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# SOME ADDITIONAL NOTES ON THE SPECTRA OF NON-NEGATIVE SYMMETRIC $5 \times 5$ MATRICES*

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**Abstract.** The Symmetric Non-negative Inverse Eigenvalue Problem (SNIEP) asks when is a list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of real, monotonically decreasing numbers, the spectrum of an $n \times n$, symmetric, non-negative matrix $A$. In that case, we say $\sigma$ is realizable and $A$ is a realizing matrix. Here, we consider the case $n = 5$, the lowest value of $n$ for which the problem is unsolved. Let $s_1(\sigma) = \sum_{i=1}^{5} \lambda_i$ and $s_3(\sigma) = \sum_{i=1}^{5} {\lambda_i}^3$. It is known that to complete the solution for $n = 5$, it remains to consider the case $\lambda_3 > s_1(\sigma)$, so let $y = \lambda_3 - s_1(\sigma)$ and assume $y \geq 0$. We prove that if $\sigma$ is realizable, then $s_3(\sigma) \geq s_1(\sigma)^3 + 6s_1(\sigma)y(s_1(\sigma) + y)$. This strengthens the inequality $s_3(\sigma) \geq s_1(\sigma)^3$ obtained by Loewy and Spector, which in turn strengthens the inequality $25s_3(\sigma) \geq s_1(\sigma)^3$, one of the Johnson–Loewy–London inequalities. As an application of the new inequality, we show that certain lists previously unknown as far as their realizability is concerned are not realizable.

**Key word.** Symmetric Non-negative Inverse Eigenvalue Problem.

**AMS subject classifications.** 15A29, 15B48.

**1. Introduction.** Given a list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers, the problem of determining whether $\sigma$ is the spectrum of a non-negative $n \times n$ matrix is called the Non-negative Inverse Eigenvalue Problem. If $\sigma$ consists of real numbers, this problem is called the Real Non-negative Inverse Eigenvalue Problem, and if we also require the matrix to be symmetric, this problem is called the Symmetric Non-negative Inverse Eigenvalue Problem (SNIEP). For each of these problems, if $\sigma$ is the spectrum of matrix $A$ in the corresponding class, we say that $\sigma$ is realizable and $A$ is a realizing matrix. Currently, all three problems are open and seem to be very difficult, for any $n \geq 5$. The survey paper by Johnson, Marijuán, Paparella and Pisonero [3] contains an extensive list of references for these problems.

One well-known necessary condition for the realizability of $\sigma$ is the so-called Perron–Frobenius condition [1, 11], namely, $\max_{1 \leq i \leq n} |\lambda_i| \in \sigma$. To state additional necessary conditions, we define the moments of $\sigma$. Given any positive integer $k$, we define its $k$th moment by

$$s_k(\sigma) = \sum_{i=1}^{n} {\lambda_i}^k.$$

Then, for $\sigma$ to be realizable, clearly all of its moments must be non-negative. There are stronger necessary conditions involving these moments, the Johnson–Loewy–London (JLL) conditions [2, 6]:

$$n^{i-1} s_{ij}(\sigma) \geq s_j(\sigma)^i \text{ for any positive integers } i \text{ and } j.$$

In this paper, we focus our attention on SNIEP for $n = 5$, so from now on we will assume that $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ consists of real numbers. Moreover, without loss of generality, we will assume throughout

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that the numbers in $\sigma$ are arranged in monotonically decreasing order. Also, an obvious necessary condition for the realizability of $\sigma$ is that $\lambda_1 \geq -\lambda_5$, so we shall assume throughout that this condition holds.

As indicated, the SNIEP for $n = 5$ is not fully solved, but significant progress has been made in recent years. One important parameter in the investigation is $s_1(\sigma)$. When $s_1(\sigma) = 0$, the problem has been solved by Spector [12], and when $s_1(\sigma) \geq \frac{1}{2}\lambda_1$ by Loewy and Spector [8].

The solution of the problem is also known whenever $\lambda_3 \leq s_1(\sigma)$. Our goal here is to consider the case that this inequality does not hold. Loewy and Spector [9] proved that if $\lambda_3 \geq s_1(\sigma)$, then $s_3(\sigma) \geq s_1(\sigma)^3$, thus improving, for $n = 5$, the JLL inequality involving $s_3(\sigma)$ and $s_1(\sigma)$. Here, we further improve the inequality, in terms of the non-negative parameter $y = \lambda_3 - s_1(\sigma)$.

In Section 2, we will prove our main result, which states that, when $y \geq 0$, we have $s_3(\sigma) \geq s_1(\sigma)^3 + 6(s_1(\sigma)^2 y + s_1(\sigma) y^2)$. In Section 3, we apply this result to show, in 2 examples, that certain lists, previously unknown as far as realizability is concerned, are in fact not realizable. In the first example, we consider the simplex $\mathbb{U}$ defined as the convex hull of the points

$$\mathbf{c} = (1,1,1,-1,-1)\,, \quad \mathbf{d} = (1,1,0,-1,-1)\,, \quad \mathbf{e} = (1,0,0,0,-1)\,,$$
$$\mathbf{i} = \left(1,\frac{1}{2},\frac{1}{2},-1,-1\right), \quad \mathbf{l} = \left(1,0,0,-\frac{1}{2},-\frac{1}{2}\right).$$

The unknown region for the SNIEP when $n = 5$ is contained in $\mathbb{U}$, which has been discussed by McDonald and Neumann [10], Loewy and McDonald [7], and Loewy and Spector [9]. The point $\mathbf{i}$ is not realizable. Every point $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ in $\mathbb{U}$ satisfies $y = \lambda_3 - s_1(\sigma) \geq 0$, with strict inequality holding when the coefficient of $\mathbf{i}$ is positive. So, we can apply our main result and obtain a sufficient condition for the nonrealizabilty of a point in $\mathbb{U}$.

In the second example, we consider lists $\sigma$ of the form $(1, a, a, b, b)$. They have attracted a lot of attention, in particular the papers by Johnson, Marijuán and Pisonero [4], Knudsen and McDonald [5], Loewy and McDonald [7], Loewy and Spector [9], and McDonald and Neumann [10]. Here, as well, we obtain a sufficient condition for nonrealizability, narrowing the unknown region.

We will use the following notation. $\mathcal{S}_5$ ($\mathcal{S}_5^+$, respectively) denotes the set of all $5 \times 5$ real, symmetric (and non-negative, respectively) matrices. Given $A \in \mathcal{S}_5$ and $\alpha \subset \{1,2,3,4,5\}$ denote by $A[\alpha]$ the principal submatrix of $A$ based on row and column indices in $\alpha$. The spectral radius of $A$ is denoted by $\rho(A)$ and its trace by $tr(A)$. In case $A$ is also non-negative, with spectrum $\sigma$ as defined earlier, then clearly $\rho(A) = \lambda_1$.

**2. Main theorem.** In this section, we prove our main result.

THEOREM 2.1. *Let $M \in \mathcal{S}_5^+$ be a matrix with spectrum $\sigma(M) := \sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, where the elements of $\sigma$ are arranged in monotonically decreasing order. Let $y = \lambda_3 - s_1(\sigma)$, and suppose that $y$ is non-negative. Then, $s_3(\sigma) \geq s_1(\sigma)^3 + 6(s_1(\sigma)^2 y + s_1(\sigma) y^2)$.*

REMARK. Substituting for $y$, the inequality obtained in Theorem 2.1 can be written as follows:

$$s_3(\sigma) \geq s_1(\sigma)^3 + 6 s_1(\sigma)(\lambda_3 - s_1(\sigma))\lambda_3.$$

Note that in case $y = 0$, the theorem reduces to Theorem 1 of [9], so from now on we can assume that $y > 0$. Before proving Theorem 2.1, we prove several lemmas. In order to simplify some notation, and since the

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matrix $M$ is fixed throughout this section, we write $s_1, s_3, \rho$ for $s_1(\sigma), s_3(\sigma), \rho(M)$, respectively. Define the following set:

$$\mathcal{M}(M) = \left\{G \in \mathcal{S}_5^+ : \textit{the spectrum of } G \textit{ is } \sigma\right\}.$$

LEMMA 2.1. *Let $G \in \mathcal{M}(M)$ and let*

$$P = \begin{bmatrix} p_{11} & p_{12} & 0 \\ p_{12} & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}$$

*be permutationally similar to $G[\alpha]$ for some $\alpha \subset \{1,2,3,4,5\}$. Then,*

$$p_{12}^2 \geq p_{11}p_{22} + (s_1+y)^2 - (s_1+y)(p_{11}+p_{22}).$$

*Proof.* The eigenvalues of $P$ are $\frac{1}{2}(p_{11}+p_{22}+\sqrt{(p_{11}-p_{22})^2+4p_{12}^2})$, $\frac{1}{2}(p_{11}+p_{22}-\sqrt{(p_{11}-p_{22})^2+4p_{12}^2})$, and $p_{33}$. By Cauchy's interlacing inequalities, we have $\rho(P) \geq \lambda_3 = s_1 + y$. We note that $p_{33} \leq tr(P) \leq tr(G) = s_1 < \lambda_3$, implying that $\rho(P) = \frac{1}{2}(p_{11}+p_{22}+\sqrt{(p_{11}-p_{22})^2+4p_{12}^2})$. Therefore,

$$p_{11}+p_{22}+\sqrt{(p_{11}-p_{22})^2+4p_{12}^2} \geq 2(s_1+y),$$

so

$$\sqrt{(p_{11}-p_{22})^2+4p_{12}^2} \geq 2(s_1+y) - p_{11} - p_{22} \geq 0,$$

and the result follows. ∎

LEMMA 2.2. *Let $G \in \mathcal{M}(M)$ and let*

$$P = \begin{bmatrix} p_{11} & p_{12} & 0 \\ p_{12} & 0 & p_{23} \\ 0 & p_{23} & 0 \end{bmatrix}$$

*be permutationally similar to $G[\alpha]$ for some $\alpha \subset \{1,2,3,4,5\}$. Then,*

$$p_{12}^2 \geq \frac{(s_1+y-p_{11})((s_1+y)^2-p_{23}^2)}{s_1+y}.$$

*Proof.* Let $f(x)$ denote the characteristic polynomial of $P$. Then,

$$f(x) = x^3 - p_{11}x^2 - (p_{12}^2+p_{23}^2)x + p_{11}p_{23}^2,$$

so $f(x) = (x-p_{11})(x^2-p_{23}^2) - p_{12}^2 x$.

As in Lemma 2.1, we have $\rho(P) \geq \lambda_3 = s_1 + y$. Let $\eta_2 \geq \eta_3$ be the second and third eigenvalues of $P$. By Perron–Frobenius, we have $\rho(P) + \eta_3 \geq 0$, implying that $\eta_2 \leq p_{11} < s_1 + y$. Hence, $f(s_1+y) \leq 0$, implying the result. ∎

We now turn to the proof of Theorem 2.1.

*Proof.* We show first that the proof can be reduced to the case that $M$ takes one of two given patterns, to be defined.

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Recall that we have $y > 0$, so $\lambda_3 > s_1$. If $s_1 = 0$, the theorem states that $s_3 \geq 0$, which certainly holds. Hence, we may assume that $s_1 > 0$. The matrix $M$ must be irreducible, for if this is not the case, it has been shown in [8] that $\lambda_3 \leq s_1$, a contradiction. We must have $\lambda_2 + \lambda_4 < 0$. Otherwise,

$$s_1 = (\lambda_1 + \lambda_5) + (\lambda_2 + \lambda_4) + \lambda_3 \geq (\lambda_1 + \lambda_5) + \lambda_3 > \lambda_3,$$

a contradiction. In particular, we conclude that $\lambda_4, \lambda_5$ are negative and $|\lambda_4| > |\lambda_2|$.

Define the following set:

$$\mathcal{W}(M) = \left\{ W \in \mathcal{S}_5^+ : \lambda_1(W) = \rho, \lambda_3(W) = \lambda_3, \text{ and } s_1(W) = s_1 \right\}.$$

Note that for any $W \in \mathcal{W}(M)$, we have $y(W) := \lambda_3(W) - s_1(W) = y$. Also, $\mathcal{W}(M)$ is a compact set. Define on it the function

$$h(W) = s_3(W) - s_1(W)^3 - 6(s_1(W)^2 y(W) + s_1(W) y(W)^2).$$

Note that only the first summand of $h(W)$ actually depends on $W$, while the other two summands are constant throughout $\mathcal{W}(M)$. This function attains its minimum on $\mathcal{W}(M)$, say at a matrix $W_1$. We claim that $W_1$ is not orthogonally similar to a positive matrix. Indeed, suppose that this is not the case, and let the spectrum of $W_1$ be given by $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$, where the elements in $\eta$ are assumed to be arranged in decreasing order. Then, there exists $\epsilon > 0$ sufficiently small such that $(\eta_1, \eta_2 + \epsilon, \eta_3, \eta_4 - \epsilon, \eta_5)$ is the spectrum of a matrix $W_{1,\epsilon} \in \mathcal{W}(M)$. However,

$$s_3(W_{1,\epsilon}) = s_3(W_1) + 3\epsilon(\eta_2^2 - \eta_4^2) + O(\epsilon^2),$$

implying that $h(W_{1,\epsilon}) < h(W_1)$, a contradiction.

Hence, we may also assume that $M$ is not orthogonally similar to a positive matrix. A discussion identical to the one in [8] (and also in [9]) leads us to conclude that it suffices to prove the theorem for matrices of the following two patterns:

$$H = \begin{pmatrix} + & + & + & 0 & 0 \\ + & 0 & 0 & + & + \\ + & 0 & + & 0 & + \\ 0 & + & 0 & * & + \\ 0 & + & + & + & 0 \end{pmatrix} \quad C = \begin{pmatrix} + & + & + & 0 & 0 \\ + & * & 0 & 0 & + \\ + & 0 & * & + & 0 \\ 0 & 0 & + & * & + \\ 0 & + & 0 & + & * \end{pmatrix},$$

where $+$ indicates a positive element and $*$ indicates a zero or a positive element. We will write $B$ and $A$ for $M$ when we consider pattern C, respectively H. We keep other notations.

*Pattern C.* Let

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 & b_{25} \\ b_{13} & 0 & b_{33} & b_{34} & 0 \\ 0 & 0 & b_{34} & b_{44} & b_{45} \\ 0 & b_{25} & 0 & b_{45} & b_{55} \end{pmatrix},$$

where all the $b_{ij}$'s are positive, except for $b_{22}, b_{33}, b_{44}, b_{55}$, which can be zero.

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Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be the spectrum of $B$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$. We may apply Lemma 2.1 to the principal submatrices $B[145]$, $B[234]$, $B[235]$, $B[124]$, and $B[135]$, leading to the following lower bounds for all off-diagonal entries of B:

$$b_{45}^2 \geq b_{44}b_{55} + (b_{44}+b_{55})(b_{11}+b_{22}+b_{33}+y) + (b_{11}+b_{22}+b_{33}+y)^2, \tag{2.1}$$

$$b_{34}^2 \geq b_{33}b_{44} + (b_{33}+b_{44})(b_{11}+b_{22}+b_{55}+y) + (b_{11}+b_{22}+b_{55}+y)^2, \tag{2.2}$$

$$b_{25}^2 \geq b_{22}b_{55} + (b_{22}+b_{55})(b_{11}+b_{33}+b_{44}+y) + (b_{11}+b_{33}+b_{44}+y)^2, \tag{2.3}$$

$$b_{12}^2 \geq b_{11}b_{22} + (b_{11}+b_{22})(b_{33}+b_{44}+b_{55}+y) + (b_{33}+b_{44}+b_{55}+y)^2, \tag{2.4}$$

$$b_{13}^2 \geq b_{11}b_{33} + (b_{11}+b_{33})(b_{22}+b_{44}+b_{55}+y) + (b_{22}+b_{44}+b_{55}+y)^2. \tag{2.5}$$

Let $q(\sigma) := \frac{1}{3}(s_3 - s_1^3) = \frac{1}{3}(\mathrm{tr}(B^3) - \mathrm{tr}(B)^3)$. Our goal is to get a lower bound for this function. It is straightforward to show that

$$\begin{aligned} q(\sigma) =& b_{11}(b_{12}^2+b_{13}^2) + b_{22}(b_{12}^2+b_{25}^2) + b_{33}(b_{13}^2+b_{34}^2) + b_{44}(b_{34}^2+b_{45}^2) \\ &+ b_{55}(b_{25}^2+b_{45}^2) - \sum_{i=1}^{5} b_{ii}^2(s_1 - b_{ii}) - 2b_{11}(b_{22}b_{33} + b_{22}b_{44} + b_{22}b_{55} \\ &+ b_{33}b_{44} + b_{33}b_{55} + b_{44}b_{55}) - 2b_{22}(b_{33}b_{44} + b_{33}b_{55} + b_{44}b_{55}) - 2b_{33}b_{44}b_{55}. \end{aligned} \tag{2.6}$$

Substituting the lower bounds for the off-diagonal entries of $B$ given by (2.1), (2.2), (2.3), (2.4), (2.5) into (2.6), we get the following lower bound for $s_3 - s_1^3 = 3q(\sigma)$:

$$s_3 - s_1^3 \geq ty + uy^2,$$

where $t, u$ do not depend on $y$, but only on the entries of $B$, and are given by $u = 6s_1$ and

$$\begin{aligned} t =& 6\sum_{i=1}^{5} b_{ii}^2 + b_{11}(18b_{22} + 18b_{33} + 24b_{44} + 24b_{55}) \\ &+ b_{22}(24b_{33} + 24b_{44} + 18b_{55}) + b_{33}(18b_{44} + 24b_{55}) + 18b_{44}b_{55}. \end{aligned}$$

It remains to show that $t \geq 6s_1^2$. Indeed, we have $t - 6s_1^2 = b_{11}(6b_{22} + 6b_{33} + 12b_{44} + 12b_{55}) + b_{22}(12b_{33} + 12b_{44} + 6b_{55}) + b_{33}(6b_{44} + 12b_{55}) + 6b_{44}b_{55} \geq 0$, completing the proof for pattern C.

*Pattern H.* Let $q(\sigma)$ be as defined in pattern C and let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{12} & 0 & 0 & a_{24} & a_{25} \\ a_{13} & 0 & a_{33} & 0 & a_{35} \\ 0 & a_{24} & 0 & a_{44} & a_{45} \\ 0 & a_{25} & a_{35} & a_{45} & 0 \end{pmatrix},$$

where all the $a_{ij}$'s are positive, except for $a_{44}$, which can be zero. Let $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_5)$ be the spectrum of $A$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$. Note that due to the symmetry of pattern H, we may assume without loss of generality that

$$a_{11} \geq a_{33}.$$

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We may apply Lemma 2.1 to the principal submatrices $A[134]$, $A[145]$, and $A[234]$, leading to the following lower bounds:

$$a_{13}^2 \geq a_{11}a_{33} + (a_{11}+a_{33})(a_{44}+y) + (a_{44}+y)^2 = a_{11}a_{33} + (a_{44}+y)(s_1+y), \tag{2.7}$$

$$a_{45}^2 \geq a_{44}(a_{11}+a_{33}+y) + (a_{11}+a_{33}+y)^2 = (a_{11}+a_{33}+y)(s_1+y), \tag{2.8}$$

$$a_{24}^2 \geq a_{44}(a_{11}+a_{33}+y) + (a_{11}+a_{33}+y)^2 = (a_{11}+a_{33}+y)(s_1+y). \tag{2.9}$$

Note that the lower bound of $a_{45}^2$ is identical to that of $a_{24}^2$, so it is also a lower bound of $a_{24}a_{45}$. We may apply Lemma 2.2 to the principal submatrices $A[125]$ and $A[235]$, leading to the following lower bounds:

$$a_{12}^2 \geq \frac{(s_1+y-a_{11})((s_1+y)^2-a_{25}^2)}{s_1+y}, \tag{2.10}$$

$$a_{35}^2 \geq \frac{(s_1+y-a_{33})((s_1+y)^2-a_{25}^2)}{s_1+y}. \tag{2.11}$$

Our goal is to get a lower bound for $q(\sigma)$. It is straightforward to show that

$$\begin{aligned} q(\sigma) = {} & a_{11}^2(-a_{33}-a_{44}) + a_{11}(a_{12}^2+a_{13}^2-a_{33}^2-2a_{33}a_{44}-a_{44}^2) \\ & - a_{33}^2a_{44} + a_{33}(a_{13}^2+a_{35}^2-a_{44}^2) + a_{44}(a_{24}^2+a_{45}^2) + 2a_{24}a_{45}a_{25}. \end{aligned}$$

Substituting the lower bound for $a_{13}^2$ from (2.7) into $q(\sigma)$, and also using $a_{11}+a_{33}+2a_{44} = s_1+a_{44}$, we obtain the following lower bound:

$$q(\sigma) \geq (a_{11}+a_{33})y^2 + (a_{11}+a_{33})(s_1+a_{44})y + a_{11}a_{12}^2 + a_{33}a_{35}^2 + a_{44}(a_{24}^2+a_{45}^2) + 2a_{24}a_{45}a_{25}. \tag{2.12}$$

Substituting into the summand $a_{44}a_{45}^2$ the lower bound for $a_{45}^2$, given by (2.8), we obtain from (2.12) after some rearrangement,

$$q(\sigma) \geq s_1^2y + s_1y^2 + a_{11}a_{12}^2 + a_{33}a_{35}^2 + a_{44}a_{24}^2 + 2a_{24}a_{45}a_{25} + 2(a_{11}+a_{33})a_{44}y + (a_{11}+a_{33})a_{44}s_1. \tag{2.13}$$

The proof now splits into two cases.

*Case* 1. We assume that at least one of $a_{12}, a_{35}$ is bounded above by $y$. Suppose first that $a_{12} \leq y$. Then it follows from (2.10) that

$$(s_1+y)y^2 \geq (s_1+y)^3 - a_{11}(s_1+y)^2 - a_{25}^2(s_1+y) + a_{11}a_{25}^2,$$

which, upon expanding $(s_1+y)^3$ as $(s_1+y)(y^2+2s_1y+s_1^2)$ and some simple manipulations, yields

$$(a_{33}+a_{44}+y)a_{25}^2 \geq 2(s_1+y)s_1y + (s_1+y)s_1^2 - a_{11}(s_1+y)^2 = (s_1+y)s_1y + (a_{33}+a_{44})(s_1+y)^2,$$

implying

$$(s_1+y)a_{25}^2 \geq (s_1+y)s_1y + (a_{33}+a_{44})(s_1+y)^2. \tag{2.14}$$

Now suppose that $a_{35} \leq y$. A similar computation, using (2.11), yields

$$(s_1+y)a_{25}^2 \geq (s_1+y)s_1y + (a_{11}+a_{44})(s_1+y)^2. \tag{2.15}$$

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Summarizing, since by the assumption of this case at least one of (2.14), (2.15) must hold, we have

$$a_{25}^2 \geq s_1 y + (s_1 + y)(\omega + a_{44}),$$

where $\omega \in \{a_{11}, a_{33}\}$.

In light of (2.13), it suffices to show that $2a_{24}a_{45}a_{25} \geq s_1^2 y + s_1 y^2$. Using the lower bounds for $a_{24}a_{45}$ (which, as indicated, is identical to the right-hand side of (2.8) or (2.9)) and $a_{25}$, it suffices to show that

$$2(a_{11} + a_{33} + y)(s_1 + y)\sqrt{s_1 y + (s_1 + y)(\omega + a_{44})} \geq s_1^2 y + s_1 y^2 = s_1 y(s_1 + y),$$

which is equivalent to showing that

$$2(a_{11} + a_{33} + y)\sqrt{s_1 y + (s_1 + y)(\omega + a_{44})} \geq s_1 y,$$

and the latter is equivalent to

$$4(a_{11} + a_{33} + y)^2(s_1 y + (s_1 + y)(\omega + a_{44})) - s_1^2 y^2 \geq 0. \tag{2.16}$$

If $a_{11} + a_{33} \geq \frac{1}{8}s_1$, then $4(a_{11} + a_{33} + y)^2 \geq 8(a_{11} + a_{33})y \geq s_1 y$, so (2.16) holds. Otherwise, we have $a_{44} \geq \frac{7}{8}s_1$, so the left-hand side of (2.16) is bounded below by $4y^2(s_1 y + \frac{7}{8}s_1^2) - s_1^2 y^2$, and since this is clearly positive, (2.16) holds. This concludes the proof in Case 1.

*Case* 2. We may assume now that $a_{12} \geq y$ and $a_{35} \geq y$. Define now

$$q_1(\sigma) = q(\sigma) - (s_1^2 y + s_1 y^2) \ \ \textit{and} \ \ q_2(\sigma) = q(\sigma) - (s_1^2 y + 2s_1 y^2).$$

We substitute the lower bound $y$ for $a_{12}$ and $a_{35}$ into (2.13), and in addition put the lower bound for $a_{24}^2$ from (2.9) into the summand $a_{44}a_{24}^2$ of that equation, and get, after some simple manipulation,

$$\begin{aligned} q_1(\sigma) \geq\, & (a_{11} + a_{33})y^2 + a_{44}(a_{44}(a_{11} + a_{33} + y) + (a_{11} + a_{33} + y)^2) \\ & + 2a_{24}a_{45}a_{25} + 2(a_{11} + a_{33})a_{44}y + (a_{11} + a_{33})a_{44}^2 + (a_{11} + a_{33})^2 a_{44} = s_1 y^2 \\ & + 2(a_{11} + a_{33})a_{44}^2 + 2(a_{11} + a_{33})^2 a_{44} + a_{44}^2 y + 4(a_{11} + a_{33})a_{44}y + 2a_{24}a_{45}a_{25}, \end{aligned}$$

implying that

$$q_2(\sigma) \geq 2(a_{11} + a_{33})a_{44}^2 + 2(a_{11} + a_{33})^2 a_{44} + a_{44}^2 y + 4(a_{11} + a_{33})a_{44}y + 2a_{24}a_{45}a_{25}. \tag{2.17}$$

Suppose that $a_{44} \geq y$. Then, partially replacing $a_{44}$ by $y$ in (2.17), we get

$$\begin{aligned} q_2(\sigma) \geq\, & 2(a_{11} + a_{33})a_{44}^2 + 2(a_{11} + a_{33})^2 a_{44} + a_{44}^2 y \geq 2(a_{11} + a_{33})a_{44}y \\ & + (a_{11} + a_{33})^2 a_{44} + (a_{11} + a_{33})^2 y + a_{44}^2 y = ((a_{11} + a_{33}) + a_{44})^2 y \\ & + (a_{11} + a_{33})^2 a_{44} \geq s_1^2 y, \end{aligned}$$

so, by the definition of $q_2(\sigma)$, the result follows. Hence, we may assume throughout that $y \geq a_{44}$.

Suppose that $a_{25} \geq \frac{1}{2}y$. Putting this lower bound, as well as the lower bound for $a_{24}a_{45}$, into the summand $2a_{24}a_{45}a_{25}$ of (2.17), the result follows, note first that

$$a_{24}a_{45} \geq (a_{11} + a_{33} + y)(s_1 + y) \geq (a_{11} + a_{33} + a_{44})(s_1 + y) \geq s_1^2,$$

and also that $2a_{24}a_{45}a_{25} \geq 2s_1^2 \frac{1}{2} y = s_1^2 y$. Hence, we may also assume throughout that $y \geq 2a_{25}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 37, pp. 1-13, January 2021.



Next we claim that if $2a_{44} \geq y$ or $2a_{44} \geq a_{11}+a_{33}$, the result follows. Indeed, we have

$$s_1^2 y = ((a_{11}+a_{33})^2 + 2(a_{11}+a_{33})a_{44} + a_{44}^2)y,$$

so, by (2.17)

$$q_2(\sigma) - s_1^2 y \geq (a_{11}+a_{33})(2a_{44}^2 + 2(a_{11}+a_{33})a_{44} + 2a_{44}y - (a_{11}+a_{33})y)$$

and the claim holds.

Note that the statement $2a_{44} \geq a_{11}+a_{33}$ is equivalent to $a_{11}+a_{33} \leq \frac{2}{3}s_1$. Based on the previous discussions, we may assume throughout that

$$2a_{44}, 2a_{25} \leq y \leq a_{12}, a_{35} \quad \textit{and} \quad \frac{2}{3}s_1 \leq a_{11}+a_{33}.$$

To conclude the proof in Case 2, our starting point now is (2.13), which we can rewrite in terms of $q_1(\sigma)$ as

$$\begin{aligned} q_1(\sigma) \geq F(A) :=\, & a_{11}a_{12}^2 + a_{33}a_{35}^2 + a_{44}a_{24}^2 + 2a_{24}a_{45}a_{25} \\ & + 2(a_{11}+a_{33})a_{44}y + (a_{11}+a_{33})a_{44}s_1 = a_{11}a_{12}^2 + a_{33}a_{35}^2 + a_{44}a_{24}^2 \\ & + 2a_{24}a_{45}a_{25} + (a_{11}+a_{33})a_{44}y + (a_{11}+a_{33})a_{44}(s_1+y). \end{aligned}$$

It suffices to show that

$$F(A) \geq s_1^2 y + s_1 y^2 = s_1 y(s_1+y).$$

We introduce the following notation:

$$a_{11} = \alpha s_1,\ a_{33} = \beta s_1 \text{ and } a_{25} = \gamma y.$$

By our assumptions, we have $\beta \leq \alpha$ (as we assume that $a_{33} \leq a_{11}$), $\gamma \leq \frac{1}{2}$, and $\frac{2}{3} \leq \alpha+\beta \leq 1$. Also, $a_{44} = (1-\alpha-\beta)s_1$. Substituting the lower bounds for $a_{12}^2$ and $a_{35}^2$ given by (2.10) and (2.11), respectively, and the common lower bound for $a_{24}^2$, $a_{45}^2$, and $a_{24}a_{45}$ given by (2.9) (or (2.8)) into $F(A)$, we obtain

$$\begin{aligned} F(A) \geq\, & \frac{(a_{11}(s_1+y) - a_{11}^2 + a_{33}(s_1+y) - a_{33}^2)((s_1+y)^2 - \gamma^2 y^2)}{s_1+y} \\ & + a_{44}(a_{11}+a_{33}+y)(s_1+y) + 2\gamma(a_{11}+a_{33}+y)(s_1+y)y \\ & + (a_{11}+a_{33})a_{44}(s_1+y) + (a_{11}+a_{33})a_{44}y \\ =\, & \frac{((\alpha+\beta)s_1(s_1+y) - (\alpha^2+\beta^2)s_1^2)((s_1+y)^2 - \gamma^2 y^2)}{s_1+y} \\ & + (1-\alpha-\beta)s_1((\alpha+\beta)s_1+y)(s_1+y) + 2\gamma((\alpha+\beta)s_1+y)(s_1+y)y \\ & + (\alpha+\beta)(1-\alpha-\beta)s_1^2(s_1+y) + (\alpha+\beta)(1-\alpha-\beta)s_1^2 y. \end{aligned} \tag{2.18}$$

In light of (2.18), it suffices to show that

$$\begin{aligned} F_1(A) :=\, & ((\alpha+\beta)s_1(s_1+y) - (\alpha^2+\beta^2)s_1^2)((s_1+y)^2 - \gamma^2 y^2) \\ & + (1-\alpha-\beta)s_1((\alpha+\beta)s_1+y)(s_1+y)^2 + 2\gamma((\alpha+\beta)s_1+y)(s_1+y)^2 y \\ & + (\alpha+\beta)(1-\alpha-\beta)s_1^2(s_1+y)^2 + (\alpha+\beta)(1-\alpha-\beta)s_1^2 y(s_1+y) \\ & - s_1 y(s_1+y)^2 \geq 0. \end{aligned}$$

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It it straightforward to check that the summands of $F_1(A)$ can be arranged so that

$$F_1(A) = f(\alpha,\beta)s_1^4 + g(\alpha,\beta,\gamma)s_1^3 y + h(\alpha,\beta,\gamma)s_1^2 y^2 + p(\alpha,\beta,\gamma)s_1 y^3 + 2\gamma y^4,$$

where

- $f(\alpha,\beta) = 3(\alpha+\beta) - 3(\alpha^2+\beta^2) - 4\alpha\beta$,
- $g(\alpha,\beta,\gamma) = 2(\alpha+\beta)\gamma + 7(\alpha+\beta) - 7(\alpha^2+\beta^2) - 10\alpha\beta$,
- $h(\alpha,\beta,\gamma) = (\alpha^2+\beta^2-\alpha-\beta)\gamma^2 + (2+4\alpha+4\beta)\gamma + 4(\alpha+\beta) - 4(\alpha^2+\beta^2) - 6\alpha\beta$,
- $p(\alpha,\beta,\gamma) = -(\alpha+\beta)\gamma^2 + (4+2\alpha+2\beta)\gamma = 4\gamma + (\alpha+\beta)(2\gamma-\gamma^2)$.

It remains to prove the non-negativity of the functions $f$, $g$, $h$, and $p$. The non-negativity of the latter is clear, since we have $0 \le \gamma \le \frac{1}{2}$. In each of the remaining 3 cases to be checked, we have the same constraints, namely

$$\frac{2}{3} \le \alpha+\beta \le 1 \text{ and } 0 \le \beta \le \alpha,$$

so the boundary of the domain of the function is given by the 4 lines

$$\alpha+\beta = \frac{2}{3},\ \alpha+\beta = 1,\ \beta = 0 \text{ and } \beta = \alpha.$$

*Non-negativity of f.* We check first for stationary points in the interior of the domain. The relevant equations are

$$\frac{\partial f(\alpha,\beta)}{\partial\alpha} = 3 - 6\alpha - 4\beta = 0,$$
$$\frac{\partial f(\alpha,\beta)}{\partial\beta} = 3 - 6\beta - 4\alpha = 0,$$

implying (subtract the 2 equations) that $\alpha = \beta$. So, there are no stationary points.

Next, we check the boundary. We have $f(\alpha,0) = 3\alpha(1-\alpha) \ge 0$, $f(\alpha,\alpha) = 2\alpha(3-5\alpha) \ge 0$. The latter inequality holds because, when $\alpha = \beta$, we must have $\alpha \le \frac{1}{2}$. $f(\alpha,1-\alpha) = 2\alpha(1-\alpha) \ge 0$. $f(\alpha,\frac{2}{3}-\alpha) = \frac{2}{3}(1-\alpha)(3\alpha+1) \ge 0$. This establishes the non-negativity of $f$ in the domain.

*Non-negativity of g.* Note that the first summand of $g(\alpha,\beta,\gamma)$, namely $2(\alpha+\beta)\gamma$, is clearly non-negative, so it suffices to prove the non-negativity of $g_1(\alpha,\beta) := 7(\alpha+\beta) - 7(\alpha^2+\beta^2) - 10\alpha\beta$. This is the case because $g_1(\alpha,\beta) = 2f(\alpha,\beta) + (\alpha+\beta) - (\alpha+\beta)^2 \ge 0$.

*Non-negativity of h.* Consider the first two summands of $h$. We have

$$(\alpha^2+\beta^2-\alpha-\beta)\gamma^2 + (2+4\alpha+4\beta)\gamma = \gamma(2 + (\alpha+\beta)(4-\gamma) + (\alpha^2+\beta^2)\gamma) \ge 0,$$

because $0 \le \gamma \le \frac{1}{2}$. So, it suffices to prove the non-negativity of $h_1(\alpha,\beta) := 4(\alpha+\beta) - 4(\alpha^2+\beta^2) - 6\alpha\beta$. This is the case because $h_1(\alpha,\beta) = f(\alpha,\beta) + (\alpha+\beta) - (\alpha+\beta)^2 \ge 0$. This concludes the proof of the theorem. $\square$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 37, pp. 1-13, January 2021.



**3. Examples.**

Example 3.1. Consider the realizability of a list $\sigma$ in the simplex $\mathbb{U}$. We apply Theorem 2.1 and obtain a sufficient condition for nonrealizability of $\sigma$. A general point in $\mathbb{U}$ may be written as the following convex combination:

$$\sigma = \sigma(t,x,w,z) = t\mathbf{c} + x\mathbf{i} + w\mathbf{d} + z\mathbf{e} + (1-t-x-w-z)\,\mathbf{l}, \tag{3.19}$$

so we define

$$T = \{(t,x,w,z) \in \mathbb{R}^4 \mid t,x,w,z \geq 0 \text{ and } t+x+w+z \leq 1\}.$$

Theorem 3.1. *Let $\sigma = \sigma(t,x,w,z)$ be given by (3.19). Then, if*

$$x > \frac{\sqrt{5t^2-6t+5}-t-1}{2},$$

*$\sigma$ is not realizable.*

*Proof.* It follows from the definition of $\mathbf{c}$, $\mathbf{i}$, $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{l}$ that

$$\sigma = \left(1, t+w+\frac{1}{2}x, t+\frac{1}{2}x, -\frac{1}{2}(1+t+x+w-z), -\frac{1}{2}(1+t+x+w+z)\right).$$

Consequently, we have $s_1(\sigma) = t$ and $y = \lambda_3(\sigma) - s_1(\sigma) = \frac{x}{2}$. Let

$$g(\sigma) = g(t,x,w,z) := s_3(\sigma) - s_1(\sigma)^3 - 6s_1(\sigma)y(s_1(\sigma)+y).$$

Then, it can be checked that

$$\begin{aligned} g(t,x,w,z) = {} & \frac{3t^3}{4} + \frac{(-3x+9w-3)t^2}{4} \\ & + \frac{(9w^2+(6x-6)w-3x^2-3z^2-6x-3)t}{4} + \frac{3w^3}{4} + \frac{(3x-3)w^2}{4} \\ & - \frac{(3z^2+6x+3)w}{4} - \frac{3xz^2}{4} - \frac{3x^2}{4} - \frac{3z^2}{4} - \frac{3x}{4} + \frac{3}{4}. \end{aligned}$$

Note that all summands where $z$ appears are negative, hence $g(t,x,w,z) \leq g_1(t,x,w)$, where $g_1(t,x,w) := g(t,x,w,0)$. We have

$$\begin{aligned} g_1(t,x,w) = {} & \frac{3t^3}{4} + \frac{(-3x+9w-3)t^2}{4} \\ & + \frac{(9w^2+(6x-6)w-3x^2-6x-3)t}{4} + \frac{3w^3}{4} + \frac{(3x-3)w^2}{4} \\ & - \frac{(6x+3)w}{4} - \frac{3x^2}{4} - \frac{3x}{4} + \frac{3}{4}. \end{aligned}$$

Partially differentiating with respect to $w$, we get

$$\frac{\partial g_1(t,x,w)}{\partial w} = \frac{3(t+w-1)(3t+2x+3w+1)}{4} \leq 0,$$

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where the inequality holds because $0 \leq t + w \leq 1$. Hence, $g_1(t,x,w) \leq g_2(t,x)$, where $g_2(t,x) := g_1(t,x,0)$. We have

$$\begin{aligned} g_2(t,x) &= \frac{3t^3}{4} - \frac{(3x+3)t^2}{4} - \frac{(3x^2+6x+3)t}{4} - \frac{3x^2}{4} - \frac{3x}{4} + \frac{3}{4} \\ &= \frac{3(t+1)(t^2 - tx - x^2 - 2t - x + 1)}{4}. \end{aligned}$$

Solving $t^2 - tx - x^2 - 2t - x + 1 = 0$ for $x$ in terms of $t$, we obtain the 2 solutions

$$x_1 = \frac{\sqrt{5t^2 - 6t + 5} - t - 1}{2} \quad \text{and} \quad x_2 = \frac{-\sqrt{5t^2 - 6t + 5} - t - 1}{2},$$

so $g(\sigma) < 0$ for $x > \frac{\sqrt{5t^2-6t+5}-t-1}{2}$, and the theorem holds. □

Note that $5t^2 - 6t + 5 - (t+1)^2 = 4(1-t)^2$, so $x_1$ is 0 for $t = 1$ and positive for $0 \leq t < 1$.

Note that in [9], Theorem 2, Loewy and Spector also considered the realizability of $\sigma$ in $\mathbb{U}$ and showed, using the same notation as above, that if $x > \frac{\sqrt{5t^2+6t+5}-3t-1}{2}$, then $\sigma$ is not realizable. Theorem 3.1 strengthens this result. To see this, it suffices to show that $-t + \sqrt{5+6t+5t^2} - \sqrt{5-6t+5t^2} \geq 0$ for $0 \leq t \leq 1$ and this is equivalent to showing $t + \sqrt{5-6t+5t^2} \leq \sqrt{5+6t+5t^2}$. The latter holds with equality for $t = 0$ and strict inequality for $0 < t < 1$. For example, when $t = \frac{1}{2}$, the lower bound for $x$ in Theorem 3.1 is 0.1513878190, while the lower bound given by Theorem 2 in [9] is given by 0.270690632. Thus, points in $\mathbb{U}$ previously in the unknown region turn out to be non-realizable.

Example 3.2. Let us now consider lists of the form $\sigma = (1, a, a, b, b)$, where $1 \geq a > 0 > b \geq -1$. Since only two parameters are involved, we can consider them in the $(a,b)$ plane. A lot of effort has been devoted to the analysis of this special case, but there is no complete solution.

To describe the current state of affairs, we introduce a few definitions. Let $\mathcal{D}$ be the curved quadrangle in the $(a,b)$ plane given by the points that satisfy the following four inequalities:

- $1 + a + 2b \leq 0$,
- $5 + 2b - 7a \geq 0$,
- $1 + a^3 + 2b^3 - (1 + a + 2b)^3 \geq 0$,
- $4ab + 1 \leq 0$.

The boundary curves, namely

$$1 + a + 2b = 0, \quad 5 + 2b - 7a = 0, \quad 1 + a^3 + 2b^3 - (1+a+2b)^3 = 0, \quad 4ab + 1 = 0,$$

are called MN, LS, JMP, and LM, respectively. Also, denote by $W_1$, $W_2$, $W_3$, $W_4$, the points of intersection of LM and MN, MN and LS, LS and JMP, JMP and LM, respectively. A computation using Maple gives

$$W_1 = \left(\frac{\sqrt{3}-1}{2}, -\frac{\sqrt{3}+1}{4}\right), \; W_2 = \left(\frac{1}{2}, -\frac{3}{4}\right),$$

$$W_3 = (0.479159\cdots, -0.822941\cdots), \; W_4 = \left(\frac{\sqrt{5}-1}{4}, -\frac{\sqrt{5}+1}{4}\right).$$

See Figure 1 for details. It follows from McDonald and Neumann [10] that points on and above the line MN are realizable and from Loewy and Spector [9] that points strictly to the right of the LS line and below the

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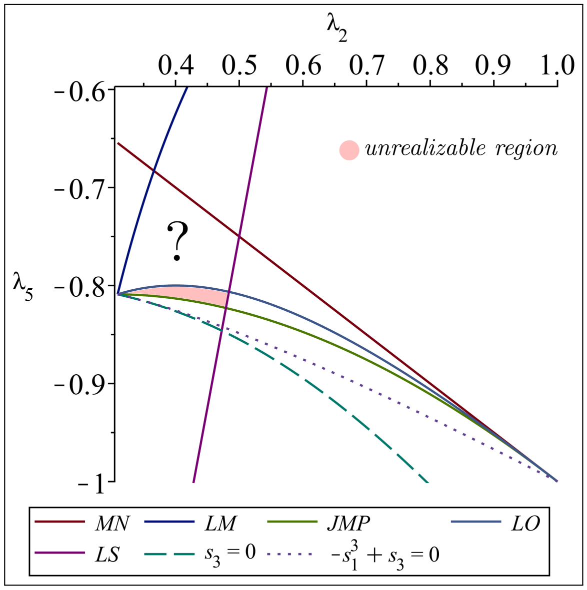


Figure 1: Two double eigenvalues

MN line are not realizable. It follows from Johnson, Marijuán and Pisonero [4] that points strictly below the JMP curve are not realizable and from Loewy and McDonald [7] that points on and to the left of the LM curve are realizable.

Consequently, the points in the $(a, b)$ plane which were not known to be realizable or nonrealizable are exactly the points in the interior of $\mathcal{D}$, as well as the points on the boundary which lie on LS or JMP (except $W_2$ and $W_4$). We apply Theorem 2.1 to show that part of the unknown region is not realizable. In Figure 1, the currently remaining unknown region is marked by "?", while the region marked as "unrealizable" was previously part of the unknown region and is now ruled out by Theorem 2.1.

Define

$$v(a,b) := s_3(\sigma) - s_1(\sigma)^3 - 6s_1(\sigma)y(s_1(\sigma) + y),$$
$$u(a,b) := 1 + a^3 + 2b^3 - (1 + a + 2b)^3.$$

Note that in the interior of $\mathcal{D}$, we have $1 + a + 2b < 0$, which is equivalent to $s_1(\sigma) < a$, so we can apply Theorem 2.1. It is straightforward to check that $s_3(\sigma) = 1 + 2a^3 + 2b^3$, $y = a - s_1(\sigma) = -1 - a - 2b > 0$, and

(3.20) $$v(a,b) = 6(a^3 - b^3 + 2a^2b + a^2 - 2b^2 - b).$$

Theorem 1 of [4] states that if $u(a, b) < 0$, then the list $\sigma$ is not realizable. The equation $u(a, b) = 0$ gives the JMP curve.

Lemma 3.1. *We have $v(a, b) \leq 0$ on the curve JMP and equality holds only at the point $W_4$.*

*Proof.* Applying the division algorithm (with respect to $a$), we obtain

$$v(a,b) = q(a,b)u(a,b) + r(a,b),$$

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where

$$q(a,b) = -\frac{2a}{2b+1},\ r(a,b) = -\frac{6b(b+1)^2(1+2a+2b)}{2b+1}.$$

Since $u(a,b)$ vanishes on JMP, we may without loss of generality replace $v(a,b)$ by the remainder $r(a,b)$. Moreover, $b$ and $2b+1$ are negative on JMP, so it suffices to show that $1+2a+2b \geq 0$, with equality holding only at $W_4$. The equality at $W_4$ does indeed hold. Suppose that there is a point on JMP where $1+2a+2b < 0$. That expression is positive at $W_3$, so it must vanish at a point on JMP other than $W_4$. Solving $1+2a+2b = 0$ for $b$, we get $b = -\frac{1}{2} - a$. Substituting it into the equation of JMP, namely $u(a,b) = 0$, we obtain the equation $4a^2 + 2a - 1 = 0$, and the only feasible solution is $a = \frac{\sqrt{5}-1}{4}$, which is the first coordinate of $W_4$. This concludes the proof of the lemma. ∎

Next, we check that $v(a,b)$ is a monotone increasing function of $b$ in $\mathcal{D}$. We have $\frac{\partial v(a,b)}{\partial b} = 12a^2 - 18b^2 - 24b - 6 = 12a^2 - 6(b+1)(3b+1)$. In $\mathcal{D}$, we have $b+1 > 0$ and $3b+1 < 0$, so the partial derivative is positive.

Corollary 3.1. *Let LO be the curve obtained by intersecting $\mathcal{D}$ with the solution set of $v(a,b) = 0$. Then, LO lies strictly above JMP, with the exception of the point $W_4$, where the two curves coincide. Consequently, all the points strictly below LO and on or above JMP are not realizable.*

As an example, consider $a = 0.4$. Then the points corresponding to this $a$ on the JMP and LO lines are $(0.4, -0.8132206607)$ and $(0.4, -0.8)$, respectively.

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