changing some zero entries of a Fiedler companion matrix $F_{i_1,\ldots,i_n}$ by nonzero constants, the extra nonzero entries are always restricted to the submatrix corresponding to the i_1-block*". They partially confirmed that supposition.

Theorem 3.9. *[3, Theorem 5.4] Let A be a matrix obtained from the Fiedler companion matrix $F_{i_1,\ldots,i_n}$ by changing zero entries that are not in the i_1-block. Then A is not companion.*

We make some progress in this problem by solving the case in which the change of zero entries in the Fiedler companion matrix is below the superdiagonal. Indeed, this is particular case of Theorem 3.7: *"$A \in \widetilde{\widehat{\mathcal{F}}}$ is companion if and only if $A \in \widehat{\mathcal{F}}$ and $s_{k-1}(A) = x_k$ for each $k \in \{1,\ldots,n\}$"*. We write this in the same language as Theorem 3.9.

Theorem 3.10. *Let A be a matrix obtained from the Fiedler companion matrix $F_{i_1,\ldots,i_n}$ by changing zero entries that are below the superdiagonal. Then A is companion if and only if A is obtained by only changing zero entries that are in the i_1-block and $s_{k-1}(A) = x_k$ for each $k \in \{1,\ldots,n\}$.*

It remains unknown if there exists a companion matrix which is obtained from some Fiedler companion matrix $F_{i_1,\ldots,i_n}$ by changing at least one zero entry of the i_1-block and at least one zero entry above the superdiagonal. We have computational evidence that this is not the case for $n \leq 7$.

3.4. Parameterization of the companions of $\widetilde{\mathcal{C}} \setminus \widetilde{\mathcal{G}}$. When we have a matrix $C_{i_1,\ldots,i_n} \in \mathcal{C} \setminus \mathcal{G}$, the set of companion matrices which are superpatterns of $C_{i_1,\ldots,i_n}$ is not so easy to parametrize as with matrices of $\mathcal{G}$. We will show it with a specific example. Let

$$C_{7,7,9,7,8,9,10,8,9,10} = \left[\begin{array}{cccccc|c|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & x_4 & 0 & x_2 & x_1 & 1 & 0 & 0 \\ \cline{8-10} x_8 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 1 & 0 \\ x_9 & 0 & 0 & x_6 & 0 & 0 & x_3 & 0 & 0 & 1 \\ x_{10} & 0 & 0 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} U_6 & e_6 & 0 \\ \hline \multicolumn{2}{c|}{} & e_1^T \\ \cline{3-3} \multicolumn{2}{c|}{X} & U_3 \end{array}\right].$$

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A publication of the International Linear Algebra Society
Volume 37, pp. 193-210, February 2021.

The set of ULH superpatterns of $C_{7,7,9,7,8,9,10,8,9,10}$ is composed of the matrices

$$C = \left[\begin{array}{cc|c} A & e_6 & 0 \\ \hline & & e_1^T \\ \cline{3-3} & Y & B \end{array}\right], \tag{3.6}$$

where matrix A is a ULH superpattern of U_6, matrix B is a ULH superpattern of U_3, and matrix Y is a superpattern of X. By Theorem 3.4, C is companion if and only if the following conditions are met:

(i) The constant part of C is a nilpotent matrix;
(ii) A belongs to $\mathfrak{N}_L^6(i_1-1, i_2-2, \dots, i_{10}-10) = \mathfrak{N}_L^6(6,5,6,3,3,3,3,0,0,0) = \mathfrak{N}_L^6(3,5,6)$;
(iii) $B \in \mathfrak{N}_T^3(10-i_1, 10-i_2, \dots, 10-i_{10}) = \mathfrak{N}_T^3(3,3,1,3,2,1,0,2,1,0) = \mathfrak{N}_T^3(1,2,3)$.

We wish to parameterize the companions matrices of type (3.6). We divide it in three cases:

(1) $A = U_6$ and $B = U_3$. So we consider the companion matrices of type

$$C_1 = \left[\begin{array}{cc|c} U_6 & e_6 & 0 \\ \hline & & e_1^T \\ \cline{3-3} & Y & U_3 \end{array}\right].$$

for some superpattern Y of X. The solution is given in Lemma 3.6: C_1 is a companion matrix if and only if $s_{k-1}(C_1) = x_k$ for each $k \in \{1, \dots, n\}$. So

$$Y = \begin{bmatrix} h & f & d & x_4 & a & x_2 & x_1 \\ x_8 & i & g & x_5 & b & -a & 0 \\ x_9 & j & -h-i & x_6 & e & c & x_3 \\ x_{10} & 0 & -j & x_7 & -f-g & -d-e & -b-c \end{bmatrix} \quad \text{where } a, b, \dots, j \in \mathbb{F}.$$

(2) $Y = X$. So we consider the companion matrices of type

$$C_2 = \left[\begin{array}{cc|c} A & e_6 & 0 \\ \hline & & e_1^T \\ \cline{3-3} & X & B \end{array}\right].$$

where A is a constant ULH matrix of order 6, and B is a constant ULH matrix of order 3. By Theorem 3.4, C_2 is a companion matrix if and only if $A \in \mathfrak{N}_L^6(3,5,6)$ and $B \in \mathfrak{N}_T^3(1,2,3)$.
By Lemma 3.5, $\mathfrak{N}_T^3(1,2,3) = \{U_3\}$. So $B = U_3$.
On the other hand, as any matrix of $\mathfrak{N}_L^6(3,5,6)$ is a ULH matrix then

$$A = \left[\begin{array}{ccc|cc|c} a_{11} & 1 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 1 & 0 & 0 \\ \cline{1-3} a_{41} & a_{42} & a_{43} & a_{44} & 1 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & 1 \\ \cline{1-5} a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array}\right],$$

where $A[1,2,3]$, $A[1,2,3,4,5]$, and $A[1,2,3,4,5,6]$ are nilpotent. The ULH submatrices $A[1]$ and $A[1,2]$ have no restriction. So, assume that we have assigned arbitrary constant values to a_{11}, a_{21}, a_{22}. As

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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$A[1,2,3]$ is a nilpotent ULH matrix, then a_{31}, a_{32}, a_{33} should take values that make the characteristic polynomial of $A[1,2,3]$ to be λ^3. Corollary 4.3, that we will state later, implies that these values are unique and can be obtained in a practical way since $(1, a_{33}, a_{32}, a_{31})$ are the coordinates of λ^3 with respect to certain basis of $\mathbb{F}_3[\lambda]$. Let us go to $A[1,2,3,4]$, for which we have no further restrictions. So, assume that we have assigned arbitrary constant values to $a_{41}, a_{42}, a_{43}, a_{44}$. As $A[1,2,3,4,5]$ is a nilpotent ULH matrix, then we repeat the arguments above in order to obtain the adequate values for $a_{51}, a_{52}, a_{53}, a_{54}, a_{55}$. And, finally, $A[1,2,3,4,5,6]$ is also a nilpotent ULH matrix which we get by doing $a_{61} = \cdots = a_{66} = 0$ since $A[1,2,3,4,5]$ is nilpotent and has characteristic polynomial λ^5. Moreover, by Corollary 4.3, this solution is unique. So, doing the corresponding calculations, we obtain

$$A = \begin{bmatrix} a & 1 & 0 & 0 & 0 & 0 \\ c & b & 1 & 0 & 0 & 0 \\ -a^3-2ac-bc & -a^2-ab-b^2-c & -a-b & 1 & 0 & 0 \\ g & f & e & d & 1 & 0 \\ a_{51} & a_{52} & ae+be-de-f & -d^2-e & -d & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{with } a,b,\ldots,g \in \mathbb{F},$$

where $a_{51} = a^3e + 2ace - ag + bce - cf - dg$ and $a_{52} = a^2e + abe + b^2e - bf + ce - df - g$.

(3) $Y \neq X$ and $(A,B) \neq (U_6, U_3)$. It is possible to give a partial parametrization for this case, although we do not have the complete parametrization. In Example 4.5 of [3], the authors gave, for $C_{3,4,3,4,7,6,7}$, a partial parametrization for an analogous 7×7 case.

4. The PB-companion ULH matrices.

Recall that a matrix A of order n is said to be *PB-companion* or *companion with respect to a polynomial basis*, provided: (i) A has $n^2 - n$ entries that are constants of a field $\mathbb{F}$; (ii) the n remaining entries of A are the variables $x_1, \ldots, x_n$; and (iii) the characteristic polynomial of A is

$$\det(\lambda I_n - A) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_{n-1}p_1(\lambda) - x_n p_0(\lambda),$$

where $\{p_n(\lambda), \ldots, p_1(\lambda), p_0(\lambda)\}$ is a polynomial basis of $\mathbb{F}_n[\lambda]$.

First note that given a PB-companion matrix, any permutation of the positions of the variables $x_1, \ldots, x_n$ will give a new matrix that is also a PB-companion matrix. So, in what follows, for any PB-companion ULH matrix we will assume that the positions of the variables $x_1, \ldots, x_n$ satisfy the total ordering defined for $\mathcal{H}$ and, consequently, it belongs to $\widetilde{\mathcal{H}}$.

The first issue we are concerned about in this section is the localization of the PB-companion ULH matrices within the set $\widetilde{\mathcal{H}}$. We will see that the set composed of the PB-companion matrices of $\widetilde{\mathcal{H}}$ is greater than $\widetilde{\mathcal{C}}$ and strictly smaller that $\widetilde{\mathcal{H}}$ and that the intermediate set $\widetilde{\mathcal{D}}$ is not useful since $\widetilde{\mathcal{D}} \setminus \widetilde{\mathcal{C}}$ does not contain any PB-companion matrix. So we need a new set between $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{C}}$ in which we will locate the PB-companion ULH matrices.

Let $\mathcal{B}$ be the set of those matrices of $\mathcal{H}$ in which the variables $x_1, \ldots, x_n$ are in a t-block for some $t \in \{1, \ldots, n\}$ (there may be more than one such t for some matrices). Note that $\mathcal{B} \cap \mathcal{D} = \mathcal{C}$. Let us describe the matrices of $\mathcal{B}$ more precisely:

$$\mathcal{B} = \{B_{(i_1,j_1),\ldots,(i_n,j_n)} | B_{(i_1,j_1),\ldots,(i_n,j_n)} = H_{(i_1,j_1),\ldots,(i_n,j_n)} \text{ with } \max\{j_1,\ldots,j_n\} \leq \min\{i_1,\ldots,i_n\}\}.$$

The variables $x_1, \ldots, x_n$ are in the t-block of $B_{(i_1,j_1),\ldots,(i_n,j_n)}$ if and only if

$$\max\{j_1,\ldots,j_n\} \leq t \leq \min\{i_1,\ldots,i_n\}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 193-210, February 2021.

As the following examples show, we have $\mathcal{H} \supsetneq \mathcal{B} \supsetneq \mathcal{C}$.

$H_{(1,1),(3,3),(2,1),(5,4),(5,2)} \in \mathcal{H} \setminus \mathcal{B}$	$B_{(3,1),(4,2),(4,1),(5,2),(5,1)} \in \mathcal{B} \setminus \mathcal{C}$
$\begin{bmatrix} x_1 & 1 & 0 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & x_5 & 0 & x_4 & 0 \end{bmatrix}$	$\left[\begin{array}{cc\|c\|cc} 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline x_1 & 0 & 0 & 1 & 0 \\ x_3 & x_2 & 0 & 0 & 1 \\ x_5 & x_4 & 0 & 0 & 0 \end{array}\right]$

Let us see that $\widetilde{\mathcal{B}}$ is the right place where we should look for PB-companion ULH matrices.

THEOREM 4.1. *If a matrix in $\widetilde{\mathcal{H}}$ is a PB-companion matrix, then it belongs to $\widetilde{\mathcal{B}}$.*

Proof. Let the matrix A be a superpattern of $H_{(i_1,j_1),\dots,(i_n,j_n)}$ which is not in $\widetilde{\mathcal{B}}$. Then

$$\max\{j_1,\dots,j_n\} = j_r > i_s = \min\{i_1,\dots,i_n\}.$$

for some $r,s \in \{1,\dots,n\}$. Therefore, $A[j_s,\dots,i_s]$ contains x_s, $A[j_r,\dots,i_r]$ contains x_r, and both matrices are disjoint. So in $P_A(\lambda)$ the term $x_s x_r \lambda^{n-(i_s-j_s+1)-(i_r-j_r+1)}$ appears, and this term cannot cancel out. We conclude that A is not a companion matrix. ∎

In the proof of the previous result, we have seen that in the characteristic polynomial of a matrix of $\widetilde{\mathcal{H}} \setminus \widetilde{\mathcal{B}}$ inevitably the product of two variables $x_r x_s$ appear. For matrices of $\widetilde{\mathcal{B}}$ no product of two variables appear, as we will show in the next Lemma. Moreover, the polynomials $p_{n-k}(\lambda)$ that accompany each variable x_k are crucial for knowing when a matrix of $\widetilde{\mathcal{B}}$ is PB-companion. We will show some useful properties of those polynomials.

LEMMA 4.2. *Let the matrix A be an ULH superpattern A of $B_{(i_1,j_1),\dots,(i_n,j_n)}$. Then the following is true:*

(i) The characteristic polynomial of A can be written as

$$P_A(\lambda) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_n p_0(\lambda), \tag{4.7}$$

where $p_n(\lambda), p_{n-1}(\lambda), \dots, p_0(\lambda)$ are monic polynomials of $\mathbb{F}_n[\lambda]$.

(ii) $\deg(p_n(\lambda)) = n$ *and* $\deg(p_{n-k}(\lambda)) = n - i_k + j_k - 1$ *for* $k = 1,\dots,n$.

(iii) $n = \deg(p_n(\lambda)) \geq \deg(p_{n-1}(\lambda)) \geq \cdots \geq \deg(p_0(\lambda))$.

(iv) If A is PB-companion, then for $k = 0,1,\dots,n$ the degree of $p_{n-k}(\lambda)$ is at least $n-k$.

(v) If A is PB-companion, then for $k = 1,\dots,n$ the variable x_k is above the k-subdiagonal.

Proof. (i) To show that the characteristic polynomial of A can be written as in (4.7) we use the same arguments that in the proof of Theorem 3.3. This leads us to conclude that

$$P_A(\lambda) = P_{A_0}(\lambda) - \sum_{k=1}^{n} x_k \, P_{A[1,\dots,j_k-1]}(\lambda) \, P_{A[i_k+1,\dots,n]}(\lambda), \tag{4.8}$$

where A_0 is the constant part of A. Since $x_1,\dots,x_n$ are in some t-block of A then for $k = 1,\dots,n$ the matrix $A[1,\dots,j_k-1]$ is empty or a submatrix of the constant matrix $A[1,\dots,t-1]$, and the matrix $A[i_k+1,\dots,n]$ is empty or a submatrix of the constant matrix $A[t+1,\dots,n]$. Since the variables $x_1,\dots,x_n$ do not appear in either $P_{A[1,\dots,j_k-1]}(\lambda)$ or $P_{A[i_k+1,\dots,n]}(\lambda)$, then we can match (4.8) and (4.7) to conclude that $p_n(\lambda) = P_{A_0}(\lambda)$ and $p_{n-k}(\lambda) = P_{A[1,\dots,j_k-1]}(\lambda) \, P_{A[i_k+1,\dots,n]}(\lambda)$ for $k = 1,\dots,n$ are monic polynomials of $\mathbb{F}_n[\lambda]$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 193-210, February 2021.

(ii) $\deg(p_n(\lambda)) = \deg(P_{A_0}(\lambda)) = n$ and for $k = 1, \ldots, n$

$$\deg(p_{n-k}(\lambda)) = \deg(P_{A[1,\ldots,j_k-1]}(\lambda)) \deg(P_{A[i_k+1,\ldots,n]}(\lambda)) = j_k - 1 + n - i_k.$$

(iii) It follows from (ii) by taking into account that the variables $x_1, \ldots, x_n$ are placed in A according to the order $\prec$, which implies that $i_1 - j_1 \leq i_2 - j_2 \leq \cdots \leq i_n - j_n$.

(iv) If A is PB-companion, then $\{p_0(\lambda), p_1(\lambda), \ldots, p_n(\lambda)\}$ is a basis of $\mathbb{F}_n[\lambda]$. Therefore, the inequality $\deg(p_{n-k}(\lambda)) \geq n - k$ for each $k = 1, \ldots, n$ follows from (iii).

(v) If x_k is on the h-subdiagonal, then by (ii) and (iv), $\deg(p_{n-k}(\lambda)) = n - h - 1 \geq n - k$. So $h < k$. ☐

As an application of Lemma 4.2, we give the following result.

Theorem 4.3. *Each matrix of $\widetilde{\mathcal{C}}$ is PB-companion.*

Proof. If A is a superpattern of $C_{(i_1,j_1),\ldots,(i_n,j_n)}$, then $j_k = i_k - k + 1$ for $k = 1, \ldots, n$. From Lemma 4.2 (i)-(ii) it follows that $P_A(\lambda) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_n p_0(\lambda)$ with $\deg(p_n(\lambda)) = n$ and

$$\deg(p_{n-k}(\lambda)) = n - i_k + j_k - 1 = n - i_k + i_k - k + 1 - 1 = n - k \quad \text{for } k = 1, \ldots, n.$$

So $\{p_n(\lambda), \ldots, p_1(\lambda), p_0(\lambda)\}$ is a basis of $\mathbb{F}_n[\lambda]$ and A is PB-companion. ☐

4.1. A criterion for PB-companion matrices. Lemma 4.2 (i) permits us to introduce a constant matrix associated to each matrix of $\widetilde{\mathcal{B}}$.

Definition 4.4. *Let A be a matrix of $\widetilde{\mathcal{B}}$ whose characteristic polynomial is, as in (4.7),*

$$P_A(\lambda) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_{n-1} p_1(\lambda) - x_n p_0(\lambda).$$

We will denote by $\mathfrak{M}_A$ the constant matrix $[p_{ij}]_{i,j=0}^n$ of order $n+1$ such that the entries on its rows are the coefficients of $p_i(\lambda) = p_{i0} + p_{i1}\lambda + \cdots + p_{in}\lambda^n$ for $i = 0, 1, \ldots, n$.

Theorem 4.5. *If a matrix in $\widetilde{\mathcal{H}}$ is a PB-companion matrix, then it belongs to $\widetilde{\mathcal{B}}$. Moreover, a matrix $A \in \widetilde{\mathcal{B}}$ is PB-companion if and only if $\mathfrak{M}_A$ is nonsingular.*

Proof. The first sentence is Theorem 4.1. Now, let A be a matrix of $\widetilde{\mathcal{B}}$. Then A is PB-companion if and only if $\{p_0(\lambda), p_1(\lambda), \ldots, p_n(\lambda)\}$ is a basis of $\mathbb{F}_n[\lambda]$ if and only if $\mathfrak{M}_A$ is nonsingular. ☐

Example 4.6. We wish to determine if

$$A = \left[\begin{array}{ccccc|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & -2 & 1 & 0 & 0 & 0 \\ 4 & 0 & 2 & -1 & 1 & 0 & 0 \\ \hline x_5 & 1 & -3 & x_2 & x_1 & 1 & 0 \\ x_7 & x_6 & 2 & 5 & x_3 & 3 & 1 \\ 1 & 0 & 4 & x_4 & 1 & 4 & 2 \end{array}\right],$$

is a PB-companion matrix. Note that A is a superpattern of $B_{(5,5),(5,4),(6,5),(7,4),(5,1),(6,2),(6,1)}$. The characteristic polynomial of A is

$$P_A(\lambda) = p_7(\lambda) - x_1 p_6(\lambda) - \cdots - x_6 p_1(\lambda) - x_7 p_0(\lambda),$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 193-210, February 2021.

with

$$\begin{aligned}
x_7 &\to \quad p_0(\lambda) = P_{A[7]}(\lambda) = -2+\lambda;\\
x_6 &\to \quad p_1(\lambda) = P_{A[1]}(\lambda)\,P_{A[7]}(\lambda) = 4-4\lambda+\lambda^2;\\
x_5 &\to \quad p_2(\lambda) = P_{A[6,7]}(\lambda) = 2-5\lambda+\lambda^2;\\
x_4 &\to \quad p_3(\lambda) = P_{A[1,2,3]}(\lambda) = 17-7\lambda-3\lambda^2+\lambda^3;\\
x_3 &\to \quad p_4(\lambda) = P_{A[1,2,3,4]}(\lambda)\,P_{A[7]}(\lambda) = -2-39\lambda+44\lambda^2-8\lambda^3-4\lambda^4+\lambda^5;\\
x_2 &\to \quad p_5(\lambda) = P_{A[1,2,3]}(\lambda)\,P_{A[6,7]}(\lambda) = 34-99\lambda+46\lambda^2+10\lambda^3-8\lambda^4+\lambda^5;\\
x_1 &\to \quad p_6(\lambda) = P_{A[1,2,3,4]}(\lambda)\,P_{A[6,7]}(\lambda) = 2+35\lambda-123\lambda^2+76\lambda^3-7\lambda^5+\lambda^6;\\
1 &\to \quad p_7(\lambda) = P_{A_0}(\lambda) = 208-317\lambda+168\lambda^2-129\lambda^3+73\lambda^4-7\lambda^6+\lambda^7.
\end{aligned}$$

The matrix $\mathfrak{M}_A = [p_{ij}]_{i,j=0}^{7}$ is

	λ^0	λ^1	λ^2	λ^3	λ^4	λ^5	λ^6	λ^7
$p_0(\lambda)$	-2	1	0	0	0	0	0	0
$p_1(\lambda)$	4	-4	1	0	0	0	0	0
$p_2(\lambda)$	2	-5	1	0	0	0	0	0
$p_3(\lambda)$	17	-7	-3	1	0	0	0	0
$p_4(\lambda)$	-2	-39	44	-8	-4	1	0	0
$p_5(\lambda)$	34	-99	46	10	-8	1	0	0
$p_6(\lambda)$	2	35	-123	76	0	-7	1	0
$p_7(\lambda)$	208	-317	168	-129	73	0	-7	1

By Theorem 4.5, A is PB-companion since $\mathfrak{M}_A[0,1,2]$, $\mathfrak{M}_A[3]$, $\mathfrak{M}_A[4,5]$, $\mathfrak{M}_A[6]$, and $\mathfrak{M}_A[7]$ (the gray blocks in the diagonal) are nonsingular.

In general, to determine if a given matrix A of $\widetilde{\mathcal{B}}$ is PB-companion requires some cumbersome calculations to obtain the polynomials that go into the characteristic polynomial of A. Nevertheless, in some cases, the calculations are going to simplify so much that we will conclude easily if A is or it is not PB-companion. This is because we do not need to known all the entries on the matrix $\mathfrak{M}_A$ to conclude that it is nonsingular. We only need to known the submatrices that were marked in gray in the previous example.

In what follows, we will use the term *concatenation* as the operation of joining two or more tuples. More precisely, the concatenation of an n-tuple $(a_1,\ldots,a_n)$ with the m-tuple $(b_1,\ldots,b_m)$ is denoted by $(a_1,\ldots,a_n)^\frown(b_1,\ldots,b_m)$ and is equal to the $(n+m)$-tuple $(a_1,\ldots,a_n,b_1,\ldots,b_m)$.

Theorem 4.7. *Let A be a matrix of $\widetilde{\mathcal{B}}$ with*

$$P_A(\lambda) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_n p_0(\lambda),$$

and let the concatenation associated to A be

$$(0,1,\ldots,n) = (b_1,\ldots,e_1)^\frown \ldots ^\frown (b_s,\ldots,e_s).$$

where $e_1,\ldots,e_s$ are the $t\in\{0,1,\ldots,n\}$ such that the degree of $p_t(\lambda)$ is t.

Then A is PB-companion if and only if $\mathfrak{M}_A[b_h,\ldots,e_h]$ is nonsingular for each $h\in\{1,\ldots,s\}$.

Proof. From Lemma 4.2 (iii) and (iv), if A is PB-companion, then $\mathfrak{M}_A$ is a lower triangular block matrix with $\mathfrak{M}_A[b_1,\ldots,e_1],\ldots,\mathfrak{M}_A[b_s,\ldots,e_s]$ as the blocks on its diagonal. From Theorem 4.5, A is PB-companion if and only if $\mathfrak{M}_A$ is nonsingular if and only if $\mathfrak{M}_A[b_h,\ldots,e_h]$ is nonsingular for each $h=1,\ldots,s$. ∎

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A publication of the International Linear Algebra Society
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4.2. Concatenations with components of length at most two. In this section, we will see that the description of the PB-companion ULH matrices given in Theorem 4.7 becomes particularly simple for those matrices of $\widetilde{\mathcal{B}}$ whose associated concatenation has all its components of length one or of length at most two.

Let us start with length one, that is, those matrices $A \in \widetilde{\mathcal{B}}$ such that the concatenation associated to A is $(0, 1, \ldots, n) = (0)^\frown(1)^\frown \ldots^\frown (n)$. This concatenation occurs if and only if $\deg(p_t(\lambda)) = t$ for $t = 0, 1, \ldots, n$, which is equivalent to saying that A belongs to $\widetilde{\mathcal{C}}$. So, Theorem 4.3 can be restated as: *Each matrix of $\widetilde{\mathcal{B}}$ whose associated concatenation has only components of length one is PB-companion.*

Now we will study a more general case (with component lengths at most two).

THEOREM 4.8. *Let A be a matrix of $\widetilde{\mathcal{B}}$ with*

$$P_A(\lambda) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_n p_0(\lambda),$$

and let the concatenation associated to A be

$$(0, 1, \ldots, n) = (b_1, \ldots, e_1)^\frown \ldots^\frown (b_s, \ldots, e_s), \tag{4.9}$$

where $e_1, \ldots, e_s$ are the $t \in \{0, 1, \ldots, n\}$ such that the degree of $p_t(\lambda)$ is t.

Then A is a PB-companion matrix such that the length of the components of the concatenation associated to A is either one or two if and only if on each k-subdiagonal of A

I. the only variable is x_{k+1}; or
II. the only two variables are x_{k+1} and x_{k+2}, and $\displaystyle\sum_{r=i-k}^{i} a_{rr} \neq \sum_{r=i'-k}^{i'} a_{rr}$ if x_{k+1} is the $(i, i-k)$ entry of A and x_{k+2} is the $(i', i'-k)$ entry of A; or
III. no variable appears.

Proof. Assume that A is a PB-companion matrix of $\widetilde{\mathcal{B}}$ such that the length of the components of the concatenation associated with A is either one or two.

We will consider all the possibilities for each k-subdiagonal of A:

(1) On the k-subdiagonal of A, there are no variables. This is case III.
(2) On the k-subdiagonal of A, the variable with smallest index is x_h for some $h \leq k$. Then, x_h is on or below the h-subdiagonal, contradicting Lemma 4.2 (v).
(3) On the k-subdiagonal of A, the variable with smallest index is x_h for some $h \geq k+2$. Then, x_{k+1} is not on the k-subdiagonal, and x_{k+2} is not on the $(k+1)$-subdiagonal. Equivalently, $\deg(p_{n-k-2}(\lambda)) \neq n-k-2$ and $\deg(p_{n-k-1}(\lambda)) \neq n-k-1$. So the concatenation has a component of length at least three, contradicting the hypothesis.
(4) On the k-subdiagonal of A, the variable with smallest index is x_{k+1}. We consider three subcases:
   (i) The only variable on the k-subdiagonal is x_{k+1}. This is case I.
   (ii) The only variables on the k-subdiagonal are x_{k+1} and x_{k+2}. Three new possibilities appear:
      (a) The variable x_{k+3} is on the $(k+1)$-subdiagonal. Then, x_{k+3} is in the same situation as in (3).
      (b) The variable x_{k+3} is on the $(k+h)$-subdiagonal with $h \geq 3$. Then, x_{k+3} is in the same situation as in (2).
      (c) The variable x_{k+3} is on the $(k+2)$-subdiagonal. Then

$$\deg(p_{n-k-3}(\lambda)) = n-k-3,\ \deg(p_{n-k-2}(\lambda)) \neq n-k-2, \text{ and } \deg(p_{n-k-1}(\lambda)) = n-k-1.$$

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A publication of the International Linear Algebra Society
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Therefore, $\frown(n-k-2, n-k-1)\frown$ is a component of the concatenation of length two. As x_{k+1} is on the k-subdiagonal, then x_{k+1} is the $(i, i-k)$ entry of A for some i and

$$\begin{aligned} p_{n-k-1}(\lambda) &= P_{A[1,\ldots,i-k-1]}(\lambda)\, P_{A[i+1,\ldots,n]}(\lambda) \\ &= (\lambda^{i-k-1} - \sum_{r=1}^{i-k-1} a_{rr}\lambda^{i-k-2} + \cdots)\,(\lambda^{n-i} - \sum_{r=i+1}^{n} a_{rr}\lambda^{n-i-1} + \cdots) \\ &= \lambda^{n-k-1} - \sum_{r\in\{1,\ldots,n\}\setminus\{i-k,\ldots,i\}} a_{rr}\lambda^{n-k-2} + \cdots . \end{aligned}$$

Analogously, as x_{k+2} is on the k-subdiagonal, then x_{k+2} is the $(i', i'-k)$ entry of A for some i' and

$$p_{n-k-2}(\lambda) = \lambda^{n-k-1} - \sum_{r\in\{1,\ldots,n\}\setminus\{i'-k,\ldots,i'\}} a_{rr}\lambda^{n-k-2} + \cdots .$$

Therefore,

$$\mathfrak{M}_A[n-k-2, n-k-1] = \begin{bmatrix} p_{n-k-2,n-k-2} & p_{n-k-2,n-k-1} \\ p_{n-k-1,n-k-2} & p_{n-k-1,n-k-1} \end{bmatrix} = \begin{bmatrix} -\sum_{r\in\{1,\ldots,n\}\setminus\{i-k,\ldots,i\}} a_{rr} & 1 \\ -\sum_{r\in\{1,\ldots,n\}\setminus\{i'-k,\ldots,i'\}} a_{rr} & 1 \end{bmatrix}.$$

By Theorem 4.7, $\mathfrak{M}_A[n-k-2, n-k-1]$ is nonsingular. And this is so if and only if $\displaystyle\sum_{r=i-k}^{i} a_{rr} \neq \sum_{r=i'-k}^{i'} a_{rr}$. This is case II.

(iii) On the k-subdiagonal there are, at least, three variables x_{k+1}, x_{k+2}, and x_{k+3}. In this case, x_{k+2} is not on the $(k+1)$-subdiagonal of A and x_{k+3} is not on the $(k+2)$-subdiagonal of A. Equivalently, $\deg(p_{n-k-3}(\lambda)) \neq n-k-3$ and $\deg(p_{n-k-2}(\lambda)) \neq n-k-2$. So the concatenation has a component of length at least three. A contradiction with the hypothesis.

For the converse argument, assume that each k-subdiagonal satisfies I, or II, or III. It is useful to note that

(4.10) each variable x_h is either on the $(h-1)$ or on the $(h-2)$-subdiagonal of A.

Let us prove that the k-subdiagonal verifies II if and only if the $(k+1)$-subdiagonal verifies III. For the sufficiency assume that the k-subdiagonal satisfies II, that is, that on the k-subdiagonal the only variables are x_{k+1} and x_{k+2}. Then x_{k+2} is not on the $(k+1)$-subdiagonal, but this makes it impossible for the $(k+1)$-subdiagonal to verify I or II, so the $(k+1)$-subdiagonal verifies III. For the necessity, assume that the $(k+1)$-subdiagonal verifies III, that is, that the $(k+1)$-subdiagonal has no variables. Then x_{k+2} is on the k-subdiagonal because of (4.10), but this makes it impossible for the k-subdiagonal to verify I or III, so x_{k+1} is on the k-subdiagonal and the k-subdiagonal verifies II.

By what we have just shown, if we traverse the subdiagonals in order we will sometimes find consecutive subdiagonals that satisfy II and III, and the rest of subdiagonals must verify I. The translation into components of the concatenation (4.9) is that two consecutive subdiagonals that verify II and III correspond to a component of length two, and a subdiagonal that verifies I corresponds to a component of length one. Let us see why this is so:

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- If two consecutive subdiagonals k and $k+1$ verify II and III, then the only variables on the k-subdiagonal are x_{k+1} and x_{k+2}, and the $(k+1)$-subdiagonal has no variables. Furthermore, x_{k+3} must be in the $(k+2)$-subdiagonal because of (4.10). So

$$\deg(p_{n-k-3}(\lambda)) = n-k-3,\ \deg(p_{n-k-2}(\lambda)) \neq n-k-2,\ \text{and}\ \deg(p_{n-k-1}(\lambda)) = n-k-1,$$

which means that $\frown(n-k-2, n-k-1)\frown$ is a component of the concatenation (4.9) of length two. If x_{k+1} is on the $(i, i-k)$ entry of A and x_{k+2} is on the $(i', i'-k)$ entry of A then (as we saw above in 4iic)

$$\mathfrak{M}_A[n-k-2, n-k-1] = \begin{bmatrix} -\sum_{r\in\{1,\ldots,n\}\setminus\{i-k,\ldots,i\}} a_{rr} & 1 \\ -\sum_{r\in\{1,\ldots,n\}\setminus\{i'-k,\ldots,i'\}} a_{rr} & 1 \end{bmatrix},$$

which is nonsingular since $\displaystyle\sum_{r=i-k}^{i} a_{rr} \neq \sum_{r=i'-k}^{i'} a_{rr}$ by hypothesis.

- If the k-subdiagonal verifies I, then the only variable on the k-subdiagonal is x_{k+1}. Furthermore, x_{k+2} must be in the $(k+1)$-subdiagonal because of (4.10). So

$$\deg(p_{n-k-2}(\lambda)) = n-k-2 \text{ and } \deg(p_{n-k-1}(\lambda)) = n-k-1,$$

which means that $\frown(n-k-1)\frown$ is a component of the concatenation (4.9) of length one. The corresponding principal submatrix of $\mathfrak{M}_A$ is $\mathfrak{M}_A[n-k-1] = [1]$, since $p_0(\lambda), \ldots, p_n(\lambda)$ are monic by Theorem 4.2 (i).

Finally, by Theorem 4.7, A is PB-companion because we have shown that all the principal submatrices of $\mathfrak{M}_A$ that correspond to the components of the concatenation are nonsingular. ☐

Example 4.9. Let

$$A = \begin{bmatrix} a_{11} & 1 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 1 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 1 & 0 \\ a_{51} & x_4 & a_{53} & x_2 & x_1 & 1 \\ x_6 & a_{62} & x_5 & a_{63} & x_3 & a_{66} \end{bmatrix}.$$

Note that A is a superpattern of $B_{(5,5),(5,4),(6,5),(5,2),(6,3),(6,1)}$. By Theorem 4.8, A is a PB-companion matrix if and only if $a_{44} \neq a_{66}$ and $a_{22} \neq a_{66}$.

Let us see it in detail. If the characteristic polynomial of A is

$$P_A(\lambda) = p_6(\lambda) - x_1 p_5(\lambda) - x_2 p_4(\lambda) - x_3 p_3(\lambda) - x_4 p_2(\lambda) - x_5 p_1(\lambda) - x_6 p_0(\lambda)$$

then

$$\begin{aligned}
x_6 \to\quad & p_0(\lambda) = P_{A[\emptyset]}(\lambda) = 1. \\
x_5 \to\quad & p_1(\lambda) = P_{A[1,2]}(\lambda) = \lambda^2 + (-a_{11} - a_{22})\lambda + \cdots; \\
x_4 \to\quad & p_2(\lambda) = P_{A[1]}(\lambda)\, P_{A[6]}(\lambda) = \lambda^2 + (-a_{11} - a_{66})\lambda + \cdots; \\
x_3 \to\quad & p_3(\lambda) = P_{A[1,2,3,4]}(\lambda) = \lambda^4 + (-a_{11} - a_{22} - a_{33} - a_{44})\lambda^3 + \cdots; \\
x_2 \to\quad & p_4(\lambda) = P_{A[1,2,3]}(\lambda)\, P_{A[6]}(\lambda) = \lambda^4 + (-a_{11} - a_{22} - a_{33} - a_{66})\lambda^3 + \cdots; \\
x_1 \to\quad & p_5(\lambda) = P_{A[1,2,3,4]}(\lambda)\, P_{A[6]}(\lambda) = \lambda^5 + \cdots; \\
1 \to\quad & p_6(\lambda) = P_{A_0}(\lambda) = \lambda^6 + \cdots;
\end{aligned}$$

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A publication of the International Linear Algebra Society
Volume 37, pp. 193-210, February 2021.

And therefore the matrix $\mathfrak{M}_A$ of coefficients is

	1	λ	λ^2	λ^3	λ^4	λ^5	λ^6
$p_0(\lambda)$	1	0	0	0	0	0	0
$p_1(\lambda)$	$*$	$-a_{11}-a_{22}$	1	0	0	0	0
$p_2(\lambda)$	$*$	$-a_{11}-a_{66}$	1	0	0	0	0
$p_3(\lambda)$	$*$	$*$	$*$	$-a_{11}-a_{22}-a_{33}-a_{44}$	1	0	0
$p_4(\lambda)$	$*$	$*$	$*$	$-a_{11}-a_{22}-a_{33}-a_{66}$	1	0	0
$p_5(\lambda)$	$*$	$*$	$*$	$*$	$*$	1	0
$p_6(\lambda)$	$*$	$*$	$*$	$*$	$*$	$*$	1

Note that the concatenation associated to A is

$$(0,1,2,3,4,5,6) = (\overset{b_1,e_1}{\overset{\shortparallel}{0}})^\frown(\overset{b_2}{\overset{\shortparallel}{1}},\overset{e_2}{\overset{\shortparallel}{2}})^\frown(\overset{b_3}{\overset{\shortparallel}{3}},\overset{e_3}{\overset{\shortparallel}{4}})^\frown(\overset{b_4,e_4}{\overset{\shortparallel}{5}})^\frown(\overset{b_5,e_5}{\overset{\shortparallel}{6}}),$$

where $0,2,4,5,6$ are all $t \in \{0,1,2,3,4,5,6\}$ such that $\deg(p_t(\lambda)) = t$. So A is PB-companion if and only if $\det(\mathfrak{M}_A[1,2]) = -a_{22}+a_{66} \neq 0$ and $\det(\mathfrak{M}_A[3,4]) = -a_{44}+a_{66} \neq 0$.

Acknowledgment. The authors wish to thank the referee for the helpful comments and specially for his or her thorough revision.

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