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LINEAR PRESERVERS OF LEFT MATRIX MAJORIZATION*

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Abstract. For $X, Y \in M_{nm}(\mathbb{R})$ $(= M_{nm})$, we say that Y is left (resp. right) matrix majorized by X and write $Y \prec_\ell X$ (resp. $Y \prec_r X$) if $Y = RX$ (resp. $Y = XR$) for some row stochastic matrix R. A linear operator $T\colon M_{nm} \to M_{nm}$ is said to be a linear preserver of a given relation $\prec$ on M_{nm} if $Y \prec X$ implies that $TY \prec TX$. The linear preservers of $\prec_\ell$ or $\prec_r$ are fully characterized by A.M. Hasani and M. Radjabalipour. Here, we launch an attempt to extend their results to the case where the domain and the codomain of T are not necessarily identical. We begin by characterizing linear preservers $T : M_{p1} \to M_{n1}$ of $\prec_\ell$.

Key words. Row stochastic matrix, Doubly stochastic matrix, Matrix majorization, Weak matrix majorization, Left (right) multivariate majorization, Linear preserver.

AMS subject classifications. 15A04, 15A21, 15A51.

1. Introduction. Throughout the paper, the notation $M_{nm}(\mathbb{R})$ or, simply, M_{nm} is fixed for the space of all $n \times m$ real matrices; this is further abbreviated by M_n when $m = n$. The space M_{n1} of all $n \times 1$ real vectors is denoted by the usual notation $\mathbb{R}^n$. The collection of all $n \times n$ permutation matrices is denoted by $\mathcal{P}(n)$ and the identity matrix is denoted by I_n or, simply, I, if the size n of the matrix I is understood from the context. For $i = 1, 2, \ldots, k$, let A_i be an $m_i \times p$ matrix for some $m_i \geq 0$. (If $m_i = 0$, the matrix A_i is vacuous and should be ignored when appearing in some formula.) We use the convention $[A_1/A_2/\ldots/A_k]$ to denote the $(m_1 + m_2 + \ldots + m_k) \times p$ matrix

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}.$$

Note that $[x_1/x_2/\ldots/x_k] = [x_1, x_2, \ldots, x_k]^t$, whenever $x_1, x_2, \ldots, x_k$ are real numbers. (Throughout the paper the notation A^t stands for the transpose of a given matrix A.)

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An $n \times m$ matrix $R = [r_{ij}]$ is called *row stochastic* (resp. *row substochastic*) if $r_{ij} \geq 0$ and $\Sigma_{k=1}^{m} r_{ik}$ is equal (resp. at most equal) to 1 for all i, j. For $X, Y \in M_{nm}$, we say Y is *left* (resp. *right*) *matrix majorized* by X (in M_{nm}), and write $Y \prec_\ell X$ (resp. $Y \prec_r X$), if $Y = RX$ (resp. $Y = XR$) for some $n \times n$ (resp. $m \times m$) row stochastic matrix R. For a given relation $\prec$ on matrices, we write $X \sim Y$ if $X \prec Y \prec X$. A linear operator $T\colon M_{pq} \to M_{nm}$ is said to be a linear preserver of $\prec$ if $Y \prec X$ (in M_{pq}) implies $TY \prec TX$ (in M_{nm}). The various notions of majorization from the left and the right are defined and studied in [1], [6]-[8], [12], [16]-[17], and the characterizations of their linear preservers in [2]-[5], [9]-[11], [13]-[15],[18].

In [9]-[11], A.M. Hasani and M. Radjabalipour characterized the structure of all linear operators $T\colon M_{nm} \to M_{nm}$ preserving left (or right) matrix majorizations. In all these results, the linear operator T maps a space of matrices into itself. In the present paper, we characterize the linear preservers of $\prec$ mapping $\mathbb{R}^p$ to $\mathbb{R}^n$ when p and n are not necessarily equal. These are the first steps in extending the results of [9]-[11] to more general linear transformations. From now on, by $\prec$, we only mean $\prec_\ell$; i.e., we are fixing the following convention throughout the remainder of the paper:

$$\prec \quad \text{stands for} \quad \prec_\ell\, . \tag{1.1}$$

It is known that, for $x, y \in \mathbb{R}^n$, $x \prec y$ if and only if $\max x \leq \max y$ and $\min x \geq \min y$.

In the following Theorems 1.1 and 1.2, we state some results from [10] which we are trying to generalize in this paper.

THEOREM 1.1. *Let $n \geq 3$. Then $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of left matrix majorization if and only if T has the form $X \mapsto aPX$, for some $a \in \mathbb{R}$ and some $P \in \mathcal{P}(n)$.*

THEOREM 1.2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. Then T preserves $\prec$ if and only if T has the form $T(X) = (aI + bP)X$ for all $X \in \mathbb{R}^2$, where P is the 2×2 permutation matrix not equal to I, and $ab \leq 0$. Moreover, for any 2×2 row stochastic matrix R, there exists a 2×2 row stochastic matrix S such that $S[T] = [T]R$.*

Let, throughout the paper, $[T] = [t_{ij}]$ denote the matrix representation of an operator $T : \mathbb{R}^p \to \mathbb{R}^n$ with respect to the standard bases. Theorem 1.2 means that the matrix representation of a linear preserver of $\prec$ with respect to the standard basis of $\mathbb{R}^2$ has the form

$$[T] = \begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

for some real numbers a, b satisfying $ab \leq 0$. As an immediate corollary we have the following.

COROLLARY 1.3. *If $y_1 \leq x_1 \leq x_2 \leq y_2$ and $ab \leq 0$, then $ax_1 + bx_2$ lies between*

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the two numbers $ay_1 + by_2$ *and* $by_1 + ay_2$.

The present paper continues in three further sections. Section 2 studies some necessary or sufficient conditions for a general linear operator T to preserve $\prec$. In particular, we prove that the condition $p \leq n$ is a necessary condition. Section 3 characterizes a general linear preserver T, for which the entries of $[T]$ have the same sign and, in particular, we will show that, in case $3 \leq p \leq n < 2p$, the matrix $[T]$ has entries all necessarily of the same sign. Section 4 deals with the case $2p \leq n < p(p-1)$.

We conclude this introductory section with a trivial observation.

PROPOSITION 1.4. *A linear operator* $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ *preserves* $\prec$ *if* $p = 1$ *or* $T = 0$.

2. Size conditions. In this section, we show that the condition $p \leq n$ is necessary for a nonzero operator $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ to be a linear preserver of $\prec$. We first establish the following definition whose symbols and notation will remain fixed throughout the remainder of the paper.

DEFINITION 2.1. *The letter* T *stands for an operator* $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ *and the notation* $[T] = [t_{ij}]$ *stands for its* $n \times p$ *matrix representation with respect to the standard bases* $\{e_1, e_2, \ldots, e_p\}$ *of* $\mathbb{R}^p$ *and* $\{f_1, f_2, \ldots, f_n\}$ *of* $\mathbb{R}^n$. *We say* T *or* $[T]$ *is nonnegative* (*resp. nonpositive*) *if the entries of* $[T]$ *are all nonnegative* (*resp. all nonpositive*). *We also define*

$$e = e_1 + e_2 + \ldots + e_p$$

$$a = \max\{\max Te_1, \max Te_2, \ldots, \max Te_p\},$$

$$b = \min\{\min Te_1, \min Te_2, \ldots, \min Te_p\}$$

and

$$c = \min Te,$$

where $\max X$ *and* $\min X$ *denote the maximum and the minimum values of the components of a given real vector* X, *respectively.*

THEOREM 2.2. *Let* $T\colon \mathbb{R}^p \rightarrow \mathbb{R}^n$ *be a nonzero linear preserver of* $\prec$, *and suppose* $p \geq 2$. *Then the following assertions are true.*

(*a*) *For each* $j \in \{1, \ldots, p\}$, $a = \max Te_j$ *and* $b = \min Te_j$. *In particular, every column of* $[T]$ *contains at least one entry equal to* a *and at least one entry equal to* b.

(*b*) $b \leq 0 \leq a$; *in particular,* $b \neq a$ *and* $n \geq 2$.

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(*c*) *The operator T is nonnegative or nonpositive if and only if $ab = 0$.*

(*d*) *$p \leq n$; moreover, if a row of $[T]$ contains an entry equal to a (resp. b), then all other nonnegative (resp. nonpositive) entries of that row are zero.*

(*e*) *$b \leq c \leq a$.*

Proof. (*a*) If $i, j \in \{1, 2, ..., p\}$, we have $e_i \prec e_j \prec e_i$ and so $T(e_i) \prec T(e_j) \prec T(e_i)$ which implies that

$$\max T(e_i) \leq \max T(e_j) \leq \max T(e_i)$$

and

$$\min T(e_i) \geq \min T(e_j) \geq \min T(e_i).$$

Hence, $\min T(e_i) = \min T(e_j) = b$ and $\max T(e_i) = \max T(e_j) = a$.

(*b*) Since $0 \prec e_i$, it follows that $b = \min T(e_i) \leq 0 \leq \max T(e_i) = a$. Also, since $T \neq 0$, $b \neq a$ and hence $n \geq 2$.

(*c*) The proof is an easy consequence of (*b*).

(*d*) Since $T \neq 0$ and $p \geq 2$, it follows that $a \neq b$, and hence, $n \geq 2$. Let J be any 2-element subset of $\{1, 2, ..., p\}$. Then $\sum_{j \in J} e_j \prec e_1$, and hence,

$$b \leq \min T(\sum_{j \in J} e_j) \leq \max T(\sum_{j \in J} e_j) \leq a.$$

We conclude that if $a > 0$ (resp. $b < 0$) and if a given row of $[T]$ contains an entry equal to a (resp. b), then there are no other positive (resp. negative) entries in that row. Now assume without loss of generality that $a > 0$. Since every column of $[T]$ has at least one entry equal to a and every row of $[T]$ contains at most one entry equal to a, it follows that $p \leq n$.

(*e*) The last inequality follows from the fact that $e \prec e_1$. □

Since T is a linear preserver of $\prec$ if and only if ηT is so for some nonzero real number η, we can fix the following assumption throughout the remainder of the paper.

Assumption 2.3. *The linear operator $T : \mathbb{R}^p \to \mathbb{R}^n$ is a preserver of $\prec$ with*

$$2 \leq p \leq n \quad \text{and} \quad 0 \leq -b \leq 1 = a. \tag{2.1}$$

We will employ the notation and the assumptions established in this section and may give no further reference.

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Theorem 2.4. *Let T be as in Assumption 2.3 and let $M = \max T(e_1 - e_2)$. Then $-M \le b \le c \le 1 \le M$ and the following assertions hold.*

(a) *The matrix $[T]$ is row stochastic if and only if $c = 1$.*

(b) *If $M > 1$, then $b < 0$ and $n \ge p(p-1)$.*

(c) *If $M = 1$ and $b < 0$, then $n \ge 2p$ and, up to a row permutation, $[T] = [I/(bI + B)/E]$, where B is a $p \times p$ nonnegative matrix with zero diagonal, and E is an $(n-2p) \times p$ matrix. The matrix E is vacuous if $n = 2p$.*

Proof. Since $e \prec e_1 \prec e_1 - e_2 \sim e_2 - e_1$, it follows that $-M \le b \le c \le 1 \le M$.

(a) The necessity is trivial and the sufficiency follows from the fact that the sum of the positive entries of each row is at most 1.

(b) Assume that $b = 0$. It follows that every entry of $T(e_1 - e_2)$ is at most 1 and hence $M = 1$. Thus, if $M > 1$, then $b < 0$.

Now, suppose $M > 1$ and let $X = e_j - e_k$ for some $j \ne k$. Since $X \sim e_1 - e_2$, it follows that $-M = \min TX$ and $M = \max TX$. Hence, for every (ordered) pair of distinct integers $(j,k) \in \{1,2,\ldots,p\} \times \{1,2,\ldots,p\}$, there exists an integer i such that $t_{ij} - t_{ik} = M$. Since $e_j - e_k \sim e_j - e_k \pm e_h$ for all $h \ne j,k$, it follows that the i^{th} row of $[T]$ has exactly two nonzero entries. This implies that there are at least $p(p-1)$ rows of $[T]$ each having exactly two nonzero entries.

(c) Suppose $b < 0$ and $M = 1$. Then every row of $[T]$ containing 1 as an entry, has all other entries equal to 0. Since every column of $[T]$ has at least one entry equal to 1 and at least one entry equal to b, it follows that $n \ge 2p$ and, up to a row permutation, $[T] = [I/(bI + B)/E]$, where B is a $p \times p$ nonnegative matrix having zero diagonal, and E is an $(n-2p) \times p$ matrix. □

3. Nonnegative linear preservers. Nonnegative linear preservers of $\prec$ were characterized as those T that, after the normalization of Assumption 2.3, satisfy the condition $b = 0$. The next theorem characterizes the structure of such nonnegative operators. We will use all the notation fixed in the previous sections as well as the notation $M = \max T(e_1 - e_2)$.

Theorem 3.1. *For the linear preserver T, the following assertions hold.*

(a) *If $n < 2p$ and $p \ge 3$, then T is nonnegative.*

(b) *If T is nonnegative, then there exists an $n \times n$ permutation matrix Q such that $[T] = Q[I/W]$, where W is a (possibly vacuous) $(n-p) \times p$ matrix of one of the following forms* (i), (ii) *or* (iii)*:*

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(i) W is row stochastic;

(ii) W is row substochastic and has a zero row;

(iii) $W = [(cI)/E]$, where $0 < c < 1$ and E is an $(n-2p)\times p$ row substochastic matrix with row sums at least c.

(c) Let Q be an $n\times n$ permutation matrix, and let W be an $(n-p)\times p$ matrix of the form (i), (ii), or (iii) of part (b). Then the operator $X \mapsto Q[X/(WX)]$ from $\mathbb{R}^p$ into $\mathbb{R}^n$ is a nonnegative linear preserver of $\prec$.

Proof. (a) Suppose $p \geq 3$ and $n < 2p$. We assume that $b < 0$ and reach a contradiction. Since each column of $[T]$ contains at least one entry equal to 1 and one entry equal to b, and since each row of $[T]$ has at most one entry equal to 1 and at most one entry equal to b, it follows that there is at least one row containing both 1 and b as entries. Thus, $M > 1$ and, hence, there are $p(p-1)$ rows each having 1 and b as entries. Therefore, $2p > n \geq p(p-1)$; a contradiction.

(b) Suppose T is nonnegative. Then $M = 1$ and every row of $[T]$ containing 1 as an entry cannot have any other nonzero (positive or negative) entry. Also, since each column of $[T]$ has at least one entry equal to 1, it follows that there exists an $n\times n$ permutation matrix Q and a nonnegative $(n-p)\times p$ matrix W such that $[T] = Q[I/W]$. We assume, without loss of generality, that $Q = I$. Since $e \prec e_1$, it follows that the sum of the entries of each row of $[T]$ is at most 1 and, hence, W is row substochastic. If $c = 1$ or $c = 0$, then W is of the form (i) or (ii), respectively. So, we assume that $0 < c < 1$ and show that W is of the form (iii). Let K be a positive integer such that, up to a row permutation, the sum $w_{i1} + w_{i2} + \ldots + w_{ip}$ of the i^{th} row of W is equal to c if and only if $i \leq K$. Now, choose $k \leq K$ such that

$$\Sigma_{j=2}^{p} w_{kj} = \min W(e_2 + e_3 + \ldots + e_p).$$

Let $\varepsilon > 0$ be small enough such that $c+\varepsilon(w_{k2}+w_{k3}+\ldots+w_{kp}) = \min T(e+\varepsilon(e_2+e_3+\ldots+e_p))$. Since $e \prec e+\varepsilon(e_2+e_3+\ldots+e_p)$, it follows that $c+\varepsilon(w_{k2}+w_{k3}+\ldots+w_{kp}) \leq c$ and, hence, $w_{k2} = w_{k3} = \ldots = w_{kp} = 0$ or, equivalently, $w_{k1} = c$. By a finite induction, we deduce that every column of W has an entry equal to c and, hence, up to a row permutation, W must have a $p\times p$ submatrix cI. That is $n \geq 2p$ and $W = [cI/E]$ for some $(n-2p)\times p$ row substochastic matrix E.

(c) Assume, without loss of generality, that $Q = I$. Let W be a row substochastic matrix as in (i) or (ii) of part (b). Suppose the first row of W is zero in case (ii). Let R be an arbitrary $p\times p$ row stochastic matrix. Define S to be the 2×2 block matrix $[[R \quad 0]/[(WR) \quad V]]$, where V is an $(n-p)\times(n-p)$ matrix whose columns are all zero except for its first column which is so designed to make S row stochastic. It is easy to see that $[I/W]RX = S[I/W]X$ and the theorem is proved in cases (i) and (ii).

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Next, let W be as in (iii). We must show that the operator T with the matrix representation $[T] = [I/cI/E]$ is a preserver of $\prec$. Let $X = [x_1/x_2/\dots/x_p] \in \mathbb{R}^p$ be arbitrary and let $Y \prec X$. Define $m = \min X$ and $M = \max X$. Then

$$\begin{array}{rcl} \min(TX) & = & \min\{m, cm, \min(EX)\} \\ \max(TX) & = & \max\{M, cM, \max(EX)\}. \end{array} \tag{3.1}$$

Suppose $m \geq 0$. Then $cm \leq m$ and $cm \leq \Sigma_{j=1}^{p} t_{ij} x_j$. Thus, $cm \leq \min(EX)$ and, hence, $\min(TX) = cm$. Then $m \leq \min(Y)$ and, hence, $\min(TX) = cm \leq c\min(Y) = \min(TY)$. Similarly, one can show that $M = \max(TX)$ and that $\max(TY) \leq \max(TX)$. Therefore, $TY \prec TX$. The case $M \leq 0$, now, follows from the fact that $Y \prec X$ if and only if $-Y \prec -X$.

Finally, if $m < 0 < M$, then $m \leq cm < 0 < cM \leq M$ and $m \leq m\Sigma_j w_{ij} \leq \Sigma_j w_{ij} x_j \leq M\Sigma_j w_{ij} \leq M$. Thus, $\min(TX) = m \leq \max(TX) = M$ and, hence, $\min(TX) \leq \min(TY) \leq \max(TY) \leq \max(TX)$. $\square$

Example 3.2. *For $p = 2$ and $n = 3$, a nonnegative preserver $[T]$ is of the form*

$$Q \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \end{bmatrix}, \tag{3.2}$$

where Q is a 3×3 permutation matrix and α, β are nonnegative numbers with sum 1 or 0. Conversely, if α and β are nonnegative numbers with sum 1 or 0, then the matrix (3.2) defines a nonnegative linear preserver of $\prec$.

4. Linear preservers with $b < 0$. In Theorem 3.1, we settled the problem of characterizing linear preservers of $\prec$ in case T is nonnegative. We also showed that if $3 \leq p \leq n < 2p$, then T is nonnegative. In this section, we study the case $b < 0$. The case is divided in three subcases: (i) $p = 2 \leq n \leq 3$; (ii) $2p \leq n < p(p-1)$; and (iii) $n \geq \max\{p(p-1), 2p\}$. In the remainder of the paper, the subcases (i) and (ii) are fully settled and the subcase (iii) is left open.

To study the subcase (i), we first strengthen Theorem 1.2.

Proposition 4.1. *Fix $-1 \leq b \leq 0$. Then for any 2×2 row stochastic matrix $R = \begin{bmatrix} r & 1-r \\ s & 1-s \end{bmatrix}$ with $r, s \in [0,1]$ there exists a 2×2 row stochastic matrix R' such that*

$$R' \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} R.$$

Proof. Examine

$$R' = (1-b)^{-1} \begin{bmatrix} r - b(1-s) & 1 - r - bs \\ s - b(1-r) & 1 - s - br \end{bmatrix}. \quad \square$$

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THEOREM 4.2. *Let $b < 0$ and $p = 2$. Then, for $n = 2$,*

$$[T] = Q \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} \tag{4.1}$$

and, for $n = 3$,

$$[T] = Q \begin{bmatrix} 1 & b \\ b & 1 \\ \eta\gamma & \eta(1+b-\gamma) \end{bmatrix}, \tag{4.2}$$

where $b \leq \gamma \leq 1$, and $\eta = 0, 1$. Conversely, every matrix of the form (4.1) or (4.2) is a linear preserver of $\prec$.

Proof. By Theorem 2.2, there is at least one row of $[T]$ containing both 1 and b as entries, and hence, in view of Theorem 2.4, there are at least two such rows. This establishes the permutation Q and the first two rows of the matrices in (4.1) and (4.2). Now, assume that $[\alpha \quad \beta]$ is the last row of the matrix in (4.2). Then $b \leq \alpha \leq 1$ and $b \leq \beta \leq 1$. Assume that $\alpha + \beta \neq 1 + b$. Let $e = e_1 + e_2$ and let $0 \neq \varepsilon \in \mathbb{R}$. Then $e \prec e + \varepsilon e_1$, and hence, $Te \prec T(e + \varepsilon e_1)$. This implies that the real numbers $1 + b$ and $\alpha + \beta$ lie between the maximum and the minimum of the set $\{1 + b + \varepsilon, 1 + b + b\varepsilon, \alpha + \alpha\varepsilon + \beta\}$ for small enough values of $|\varepsilon|$. One can easily verify that this is possible only when $\alpha = 0$. Similarly $\beta = 0$ and the necessity of the condition is established.

For the sufficiency of the condition, without loss of generality, we may assume that $Q = I$. Now, the case $p = n = 2$ follows from Theorem 4.1. For $p = 2$ and $n = 3$, let R and R' be as in Proposition 4.1 and its proof. We construct the 3×3 row stochastic matrix

$$R'' = \begin{bmatrix} \multicolumn{2}{c}{R'} & 0 \\ u & v & 1-u-v \end{bmatrix},$$

such that $R''[T] = [T]R$ for the matrix $[T]$ defined in (4.2). If $\eta = 0$, then we can choose $u = v = 0$. If $\eta = 1$, then it is sufficient to find $u, v \in [0, 1]$ such that $u + v \leq 1$ and

$$G(u, v) = u(1-\gamma) + v(b-\gamma) - (1+b)s + (1+s-r)\gamma = 0, \tag{4.3}$$

where r, s are the entries of the first column of R. Let $K(b, \gamma, r, s) = G(0, 0) = -s(1+b) + (1+s-r)\gamma$ and observe that $G(0, 1) = K(b, \gamma, r, s) + b - \gamma \leq G(u, v) \leq K(b, \gamma, r, s) + 1 - \gamma = G(1, 0)$, whenever $u, v \in [0, 1]$ and $u + v \leq 1$. It is now easy to see that $K(b, \gamma, r, s) + b - \gamma \leq 0 \leq K(b, \gamma, r, s) + 1 - \gamma$, whenever $-1 \leq b \leq 0$, $b \leq \gamma \leq 1$, $0 \leq r \leq 1$ and $0 \leq s \leq 1$. Hence, equation (4.3) has the desired solution. $\square$

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Now that we have settled the subcase (i), we turn to the subcase (ii). First we need some lemmas.

LEMMA 4.3. *Suppose $b < 0$ and $2p \leq n < p(p-1)$. Then $[T]$ has a block of the form bI.*

Proof. We first show that $[T]$ contains at least one row of the form $(be_j)^t$ for some $j = 1, \ldots, p$. If not, choose an arbitrary pair (j, k) of distinct integers in $\{1, 2, \ldots, p\}$ and let $J = \{1, 2, \ldots, p\} \backslash \{j, k\}$. It is clear that $e_j + \varepsilon \sum_{q \in J} e_q \sim e_j$ whenever $0 < \varepsilon < 1$. Then, given $0 < \varepsilon < 1$, there exists $1 \leq i \leq n$ such that $t_{ij} + \varepsilon \sum_{q \in J} t_{iq} = b$. Since n is finite, there exist $0 < \varepsilon_1 < \varepsilon_2 < 1$ for which the corresponding integers coincide; i.e., there exists i such that $t_{ij} + \varepsilon_1 \sum_{q \in J} t_{iq} = t_{ij} + \varepsilon_2 \sum_{q \in J} t_{iq} = b$. Hence, $t_{ij} = b$ and $t_{ik} = 0$ for all $k \in J$. Then to each pair (j, k) as above there corresponds a positive integer $i \leq n$ such that $t_{ij} = b$ and $t_{ik} > 0$. Since the correspondence is one to one, it follows that $n \geq p(p-1)$; a contradiction. Thus, $[T]$ contains a row equal to $b(e_j)^t$ for some $j \in \{1, 2, \ldots, p\}$.

Since $e + e_k \sim e + e_j$ for all $k \in \{1, 2, \ldots, p\}$, it follows that to each $k \in \{1, 2, \ldots, p\}$, there corresponds an integer h such that $t_{hk} + b \leq t_{hk} + \Sigma_{q=1}^{p} t_{hq} = \min T(e + e_k) = \min T(e + e_j) = 2b$. Hence, $t_{hk} = b$ and the remaining entries of the h^{th} row of $[T]$ are zero. Thus, $[T]$ has a block bI. ☐

THEOREM 4.4. *Suppose $b < 0$ and $2p \leq n < p(p-1)$. Let A_i (resp. B_i) denote the sum of the positive (resp. the negative) entries of the i^{th} row of $[T]$. Then, up to a row permutation, $[T] = [I/bI/E]$ and $\min\{B_i + bA_i : i = 1, 2, \ldots, n\} = b$.*

Proof. Note that, necessarily, $p \geq 4$. It follows from Theorem 2.4(c) and Lemma 4.3 that, up to a row permutation, $[T] = [I/bI/E]$. Obviously, $B_i + bA_i = b$ for $i = 1, 2, \ldots, 2p$, and thus, $\min\{B_i + bA_i : i = 1, 2, \ldots, n\} \leq b$. Assume, if possible, that

$$B_h + bA_h = \min\{B_i + bA_i : i = 1, 2, \ldots, n\} < b,$$

for some $h > 2p$. Define $X = [x_1, x_2, \ldots, x_p]^t \in \mathbb{R}^p$ by $x_j = 1$ if $t_{hj} < 0$ and $x_j = b$, otherwise. Then $\Sigma_{j=1}^{p} t_{ij} x_j \geq B_i + bA_i \geq B_h + bA_h = \Sigma_{j=1}^{p} t_{hj} x_j$ for $i = 1, 2, \ldots, n$. Thus, $\min TX = B_h + bA_h < b$. Fix $(j, k) \in \{1, 2, \ldots, p\} \times \{1, 2, \ldots, p\}$ with $j \neq k$. Observe that $e_j + be_k \sim X$. Hence, $\min T(e_j + be_k) = B_h + bA_h$, which implies that there exists a positive integer $q \leq n$ such that $t_{qj} + bt_{qk} = B_h + bA_h$. We claim $t_{qj} < 0$ and $t_{qk} > 0$.

Assume, if possible, that $t_{qk} \leq 0$. Then $bt_{qk} \geq 0 \geq bA_q$ and $t_{qj} \geq B_q$. Hence, $B_h + bA_h = t_{qj} + bt_{qk} \geq B_q + bA_q \geq B_h + bA_h$. It follows that $t_{qk} = A_q = 0$ and $t_{qj} = B_q \geq b$. Therefore, $b > B_h + bA_h = B_q \geq b$; a contradiction. Thus $t_{qk} > 0$.

Next, we assume that $t_{qj} \geq 0$ and reach a contradiction. In this case, $B_h + bA_h =$

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$t_{qj} + bt_{qk} \geq B_q + bA_q \geq B_h + bA_h$. Hence, $t_{qj} = B_q = 0$ and $A_q = t_{qk}$. Thus, $b > B_h + bA_h = bA_q$ or, equivalently, $A_q > 1$; a contradiction. Thus, $t_{qj} < 0$.

Since $b < 0$, it follows that $B_q + bA_q \leq t_{qj} + bt_{qk} = B_h + bA_h \leq B_q + bA_q$. Hence, $B_q = t_{qj}$, $A_q = t_{qk}$ and, consequently, $t_{qr} = 0$ for all $r \in \{1, 2, \ldots, p\} \backslash \{j, k\}$. Since there are $p(p-1)$ distinct pairs like (j, k), it follows that $n \geq p(p-1)$; a contradiction. Hence, $\min(B_i + bA_i) = b$.□

In the following, we prove the converse of Theorem 4.4; in fact, we prove more.

THEOREM 4.5. *Suppose* $-1 \leq b < 0$ *and let* I *be the* $p \times p$ *identity matrix. Let* $E = [e_{ij}]$ *be an* $m \times p$ *matrix for some nonnegative integer* m *such that, if* $m \geq 1$, *then* $\min\{B_i + bA_i : i = 1, 2, \ldots, m\} = b$, *where* A_i (*resp.* B_i) *is the sum of the positive* (*resp. negative*) *entries of the* i^{th} *row of* E. (*Note that* E *is vacuous if* $m = 0$.) *Then the operator represented by the* $(2p+m) \times p$ *matrix* $Q[I/bI/E]$ *with respect to the standard bases of* $\mathbb{R}^p$ *and* $\mathbb{R}^{2p+m}$ *is a linear preserver of* $\prec$ *for any* $(2p+m) \times (2p+m)$ *permutation matrix* Q.

Proof. Assume, without loss of generality, that $Q = I$. Let $\tau : \mathbb{R}^p \to \mathbb{R}^{2p+m}$ (resp. $\tau_0 : \mathbb{R}^p \to \mathbb{R}^{2p}$) be the operator represented by the matrix $[I/bI/E]$ (resp. $[I/bI]$). We claim that $\max \tau X = \max \tau_0 X$ and $\min \tau X = \min \tau_0 X$ for all $X \in \mathbb{R}^p$. Fix $X \in \mathbb{R}^p$ and write $m_1 = \min X$ and $m_2 = \max X$. To prove the claim, it suffices to show that $\max EX \leq \max\{m_2, bm_1\}$ and $\min EX \geq \min\{m_1, bm_2\}$.

For a real number u, define $u^+ = 2^{-1}(|u| + u)$ and $u^- = 2^{-1}(|u| - u)$. Thus, for the i^{th} component $(EX)_i$ of EX, we have

$$\begin{aligned} (EX)_i &= \Sigma_j e_{ij} x_j \geq -\Sigma_j e_{ij}^+ x_j^- - \Sigma_j e_{ij}^- x_j^+ \\ &\geq m_1 \Sigma_j e_{ij}^+ - m_2 \Sigma_j e_{ij}^- \geq m_1 A_i + m_2 B_i, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} (EX)_i &= \Sigma_j e_{ij} x_j \leq \Sigma_j e_{ij}^+ x_j^+ + \Sigma_j e_{ij}^- x_j^- \\ &\leq m_2 \Sigma_j e_{ij}^+ - m_1 \Sigma_j e_{ij}^- \leq m_2 A_i + m_1 B_i. \end{aligned} \tag{4.5}$$

It thus suffices to show

$$m_1 A_i + m_2 B_i \geq \min\{m_1, bm_2\}$$

and

$$m_2 A_i + m_1 B_i \leq \max\{m_2, bm_1\}$$

whenever $m_1 \leq m_2$ and the variables A_i and B_i satisfy $0 \leq A_i \leq 1$, $b \leq B_i \leq 0$, and $B_i + bA_i \geq b$. Since this is a linear programming problem, it suffices to verify the inequalities for the three vertices $(A_i, B_i) = (0, 0), (1, 0), (0, b)$. The first case uses the assumption $m_1 \leq m_2$, and the last two cases are trivial.

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Thus, $\max(\tau X) = \max(\tau_0 X)$ and $\min(\tau X) = \min(\tau_0 X)$. Therefore, τ is a linear preserver of $\prec$ if and only if τ_0 is so. To complete the proof of the theorem, it remains to show that τ_0 is a linear preserver of $\prec$. Let R be a $p \times p$ row stochastic matrix and define

$$S = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}.$$

Then S is a $2p \times 2p$ row stochastic matrix and $S\tau_0 = \tau_0 R$, which implies that τ_0 is a linear preserver of $\prec$. Thus, τ is also a linear preserver of $\prec$. $\square$

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