Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

ELA

ON SOLUTIONS TO THE QUATERNION MATRIX EQUATION $AXB + CYD = E$*

QING-WEN WANG†, HUA-SHENG ZHANG†, AND SHAO-WEN YU†

Abstract. Expressions, as well as necessary and sufficient conditions are given for the existence of the real and pure imaginary solutions to the consistent quaternion matrix equation $AXB+CYD = E$. Formulas are established for the extreme ranks of real matrices $X_i, Y_i, i = 1, \cdots, 4$, in a solution pair $X = X_1+X_2i+X_3j+X_4k$ and $Y = Y_1+Y_2i+Y_3j+Y_4k$ to this equation. Moreover, necessary and sufficient conditions are derived for all solution pairs X and Y of this equation to be real or pure imaginary, respectively. Some known results can be regarded as special cases of the results in this paper.

Key words. Quaternion matrix equation, Extreme rank, Generalized inverse.

AMS subject classifications. 15A03, 15A09, 15A24, 15A33.

1. Introduction. Throughout this paper, we denote the real number field by R, and the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_0, a_1, a_2, a_3 \ \in R\}$$

by $H^{m\times n}$. The symbols I, A^T, $R(A)$, $N(A)$, $\dim R(A)$ stand for the identity matrix of the appropriate size, the transpose, the column right space, the row left space of a matrix A over H, and the dimension of $R(A)$, respectively. By [6], for a quaternion matrix A, $\dim R(A) = \dim N(A)$, which is called the rank of A and denoted by $r(A)$. A generalized inverse of a matrix A which satisfies $AA^-A = A$ is denoted by A^-. Moreover, R_A and L_A stand for the two projectors $L_A = I - A^-A$, $R_A = I - AA^-$ induced by A. For an arbitrary quaternion matrix $M = M_1 + M_2i + M_3j + M_4k$, we define a map $\phi(\cdot)$ from $H^{m\times n}$ to $R^{4m\times 4n}$ by

$$\phi(M) = \begin{pmatrix} M_4 & M_3 & M_2 & M_1 \\ -M_3 & M_4 & M_1 & -M_2 \\ M_2 & M_1 & -M_4 & -M_3 \\ M_1 & -M_2 & M_3 & -M_4 \end{pmatrix}. \tag{1.1}$$

*Received by the editors August 2, 2007. Accepted for publication July 21, 2008. Handling Editor: Michael Neumann.

†Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China (wqw858@yahoo.com.cn). The first author was supported by the Natural Science Foundation of China (60672160), Shanghai Pujiang Program (06PJ14039), and by the Shanghai Leading Academic Discipline Project (J50101).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

By (1.1), it is easy to verify that $\phi(\cdot)$ satisfies the following properties:
(a) $M = N \iff \phi(M) = \phi(N)$.
(b) $\phi(M+N) = \phi(M) + \phi(N)$, $\phi(MN) = \phi(M)\phi(N)$, $\phi(kM) = k\phi(M)$, $k \in R$.
(c) $\phi(M) = -T_m^{-1}\phi(M)T_n = -R_m^{-1}\phi(M)R_n = S_m^{-1}\phi(M)S_n$, where $t = m, n$,

$$R_t = \begin{pmatrix} 0 & -I_{2t} \\ I_{2t} & 0 \end{pmatrix}, T_t = \begin{pmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{pmatrix}, S_t = \begin{pmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{pmatrix}.$$

(d) $r\left[\phi(M)\right] = 4r(M)$.

Tian [17] in 2003 gave the maximal and minimal ranks of two real matrices X_0 and X_1 in complex solution $X = X_0 + iX_1$ to the classical linear matrix equation

$$AXB = C. \tag{1.2}$$

Liu [11] in 2006 investigated the extreme ranks of solution pairs X and Y, and the extreme ranks of four real matrices X_0, X_1, Y_0 and Y_1 in a pair of a complex solutions $X = X_0 + iX_1$ and $Y = Y_0 + iY_1$ to the generalized Sylvester matrix equation

$$AX + YB = C, \tag{1.3}$$

which is widely studied (see, e.g., [4], [5], [7], [8], [11], [13], [22]-[24], [33], [34]). As an extension of (1.2) and (1.3), the matrix equation

$$AXB + CYD = E \tag{1.4}$$

has been well-studied in matrix theory (see, e.g., [2], [9], [10], [12], [14], [18]-[21], [35]). For instance, Baksalary and Kala [2] gave necessary and sufficient conditions for the existence and an representation of the general solution to (1.4). Özgüler [12] investigated (1.4) over a principal ideal domain. Huang and Zeng [9] considered (1.4) over a simple Artinian ring. Wang [20]$-$[21] investigated (1.4) over an arbitrary division ring and on any regular ring with identity in 1996 and 2004, respectively. Tian [19] in 2006 established formulas for extremal ranks of the solution of (1.4) over the complex number field. Note that, to our knowledge, the real and pure imaginary solutions to (1.4) over the quaternion algebra H have not been investigated so far in the literature. Motivated by the work mentioned above, and keeping the applications and interests of quaternion matrices in view (e.g., [1], [3], [25]-[32], [36]-[37]), in this paper we consider the real and pure imaginary solutions to (1.4) over H. We first derive the formulas for extremal ranks of real matrices $X_i, Y_i, i = 1, \cdots, 4$, in a solution pair $X = X_1 + X_2 i + X_3 j + X_4 k$ and $Y = Y_1 + Y_2 i + Y_3 j + Y_4 k$ to (1.4), then use the results to derive necessary and sufficient conditions for (1.4) over H to have a real solution and a pure imaginary solution, respectively. Finally, we establish necessary and sufficient conditions for all solutions (X, Y) to (1.4) over H to be real or pure imaginary.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

2. Main results. In this section, we consider (1.4) over H, where $A = A_1 + A_2 i + A_3 j + A_4 k$, $B = B_1 + B_2 i + B_3 j + B_4 k$, $C = C_1 + C_2 i + C_3 j + C_4 k$, $D = D_1 + D_2 i + D_3 j + D_4 k$, $E = E_1 + E_2 i + E_3 j + E_4 k$ are known and $X = X_1 + X_2 i + X_3 j + X_4 k \in H^{n\times s}$, $Y = Y_1 + Y_2 i + Y_3 j + Y_4 k \in H^{l\times k}$ unknown; here $A_i, B_i, C_i, D_i, E_i, X_i$ and $Y_i, i = 1, \cdots, 4$, are real matrices with suitable sizes.

The following lemmas are due to Tian (see [15], [16] and [18]) that can be generalized to H.

LEMMA 2.1. *Let* $A \in H^{m\times k}$, $B \in H^{l\times n}$, $C \in H^{m\times i}$, $D \in H^{j\times n}, E \in H^{m\times n}$ *and* $p\,(X,Y,Z) = E - AXB - CY - ZD$. *Then*

$$\max_{X,Y,Z} r\left[p\,(X,Y,Z)\right] = \min\left\{m,\ n,\ r\begin{pmatrix} E & A & C \\ D & 0 & 0 \end{pmatrix}, r\begin{pmatrix} E & C \\ B & 0 \\ D & 0 \end{pmatrix}\right\}; \tag{2.1}$$

$$\min_{X,Y,Z} r\left[p\,(X,Y,Z)\right] = r\begin{pmatrix} E & C \\ B & 0 \\ D & 0 \end{pmatrix} - r\begin{pmatrix} E & A & C \\ B & 0 & 0 \\ D & 0 & 0 \end{pmatrix} + r\begin{pmatrix} E & A & C \\ D & 0 & 0 \end{pmatrix} - r(C) - r(D). \tag{2.2}$$

LEMMA 2.2. *Let* $A \in H^{m\times n}$, $B \in H^{m\times k}$ *and* $C \in H^{l\times n}$. *Then*

$$r\,(B, AL_C) = r\begin{pmatrix} B & A \\ 0 & C \end{pmatrix} - r(C);\ r\begin{pmatrix} C \\ R_B A \end{pmatrix} = r\begin{pmatrix} C & 0 \\ A & B \end{pmatrix} - r(B).$$

LEMMA 2.3. *Suppose that matrix equation (1.4) is consistent over* H. *Then its general solutions can be expressed as*

$$X = \widetilde{X_0} + \widetilde{S_1} L_G U R_H \widetilde{T_1} + L_A V_1 + V_2 R_B, \tag{2.3}$$

$$Y = \widetilde{Y_0} + \widetilde{S_2} L_G U R_H \widetilde{T_2} + L_C W_1 + W_2 R_D \tag{2.4}$$

where $\widetilde{X_0}$ *and* $\widetilde{Y_0}$ *are a pair of special solutions of (1.4),* $\widetilde{S_1} = (I_p, 0)$, $\widetilde{S_2} = (0, I_s)$, $\widetilde{T_1} = (I_q, 0)^T$, $\widetilde{T_2} = (0, I_t)^T$, $G = (A, C)$, $H = \left(B^T, -D^T\right)^T$, *the quaternion matrices* U, V_1, V_2, W_1 *and* W_2 *are arbitrary with suitable sizes.*

In the following theorems and corollaries, $\widetilde{X_0}, \widetilde{Y_0}, \widetilde{S_1}, \widetilde{S_2}, \widetilde{T_1}, \widetilde{T_2}, G$ and H are defined as in Lemma 2.3.

THEOREM 2.4. *Matrix equation (1.4) is consistent over* H *if and only if the matrix equation*

$$\phi(A)\left(X_{ij}\right)_{4\times 4}\phi(B) + \phi(C)\left(Y_{ij}\right)_{4\times 4}\phi(D) = \phi(E), i, j = 1, 2, 3, 4, \tag{2.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

is consistent over R. In that case, the general solution of (1.4) over H can be written as

$$
\begin{aligned}
(2.6)\qquad X &= X_1 + X_2 i + X_3 j + X_4 k \\
&= \frac{1}{4}(X_{11} + X_{22} - X_{33} - X_{44}) + \frac{1}{4}(X_{12} - X_{21} + X_{43} - X_{34})i \\
&+ \frac{1}{4}(X_{13} + X_{31} - X_{24} - X_{42})j + \frac{1}{4}(X_{41} + X_{14} + X_{32} + X_{23})k, \\
(2.7)\qquad Y &= Y_1 + Y_2 i + Y_3 j + Y_4 k \\
&= \frac{1}{4}(Y_{11} + Y_{22} - Y_{33} - Y_{44}) + \frac{1}{4}(Y_{12} - Y_{21} + Y_{43} - Y_{34})i \\
&+ \frac{1}{4}(Y_{31} + Y_{13} - Y_{24} - Y_{42})j + \frac{1}{4}(Y_{41} + Y_{14} + Y_{32} + Y_{23})k.
\end{aligned}
$$

Written in an explicit form, X_i and $Y_i, i = 1, \cdots, 4$, in (2.6) are as follows.

$$
\begin{aligned}
(2.8)\qquad X_1 &= \frac{1}{4}P_1\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_2 - \frac{1}{4}P_3\phi(X_0)Q_3 - \frac{1}{4}P_4\phi(X_0)Q_4 \\
&+ \frac{1}{4}\left(P_1R_1, P_2R_1, -P_3R_1, -P_4R_1\right) U \begin{pmatrix} L_1Q_1 \\ L_1Q_2 \\ L_1Q_3 \\ L_1Q_4 \end{pmatrix} \\
&+ \left(P_1L_{\phi(A)}, P_2L_{\phi(A)}, -P_3L_{\phi(A)}, -P_4L_{\phi(A)}\right) V + \widetilde{U} \begin{pmatrix} R_{\phi(B)}Q_1 \\ R_{\phi(B)}Q_2 \\ -R_{\phi(B)}Q_3 \\ -R_{\phi(B)}Q_4 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
(2.9)\qquad X_2 &= \frac{1}{4}P_1\phi(X_0)Q_2 - \frac{1}{4}P_2\phi(X_0)Q_1 + \frac{1}{4}P_4\phi(X_0)Q_3 - \frac{1}{4}P_3\phi(X_0)Q_4 \\
&+ \frac{1}{4}\left(P_1R_1, -P_2R_1, -P_3R_1, P_4R_1\right) U \begin{pmatrix} L_1Q_2 \\ L_1Q_1 \\ L_1Q_4 \\ L_1Q_3 \end{pmatrix} \\
&+ \left(-P_2L_{\phi(A)}, P_1L_{\phi(A)}, P_4L_{\phi(A)}, -P_3L_{\phi(A)}\right) V + \widetilde{U} \begin{pmatrix} -R_{\phi(B)}Q_2 \\ R_{\phi(B)}Q_1 \\ R_{\phi(B)}Q_4 \\ -R_{\phi(B)}Q_3 \end{pmatrix},
\end{aligned}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

$$X_3 = \frac{1}{4}P_1\phi(X_0)Q_3 - \frac{1}{4}P_2\phi(X_0)Q_4 + \frac{1}{4}P_3\phi(X_0)Q_1 - \frac{1}{4}P_4\phi(X_0)Q_2 + \frac{1}{4}\left(P_1R_1, -P_2R_1, P_3R_1, -P_4R_1\right)U\begin{pmatrix} L_1Q_3 \\ L_1Q_4 \\ L_1Q_1 \\ L_1Q_2 \end{pmatrix} + \left(P_3L_{\phi(A)}, -P_2L_{\phi(A)}, P_1L_{\phi(A)}, -P_4L_{\phi(A)}\right)V + \widetilde{U}\begin{pmatrix} R_{\phi(B)}Q_3 \\ -R_{\phi(B)}Q_2 \\ R_{\phi(B)}Q_1 \\ -R_{\phi(B)}Q_4 \end{pmatrix}, \tag{2.10}$$

$$X_4 = \frac{1}{4}P_1\phi(X_0)Q_4 + \frac{1}{4}P_2\phi(X_0)Q_3 + \frac{1}{4}P_3\phi(X_0)Q_2 + \frac{1}{4}P_4\phi(X_0)Q_1 + \frac{1}{4}\left(P_1R_1, P_2R_1, P_3R_1, P_4R_1\right)U\begin{pmatrix} L_1Q_4 \\ L_1Q_3 \\ L_1Q_2 \\ L_1Q_1 \end{pmatrix} + \left(P_4L_{\phi(A)}, P_3L_{\phi(A)}, P_2L_{\phi(A)}, P_1L_{\phi(A)}\right)V + \widetilde{U}\begin{pmatrix} R_{\phi(B)}Q_4 \\ R_{\phi(B)}Q_3 \\ R_{\phi(B)}Q_2 \\ R_{\phi(B)}Q_1 \end{pmatrix}, \tag{2.11}$$

$$Y_1 = \frac{1}{4}S_1\phi(Y_0)T_1 + \frac{1}{4}S_2\phi(Y_0)T_2 - \frac{1}{4}S_3\phi(Y_0)T_3 - \frac{1}{4}S_4\phi(Y_0)T_4 + \frac{1}{4}\left(S_1R_2, S_2R_2, -S_3R_2, -S_4R_2\right)U\begin{pmatrix} L_2T_1 \\ L_2T_2 \\ L_2T_3 \\ L_2T_4 \end{pmatrix} + \left(S_1L_{\phi(A)}, S_2L_{\phi(A)}, -S_3L_{\phi(A)}, -S_4L_{\phi(A)}\right)\widehat{U} + W\begin{pmatrix} R_{\phi(B)}T_1 \\ R_{\phi(B)}T_2 \\ -R_{\phi(B)}T_3 \\ -R_{\phi(B)}T_4 \end{pmatrix}, \tag{2.12}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

$$
\begin{aligned}
Y_2 = {} & \frac{1}{4}S_1\phi(Y_0)T_2 - \frac{1}{4}S_2\phi(Y_0)T_1 - \frac{1}{4}S_3\phi(Y_0)T_4 + \frac{1}{4}S_4\phi(Y_0)T_3 \\
& + \frac{1}{4}\left(S_1R_2, -S_2R_2, -S_3R_2, S_4R_2\right)U\begin{pmatrix} L_2T_2 \\ L_2T_1 \\ L_2T_4 \\ L_2T_3 \end{pmatrix} \\
& + \left(-S_2L_{\phi(A)}, S_1L_{\phi(A)}, S_4L_{\phi(A)}, -S_3L_{\phi(A)}\right)\widehat{U} + W\begin{pmatrix} -R_{\phi(B)}T_2 \\ R_{\phi(B)}T_1 \\ R_{\phi(B)}T_4 \\ -R_{\phi(B)}T_3 \end{pmatrix},
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
Y_3 = {} & \frac{1}{4}S_1\phi(Y_0)T_3 - \frac{1}{4}S_2\phi(Y_0)T_4 + \frac{1}{4}S_3\phi(Y_0)T_1 - \frac{1}{4}S_4\phi(Y_0)T_2 \\
& + \frac{1}{4}\left(S_1R_2, -S_2R_2, S_3R_2, -S_4R_2\right)U\begin{pmatrix} L_2T_3 \\ L_2T_4 \\ L_2T_1 \\ L_2T_2 \end{pmatrix} \\
& + \left(S_3L_{\phi(A)}, -S_4L_{\phi(A)}, S_1L_{\phi(A)}, -S_2L_{\phi(A)}\right)\widehat{U} + W\begin{pmatrix} R_{\phi(B)}T_3 \\ -R_{\phi(B)}T_4 \\ R_{\phi(B)}T_1 \\ -R_{\phi(B)}T_2 \end{pmatrix},
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
Y_4 = {} & \frac{1}{4}S_1\phi(Y_0)T_4 + \frac{1}{4}S_2\phi(Y_0)T_3 + \frac{1}{4}S_3\phi(Y_0)T_2 + \frac{1}{4}S_4\phi(Y_0)T_1 \\
& + \frac{1}{4}\left(S_1R_2, S_2R_2, S_3R_2, S_4R_2\right)U\begin{pmatrix} L_2T_4 \\ L_2T_3 \\ L_2T_2 \\ L_2T_1 \end{pmatrix} \\
& + \left(S_4L_{\phi(A)}, S_3L_{\phi(A)}, S_2L_{\phi(A)}, S_1L_{\phi(A)}\right)\widehat{U} + W\begin{pmatrix} R_{\phi(B)}T_4 \\ R_{\phi(B)}T_3 \\ R_{\phi(B)}T_2 \\ R_{\phi(B)}T_1 \end{pmatrix},
\end{aligned} \tag{2.15}
$$

where

$$
\begin{aligned}
& P_1 = (I_p, 0, 0, 0)\,,\ P_2 = (0, I_p, 0, 0)\,,\ P_3 = (0, 0, I_p, 0)\,,\ P_4 = (0, 0, 0, I_p)\,, \\
& S_1 = (I_m, 0, 0, 0)\,,\ S_2 = (0, I_m, 0, 0)\,,\ S_3 = (0, 0, I_m, 0)\,,\ S_4 = (0, 0, 0, I_m)\,, \\
& Q_1 = (I_n, 0, 0, 0)^T\,,\ Q_2 = (0, I_n, 0, 0)^T\,,\ Q_3 = (0, 0, I_n, 0)^T\,,\ Q_4 = (0, 0, 0, I_n)^T\,, \\
& T_1 = (I_q, 0, 0, 0)^T\,,\ T_2 = (0, I_q, 0, 0)^T\,,\ T_3 = (0, 0, I_q, 0)^T\,,\ T_4 = (0, 0, 0, I_q)^T\,, \\
& R_1 = \phi(\widetilde{S_1})L_{\phi(G)},\ L_1 = R_{\phi(H)}\phi(\widetilde{T_1}),\ R_2 = \phi(\widetilde{S_2})L_{\phi(G)},\ L_2 = R_{\phi(H)}\phi(\widetilde{T_2}),
\end{aligned}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

ELA

and $U, \widetilde{U}, \hat{U}, V$ *and* W *are arbitrary real matrices with compatible sizes.*

Proof. Suppose that (1.4) has a solution pair X, Y over H. Applying properties (a) and (b) of $\phi(\cdot)$ above to (1.4) yields

$$\phi(A)\phi(X)\phi(B) + \phi(C)\phi(Y)\phi(D) = \phi(E),$$

which implies that $\phi(X), \phi(Y)$ is a solution pair to (2.5). Conversely, suppose that (2.5) has a solution pair

$$\hat{X} = (X_{ij})_{4\times 4}\,,\ \hat{Y} = (Y_{ij})_{4\times 4}\,, i, j = 1, 2, 3, 4.$$

i.e.

$$\phi(A)\hat{X}\phi(B) + \phi(C)\hat{Y}\phi(D) = \phi(E).$$

Then, applying property (c) of $\phi(\cdot)$ above, it yields

$$\begin{aligned}
T_m^{-1}\phi(A)T_p\hat{X}T_q^{-1}\phi(B)T_n + T_m^{-1}\phi(C)T_p\hat{Y}T_q^{-1}\phi(D)T_n &= -T_m^{-1}\phi(E)T_n,\\
R_m^{-1}\phi(A)R_p\hat{X}R_q^{-1}\phi(B)R_n + R_m^{-1}\phi(C)R_p\hat{Y}R_q^{-1}\phi(D)R_n &= -R_m^{-1}\phi(E)R_n,\\
S_m^{-1}\phi(A)S_p\hat{X}S_q^{-1}\phi(B)S_n + S_m^{-1}\phi(C)S_p\hat{Y}S_q^{-1}\phi(D)S_n &= S_m^{-1}\phi(E)S_n.
\end{aligned}$$

Hence,

$$\begin{aligned}
\phi(A)T_p\hat{X}T_q^{-1}\phi(B) + \phi(C)T_m\hat{Y}T_q^{-1}\phi(D) &= \phi(E),\\
\phi(A)R_p\hat{X}R_n^{-1}\phi(B) + \phi(C)R_m\hat{Y}R_q^{-1}\phi(D) &= \phi(E),\\
\phi(A)S_p\hat{X}S_n^{-1}\phi(B) + \phi(C)S_m\hat{Y}S_q^{-1}\phi(D) &= \phi(E),
\end{aligned}$$

which implies that $T_p\hat{X}T_n^{-1}, T_m\hat{Y}T_q^{-1}, R_p\hat{X}R_n^{-1}, R_m\hat{Y}R_q^{-1}$ and $S_p\hat{X}S_n^{-1}, S_m\hat{Y}S_q^{-1}$ are also solutions of (2.5). Thus,

$$\frac{1}{4}(\hat{X} - T_p\hat{X}T_n^{-1} - R_p\hat{X}R_n^{-1} + S_p\hat{X}S_n^{-1}),\ \frac{1}{4}(\hat{Y} - T_m\hat{Y}T_q^{-1} - R_m\hat{Y}R_q^{-1} + S_m\hat{Y}S_q^{-1})$$

is a solution pair of (2.5), where

$$\hat{X} - T_p\hat{X}T_n^{-1} - R_p\hat{X}R_n^{-1} + S_p\hat{X}S_n^{-1} = \left(\widetilde{X_{ij}}\right)_{4\times 4}, i, j = 1, 2, 3, 4$$

and

$$\begin{aligned}
&\widetilde{X_{11}} = X_{11} + X_{22} - X_{33} - X_{44}, \widetilde{X_{12}} = X_{12} - X_{21} + X_{43} - X_{34},\\
&\widetilde{X_{13}} = X_{13} + X_{31} - X_{24} - X_{42}, \widetilde{X_{14}} = X_{41} + X_{14} + X_{32} + X_{23},\\
&\widetilde{X_{21}} = X_{21} - X_{12} + X_{34} - X_{43}, \widetilde{X_{22}} = X_{11} + X_{22} - X_{33} - X_{44},\\
&\widetilde{X_{23}} = X_{41} + X_{14} + X_{32} + X_{23}, \widetilde{X_{24}} = X_{24} + X_{42} - X_{13} - X_{31},\\
&\widetilde{X_{31}} = X_{13} + X_{31} - X_{24} - X_{42}, \widetilde{X_{32}} = X_{41} + X_{14} + X_{32} + X_{23},\\
&\widetilde{X_{33}} = X_{33} + X_{44} - X_{11} - X_{22}, \widetilde{X_{34}} = X_{21} - X_{12} + X_{34} - X_{43},\\
&\widetilde{X_{41}} = X_{41} + X_{14} + X_{32} + X_{23}, \widetilde{X_{42}} = X_{24} + X_{42} - X_{13} - X_{31},\\
&\widetilde{X_{43}} = X_{12} - X_{21} + X_{43} - X_{34}, \widetilde{X_{44}} = X_{33} + X_{44} - X_{11} - X_{22}.
\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

$\hat{Y} - T_m\hat{Y}T_q^{-1} - R_m\hat{Y}R_q^{-1} + S_m\hat{Y}S_q^{-1}$ has a form similar to $\left(\widetilde{X_{ij}}\right)_{4\times 4}$. We omit it here for simplicity.
Let

$$\begin{aligned}\widetilde{X} &= \frac{1}{4}(X_{11} + X_{22} - X_{33} - X_{44}) + \frac{1}{4}(X_{12} - X_{21} + X_{43} - X_{34})i\\ &+ \frac{1}{4}(X_{13} + X_{31} - X_{24} - X_{42})j + \frac{1}{4}(X_{41} + X_{14} + X_{32} + X_{23})k,\\ \widetilde{Y} &= \frac{1}{4}(Y_{11} + Y_{22} - Y_{33} - Y_{44}) + \frac{1}{4}(Y_{12} - Y_{21} + Y_{43} - Y_{34})i\\ &+ \frac{1}{4}(Y_{31} + Y_{13} - Y_{24} - Y_{42})j + \frac{1}{4}(Y_{41} + Y_{14} + Y_{32} + Y_{23})k.\end{aligned}$$

Then, by (1.1),

$$\begin{aligned}\phi(\widetilde{X}) &= \frac{1}{4}(\hat{X} - T_p\hat{X}T_n^{-1} - R_p\hat{X}R_n^{-1} + S_p\hat{X}S_n^{-1}),\\ \phi(\widetilde{Y}) &= \frac{1}{4}(\hat{Y} - T_m\hat{Y}T_q^{-1} - R_m\hat{Y}R_q^{-1} + S_m\hat{Y}S_q^{-1})\end{aligned}$$

is a solution pair of (2.5). Hence, by property (a) of $\phi(\cdot)$, we know that $\widetilde{X}, \widetilde{Y}$ is a solution pair of (1.4). The above discussion shows that the two matrix equations (1.4) and (2.5) have the same solvability condition and their solutions satisfy (2.6) and (2.7). Observe that X_{ij} and Y_{ij}, $i, j = 1, 2, 3, 4$ in (2.5) can be written as

$$X_{ij} = P_i\widehat{X}Q_j,\ Y_{ij} = S_i\widehat{Y}T_j.$$

From Lemma 2.3, the general solution to (2.5) can be written as

$$\begin{aligned}\widehat{X} &= \phi(X_0) + \phi(\widetilde{S_1})L_{\phi(G)}UR_{\phi(H)}\phi(\widetilde{T_1}) + 4L_{\phi(A)}V + 4\widetilde{U}^TR_{\phi(B)},\\ \widehat{Y} &= \phi(Y_0) + \phi(\widetilde{S_2})L_{\phi(G)}UR_{\phi(H)}\phi(\widetilde{T_2}) + 4L_{\phi(C)}\widehat{U} + 4W^TR_{\phi(D)}.\end{aligned}$$

Hence,

$$\begin{aligned}X_{ij} &= P_i\phi(X_0)Q_j + P_i\phi(\widetilde{S_1})L_{\phi(G)}UR_{\phi(H)}\phi(\widetilde{T_1})Q_j + 4P_iL_{\phi(A)}V_j + 4U_iR_{\phi(B)}Q_j,\\ Y_{ij} &= S_i\phi(Y_0)T_j + S_i\phi(\widetilde{S_2})L_{\phi(G)}UR_{\phi(H)}\phi(\widetilde{T_2})T_j + 4S_iL_{\phi(C)}\hat{U}_j + 4W_iR_{\phi(D)}T_j,\end{aligned}$$

where $U, U_1, \cdots, U_4 \in R^{p\times 4q}$, $V_1, \cdots, V_4 \in R^{4p\times q}$, $\hat{U}_1, \cdots, \hat{U}_4 \in R^{4p\times q}$, $W_1, \cdots, W_4 \in R^{p\times 4q}$ are arbitrary, and

$$\begin{aligned}&V = (V_1, V_2, V_3, V_4)\,, \widetilde{U}^T = \left(U_1^T, U_2^T, U_3^T, U_4^T\right),\\ &\widehat{U} = \left(\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{U}_4\right), W^T = \left(W_1^T, W_2^T, W_3^T, W_4^T\right).\end{aligned}$$

Putting them into (2.6), (2.7) yields the eight real matrices $X_1, \cdots, X_4$ and $Y_1, \cdots, Y_4$, in (2.8)-(2.15). ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

Now we consider the extreme ranks of real matrices $X_1, \cdots, X_4$ and $Y_1, \cdots, Y_4$ in the solution (X, Y) to (1.4) over H.

Theorem 2.5. *Suppose that (1.4) over H is consistent, and for $i, j = 1, 2, 3, 4$,*

$$J_i = \left\{ X_i \in R^{n\times s} \,|A(X_1 + X_2 i + X_3 j + X_4 k)B + C\left(Y_1 + Y_2 i + Y_3 j + Y_4 k\right)D = E \right\},$$

$$K_j = \left\{ Y_j \in R^{l\times k} \,|A(X_1 + X_2 i + X_3 j + X_4 k)B + C\left(Y_1 + Y_2 i + Y_3 j + Y_4 k\right)D = E \right\}.$$

Then we have the following:
(a) The maximal and minimal ranks of X_i in the solution $X = X_1 + X_2 i + X_3 j + X_4 k$ to (1.4) are given by

$$\max_{X_i\in J_i} r(X_i) = \min\left\{ p, q, p+q+r\begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} - 4(r(B) - r(A, C)),\right.$$
$$\left. p+q+r\begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} - 4r(A) - 4r\begin{pmatrix} B \\ D \end{pmatrix} \right\},$$

$$\min_{X_i\in J_i} r(X_i) = r\begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} + r\begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} - r\begin{pmatrix} 0 & 0 & \widehat{B}_i \\ 0 & 0 & \phi(D) \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix}$$

where

$$\widehat{A}_1 = \begin{pmatrix} A_2 & A_3 & -A_4 \\ -A_1 & -A_4 & -A_3 \\ A_4 & -A_1 & A_2 \\ A_3 & -A_2 & -A_1 \end{pmatrix}, \widehat{A}_2 = \begin{pmatrix} -A_1 & A_3 & -A_4 \\ -A_2 & -A_4 & -A_3 \\ -A_3 & -A_1 & A_2 \\ A_4 & -A_2 & -A_1 \end{pmatrix},$$

$$\widehat{A}_3 = \begin{pmatrix} -A_1 & A_2 & -A_4 \\ -A_2 & -A_1 & -A_3 \\ -A_3 & A_4 & A_2 \\ A_4 & A_3 & -A_1 \end{pmatrix}, \widehat{A}_4 = \begin{pmatrix} -A_1 & A_2 & A_3 \\ -A_2 & -A_1 & -A_4 \\ -A_3 & A_4 & -A_1 \\ A_4 & A_3 & -A_2 \end{pmatrix},$$

$$\widehat{B}_1 = \begin{pmatrix} -B_2 & -B_1 & -B_4 & -B_3 \\ -B_3 & -B_4 & -B_1 & B_2 \\ B_4 & B_3 & -B_2 & -B_1 \end{pmatrix}, \widehat{B}_2 = \begin{pmatrix} -B_1 & B_2 & B_3 & -B_4 \\ -B_3 & -B_4 & -B_1 & B_2 \\ B_4 & B_3 & -B_2 & -B_1 \end{pmatrix},$$

$$\widehat{B}_3 = \begin{pmatrix} -B_1 & B_2 & B_3 & -B_4 \\ -B_2 & -B_1 & -B_4 & -B_3 \\ B_4 & B_3 & -B_2 & -B_1 \end{pmatrix}, \widehat{B}_4 = \begin{pmatrix} -B_1 & B_2 & B_3 & -B_4 \\ -B_2 & -B_1 & -B_4 & -B_3 \\ -B_3 & B_4 & -B_1 & B_2 \end{pmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

(*b*) *The maximal and minimal ranks of* Y_j *in the solution* $Y = Y_1 + Y_2 i + Y_3 j + Y_4 k$ *to (1.4) are given by*

$$\max_{Y_j \in K_j} r(Y_j) = \min \left\{ s, t, s+t+r\begin{pmatrix} 0 & 0 & \widehat{D}_j \\ \widehat{C}_j & \phi(A) & \phi(E) \end{pmatrix} - 4r(B) - 4r\,(C, A) \right.$$
$$\left. s+t+r\begin{pmatrix} 0 & \widehat{D}_j \\ 0 & \phi(B) \\ \widehat{C}_j & \phi(E) \end{pmatrix} - 4r(C) - 4r\begin{pmatrix} B \\ D \end{pmatrix} \right\},$$

$$\min_{Y_j \in K_j} r(Y_j) = r\begin{pmatrix} 0 & 0 & \widehat{D}_j \\ \widehat{C}_j & \phi(A) & \phi(E) \end{pmatrix} + r\begin{pmatrix} 0 & \widehat{D}_j \\ 0 & \phi(B) \\ \widehat{C}_j & \phi(E) \end{pmatrix} - r\begin{pmatrix} 0 & 0 & \widehat{D}_j \\ 0 & 0 & \phi(B) \\ \widehat{C}_j & \phi(A) & \phi(E) \end{pmatrix}$$

where

$$\widehat{C}_1 = \begin{pmatrix} C_2 & C_3 & -C_4 \\ -C_1 & -C_4 & -C_3 \\ C_4 & -C_1 & C_2 \\ C_3 & -C_2 & -C_1 \end{pmatrix}, \widehat{C}_2 = \begin{pmatrix} -C_1 & C_3 & -C_4 \\ -C_2 & -C_4 & -C_3 \\ -C_3 & -C_1 & C_2 \\ C_4 & -C_2 & -C_1 \end{pmatrix},$$

$$\widehat{C}_3 = \begin{pmatrix} -C_1 & C_2 & -C_4 \\ -C_2 & -C_1 & -C_3 \\ -C_3 & C_4 & C_2 \\ C_4 & C_3 & -C_1 \end{pmatrix}, \widehat{C}_4 = \begin{pmatrix} -C_1 & C_2 & C_3 \\ -C_2 & -C_1 & -C_4 \\ -C_3 & C_4 & -C_1 \\ C_4 & C_3 & -C_2 \end{pmatrix},$$

$$\widehat{D}_1 = \begin{pmatrix} -D_2 & -D_1 & -D_4 & -D_3 \\ -D_3 & D_4 & -D_1 & D_2 \\ D_4 & D_3 & -D_2 & -D_1 \end{pmatrix}, \widehat{D}_2 = \begin{pmatrix} -D_1 & D_2 & D_3 & -D_4 \\ -D_3 & D_4 & -D_1 & D_2 \\ D_4 & D_3 & -D_2 & -D_1 \end{pmatrix},$$

$$\widehat{D}_3 = \begin{pmatrix} -D_1 & D_2 & D_3 & -D_4 \\ -D_2 & -D_1 & -D_4 & -D_3 \\ D_4 & D_3 & -D_2 & -D_1 \end{pmatrix}, \widehat{D}_4 = \begin{pmatrix} -D_1 & D_2 & D_3 & -D_4 \\ -D_2 & -D_1 & -D_4 & -D_3 \\ -D_3 & D_4 & -D_1 & D_2 \end{pmatrix}.$$

Proof. We only derive the maximal and minimal ranks of the matrix X_1; the other $r\,(X_i)\,, r(Y_j)$ can be established similarly. Applying (2.1) and (2.2) to (2.8), we get the following

$$\max_{X_1 \in J_1} r(X_1) = \min\left\{p, q, r(M_1), r(M_2)\right\},$$
$$\min_{X_1 \in J_1} r(X_1) = r(M_1) + r(M_2) - r(M_3) - r\,(P) - r\,(Q)\,,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

where

$$M_1 = \begin{pmatrix} \widetilde{X_0} & P & \widehat{P} \\ Q & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} \widetilde{X_0} & P \\ Q & 0 \\ \widehat{Q} & 0 \end{pmatrix}, M_3 = \begin{pmatrix} \widetilde{X_0} & P & \widehat{P} \\ Q & 0 & 0 \\ \widehat{Q} & 0 & 0 \end{pmatrix},$$

$$\widetilde{X_0} = \frac{1}{4}P_1\phi(X_0)Q_1 + \frac{1}{4}P_2\phi(X_0)Q_2 - \frac{1}{4}P_3\phi(X_0)Q_3 - \frac{1}{4}P_4\phi(X_0)Q_4,$$

$$P = \left(P_1 L_{\phi(A)}, P_2 L_{\phi(A)}, -P_3 L_{\phi(A)}, -P_4 L_{\phi(A)}\right),$$

$$\widehat{P} = \left(P_1\phi(\widetilde{S_1})L_{\phi(G)}, P_2\phi(\widetilde{S_1})L_{\phi(G)}, -P_3\phi(\widetilde{S_1})L_{\phi(G)}, -P_4\phi(\widetilde{S_1})L_{\phi(G)}\right),$$

$$Q = \begin{pmatrix} R_{\phi(B)}Q_1 \\ R_{\phi(B)}Q_2 \\ -R_{\phi(B)}Q_3 \\ -R_{\phi(B)}Q_4 \end{pmatrix}, \widehat{Q} = \begin{pmatrix} R_{\phi(H)}\phi(\widetilde{T_1})Q_1 \\ R_{\phi(H)}\phi(\widetilde{T_1})Q_2 \\ R_{\phi(H)}\phi(\widetilde{T_1})Q_3 \\ R_{\phi(H)}\phi(\widetilde{T_1})Q_4 \end{pmatrix}.$$

By Lemma 2.2, block Gaussian elimination and $AX_0B + CY_0D = E$, we have that $r(L_A) = p - r(A)$, $r(R_B) = q - r(B)$,

$$r(M_1) = \begin{bmatrix} \widetilde{X_0} & \widetilde{P} & \overline{P} & 0 \\ \widetilde{Q} & 0 & 0 & \Psi(B) \\ 0 & \Psi(A) & 0 & 0 \\ 0 & 0 & \Psi(G) & 0 \end{bmatrix} - 4r(\phi(A)) - 4r(\phi(B)) - 4r(\phi(G))$$

$$= r\begin{bmatrix} 0 & 0 & \phi(\widehat{B}_1) \\ \phi(\widehat{A}_1) & \phi(C) & \phi(E) \end{bmatrix} + p + q - 4r(B) - 4r(G)$$

where

$$\widetilde{P} = (P_1, P_2, -P_3, -P_4), \ \overline{P} = \left(P_1\phi(\widetilde{S_1}), P_2\phi(\widetilde{S_1}), -P_3\phi(\widetilde{S_1}), -P_4\phi(\widetilde{S_1})\right),$$

$$\widetilde{Q} = \begin{pmatrix} Q_1 \\ Q_2 \\ -Q_3 \\ -Q_4 \end{pmatrix}, \ \Psi(Z) = \begin{pmatrix} \phi(Z) & 0 & 0 & 0 \\ 0 & \phi(Z) & 0 & 0 \\ 0 & 0 & \phi(Z) & 0 \\ 0 & 0 & 0 & \phi(Z) \end{pmatrix}, Z = A, B, G.$$

In the same manner, we can simplify $r(M_2)$ and $r(M_3)$ as follows.

$$r(M_2) = r\begin{pmatrix} 0 & \widehat{B}_1 \\ 0 & \phi(D) \\ \widehat{A}_1 & \phi(E) \end{pmatrix} + p + q - 4r(A) - 4r(H),$$

$$r(M_3) = r\begin{pmatrix} 0 & 0 & \widehat{B}_1 \\ 0 & 0 & \phi(D) \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{pmatrix} + p + q - 4r(G) - 4r(H).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

Thus, we have the results for the extreme ranks of the matrix X_1 in (a). Similarly, applying (2.1) and (2.2) to (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) yields the other results in $(a), (b)$. □

In the following corollaries, $\widehat{A}_i, \widehat{B}_i, \widehat{C}_j$ and $\widehat{D}_j$ $(i, j = 1, 2, 3, 4)$ are defined as in Theorem 2.5.

Corollary 2.6. *Suppose that (1.4) over H is consistent. Then*
(a) *(1.4) has a real solution for X if and only if*

$$r\begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} + r\begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} = r\begin{pmatrix} 0 & 0 & \widehat{B}_i \\ 0 & 0 & \phi(D) \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix}, i = 2, 3, 4.$$

In that case, the real solution X can be expressed as $X = X_1$ in (2.8).
(b) *All the solutions of (1.4) for X are real if and only if*

$$p + q + r\begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} = 4r(B) + 4r(A, C)$$

or

$$p + q + r\begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} = 4r(A) + 4r\begin{pmatrix} B \\ D \end{pmatrix}, i = 2, 3, 4.$$

In that case, the real solutions X can be expressed as $X = X_1$ in (2.8).
(c) *(1.4) has a pure imaginary solution for X if and only if*

$$r\begin{pmatrix} 0 & 0 & \widehat{B}_1 \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{pmatrix} + r\begin{pmatrix} 0 & \widehat{B}_1 \\ 0 & \phi(D) \\ \widehat{A}_1 & \phi(E) \end{pmatrix} = r\begin{pmatrix} 0 & 0 & \widehat{B}_1 \\ 0 & 0 & \phi(D) \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{pmatrix}.$$

In that case, the pure imaginary solution can be expressed as $X = X_2 i + X_3 j + X_4 k$ where X_2, X_3 and X_4 are expressed as (2.9), (2.10) and (2.11), respectively.
(d) *All the solutions of (1.4) for X are pure imaginary if and only if*

$$p + q + r\begin{pmatrix} 0 & 0 & \widehat{B}_1 \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{pmatrix} = 4r(B) + 4r(A, C)$$

or

$$p + q + r\begin{pmatrix} 0 & \widehat{B}_1 \\ 0 & \phi(D) \\ \widehat{A}_1 & \phi(E) \end{pmatrix} = 4r(A) + 4r\begin{pmatrix} B \\ D \end{pmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

In that case, the pure imaginary solutions can be expressed as $X = X_2 i + X_3 j + X_4 k$*, where* X_2, X_3 *and* X_4 *are expressed as (2.9), (2.10) and (2.11), respectively.*

Using the same method, we can get the corresponding results on Y.

We now consider all solution pairs X and Y to (1.4) over H to be real or pure imaginary, respectively.

THEOREM 2.7. *Suppose that (1.4) over* H *is consistent,* $q\left(X_1, Y_1\right) = E - AX_1B - CY_1D$*, and* J_1*,* K_1 *as in Theorem 2.2 are two independent sets. Then*

$$\max_{\substack{X_1 \in J_1, \\ Y_1 \in K_1}} r\left[q\left(X_1, Y_1\right)\right] = \min\left\{r\left(A\right) + r\left(C\right) - r\left(A, C\right), r\left(B\right) + r\left(D\right) - r\begin{pmatrix} B \\ D \end{pmatrix}\right\}. \tag{2.16}$$

In particular:
(a) The solutions $\left(X_1, Y_1\right)$ *of (1.4) over* H *are independent, that is, for any* $X_1 \in J_1$ *and* $Y_1 \in K_1$ *the solutions* X_1 *and* Y_1 *satisfy (1.4) over* H *if and only if*

$$r\left(A, C\right) = r\left(A\right) + r\left(C\right) \quad \text{or } r\begin{pmatrix} B \\ D \end{pmatrix} = r\left(B\right) + r\left(D\right). \tag{2.17}$$

(b) Under (2.17), the general solution of (1.4) over H *can be written as the two independent forms*

$$X = X_0 + \widetilde{S_1} L_G U_1 R_H \widetilde{T_1} + L_A V_1 + V_2 R_B, \tag{2.18}$$

$$Y = Y_0 + \widetilde{S_2} L_G U_2 R_H \widetilde{T_2} + L_C W_1 + W_2 R_D, \tag{2.19}$$

where $\left(X_0, Y_0\right)$ *is a particular real solution of (1.4),* U_1, U_2, V_1, V_2, W_1 *and* W_2 *are arbitrary.*

Proof. Writing (2.3) and (2.4) as two independent matrix expressions, that is, replacing U in (2.3) and (2.4) by U_1 and U_2 respectively, then taking them into $E - AX_1B - CY_1D$ yields

$$\begin{aligned} q\left(X_1, Y_1\right) &= E - AX_0B - CY_0D - A\widetilde{S_1} L_G U_1 R_H \widetilde{T_1} B - C\widetilde{S_2} L_G U_2 R_H \widetilde{T_2} D \\ &= A\widetilde{S_1} L_G \left(-U_1 + U_2\right) R_H \widetilde{T_1} B, \end{aligned}$$

where U_1 and U_2 are arbitrary. Then by (2.1), it follows that

$$\begin{aligned} \max_{\substack{X_1 \in J_1, \\ Y_1 \in K_1}} r\left[q\left(X_1, Y_1\right)\right] &= \max_{U_1, U_2} r\left(A\widetilde{S_1} L_G \left(-U_1 + U_2\right) R_H \widetilde{T_1} B\right) \\ &= \min\left\{r\left(A\widetilde{S_1} L_G\right), r\left(R_H \widetilde{T_1} B\right)\right\} \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

where

$$r\left(A\widetilde{S_1}L_G\right) = r\begin{pmatrix} A\widetilde{S_1} \\ G \end{pmatrix} - r(G) = r(A) + r(C) - r(G),$$
$$r\left(R_H\widetilde{T_1}B\right) = r\left(\widetilde{T_1}B, H\right) - r(H) = r(B) + r(D) - r(H).$$

Therefore, we have (2.16). The result in (2.17) follows directly from (2.16) and the solutions in (2.18) and (2.19) follow from (2.3) and (2.4). □

Corollary 2.8. *Suppose that (1.4) over H is consistent, and (2.17) holds. Then* (*a*) *All solution pairs X and Y to (1.4) over H are real if and only if*

$$p+q+r\begin{pmatrix} 0 & 0 & \widehat{B}_i \\ \widehat{A}_i & \phi(C) & \phi(E) \end{pmatrix} = 4r(B) + 4r(A,C)$$

or

$$p+q+r\begin{pmatrix} 0 & \widehat{B}_i \\ 0 & \phi(D) \\ \widehat{A}_i & \phi(E) \end{pmatrix} = 4r(A) + 4r\begin{pmatrix} B \\ D \end{pmatrix}$$

and

$$s+t+r\begin{pmatrix} 0 & 0 & \widehat{D}_j \\ \widehat{C}_j & \phi(A) & \phi(E) \end{pmatrix} = 4r(B) + 4r(A,C)$$

or

$$s+t+r\begin{pmatrix} 0 & \widehat{D}_j \\ 0 & \phi(B) \\ \widehat{C}_j & \phi(E) \end{pmatrix} = 4r(C) + 4r\begin{pmatrix} B \\ D \end{pmatrix}, i,j = 2,3,4.$$

(*b*) *All solution pairs X and Y to (1.4) over H are imaginary if and only if*

$$p+q+r\begin{bmatrix} 0 & 0 & \widehat{B}_1 \\ \widehat{A}_1 & \phi(C) & \phi(E) \end{bmatrix} = 4r(B) + 4r(A,C)$$

or

$$p+q+r\begin{pmatrix} 0 & \widehat{B}_1 \\ 0 & \phi(D) \\ \widehat{A}_1 & \phi(E) \end{pmatrix} = 4r(A) + 4r\begin{pmatrix} B \\ D \end{pmatrix}$$

and

$$s+t+r\begin{pmatrix} 0 & 0 & \widehat{D}_1 \\ \widehat{C}_1 & \phi(A) & \phi(E) \end{pmatrix} = 4r(B) + 4r(C,A)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

or

$$s + t + r \begin{pmatrix} 0 & \widehat{D}_1 \\ 0 & \phi(B) \\ \widehat{C}_1 & \phi(E) \end{pmatrix} = 4r(C) + 4r \begin{pmatrix} B \\ D \end{pmatrix}.$$

REMARK 2.9. The main results of [11], [17] and [19] can be regarded as special cases of results in this paper.

Acknowledgment. The authors would like to thank Professor Michael Neumann and a referee very much for their valuable suggestions and comments, which resulted in great improvement of the original manuscript.

REFERENCES

[1] S.L. Adler. *Quaternionic Quantum Mechanics and Quantum Fields*. Oxford University Press, Oxford, 1995.
[2] J.K. Baksalary and P. Kala. The matrix equation $AXB + CYD = E$. *Linear Algebra Appl.*, 30:141–147, 1980.
[3] D.R. Farenick and B.A.F. Pidkowich. The spectral theorem in quaternions. *Linear Algebra Appl.*, 371:75–102, 2003.
[4] R.M. Guralnick. Roth's theorems for sets of matrices. *Linear Algebra Appl.*, 71:113–117, 1985.
[5] W. Gustafson. Roth's theorems over commutative rings. *Linear Algebra Appl.*, 23:245–251, 1979.
[6] T. W. Hungerford. *Algebra*. Spring-Verlag, New York, 1980.
[7] R. Hartwig. Roth's equivalence problem in unite regular rings. *Proc. Amer. Math. Soc.*, 59:39–44, 1976.
[8] R. Hartwig. Roth's removal rule revisited. *Linear Algebra Appl.*, 49:91–115, 1984.
[9] L. Huang and Q. Zeng. The solvability of matrix equation $AXB + CYD = E$ over a simple Artinian ring. *Linear Multilinear Algebra*, 38:225–232, 1995.
[10] A.P. Liao, Z.Z Bai, and Y. Lei. Best approximate solution of matrix equation $AXB+CYD = E$. *SIAM J. Matrix Anal. Appl.*, 27(3):675–688, 2006.
[11] H. Liu. Ranks of solutions of the linear matrix equation $AX + YB = C$. *Comput. Math. Appl.*, 52:861–872, 2006.
[12] A.B. Özgüle. The matrix equation $AXB + CYD = E$ over a principal ideal domain. *SIAM J. Matrix Anal. Appl.*, 12:581–591, 1991.
[13] W.E. Roth. The equation $AX - YB = C$ and $AX - XB = C$ in matrices. *Proc. Amer. Math. Soc.*, 3:392–396, 1952.
[14] Y. Tian. The solvability of two linear matrix equations. *Linear Multilinear Algebra*, 48:123–147, 2000.
[15] Y. Tian. The minimal rank of the matrix expression $A - BX - YC$. *Missouri J. Math. Sci.*, 14(1):40–48, 2002.
[16] Y. Tian. Upper and lower bounds for ranks of matrix expressions using generalized inverses. *Linear Algebra Appl.*, 355:187–214, 2002.
[17] Y. Tian. Ranks of solutions of the matrix equation $AXB = C$. *Linear Multilinear Algebra*, 51(2):111–125, 2003.
[18] Y. Tian. Ranks and independence of solutions of the matrix equation $AXB+CYD = M$. *Acta Math. Univ. Comenianae*. 1:75–84, 2006.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 343-358, August 2008

ELA

[19] Y. Tian. Ranks and independence of solutions of the matrix equations of the matrix equation $AXB + CYD = M$. *Acta Math. Univ. Comenianae*, 1:1–10, 2006.
[20] Q.W. Wang. The decomposition of pairwise matrices and matrix equations over an arbitrary skew field. *Acta Math. Sinica*, Ser. A, 39 (3):396–403, 1996. (See also MR 97i:15018.)
[21] Q.W. Wang. A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity. *Linear Algebra Appl.*, 384:43–54, 2004.
[22] Q.W. Wang, J.H. Sun, and S.Z. Li. Consistency for bi(skew)symmetric solutions to systems of generalized Sylvester equations over a finite central algebra. *Linear Algebra Appl.*, 353:169–182, 2002.
[23] Q.W. Wang, Y.G. Tian, and S.Z. Li. Roth's theorems for centroselfconjugate solutions to systems of matrix equations over a finite dimensional central algebra. *Southeast Asian Bulletin of Mathematics*, 27:929–938, 2004.
[24] Q.W. Wang and S.Z. Li. The persymmetric and perskewsymmetric solutions to sets of matrix equations over a finite central algebra. *Acta Math Sinica*, 47(1):27–34, 2004.
[25] Q.W. Wang. The general solution to a system of real quaternion matrix equations. *Comput. Math. Appl.*, 49:665–675, 2005.
[26] Q.W. Wang, Z.C. Wu, and C.Y. Lin. Extremal ranks of a quaternion matrix expression subject to consistent systems of quaternion matrix equations with applications. *Appl. Math. Comput.*, 182:1755–1764, 2006.
[27] Q.W. Wang, G.J. Song, and C.Y. Lin. Extreme ranks of the solution to a consistent system of linear quaternion matrix equations with an application. *Appl. Math. Comput.*, 189:1517–1532, 2007.
[28] Q.W. Wang, S.W. Yu, and C.Y. Lin. Extreme ranks of a linear quaternion matrix expression subject to triple quaternion matrix equations with applications. *Appl. Math. Comput.*, 195:733–744, 2008.
[29] Q.W. Wang, H.X. Chang, and Q. Ning. The common solution to six quaternion matrix equations with applications. *Appl. Math. Comput.*, 198:209–226, 2008.
[30] Q.W. Wang, H.X. Chang, and C.Y. Lin. P-(skew)symmetric common solutions to a pair of quaternion matrix equations. *Appl. Math. Comput.*, 195:721–732, 2008.
[31] Q.W. Wang and F. Zhang The reflexive re-nonnegative definite solution to a quaternion matrix equation. *Electron. J. Linear Algebra*, 17:88–101, 2008.
[32] Q.W. Wang and C.K. Li. Ranks and the least-norm of the general solution to a system of quaternion matrix equations. *Linear Algebra Appl.*, to appear. (doi:10.1016/j.laa.2008.05.031, 2008.)
[33] H.K. Wimmer. Roth's theorems for matrix equations with symmetry constraints. *Linear Algebra Appl.*, 199:357–362, 1994.
[34] H.K. Wimmer. Consistency of a pair of generalized Sylvester equations. *IEEE Trans. Automat. Control*, 39:1014–1015, 1994.
[35] G. Xu, M. Wei and D. Zheng. On solution of matrix equation $AXB + CYD = F$. *Linear Algebra Appl.*, 279:93–109, 1998.
[36] F. Zhang. Quaternions and matrices of quaternions. *Linear Algebra Appl.*, 251:21–57, 1997.
[37] F. Zhang. Geršgorin type theorems for quaternionic matrices. *Linear Algebra Appl.*, 424:139–153, 2007.