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ON (CON)SIMILARITIES AND CONGRUENCES BETWEEN A AND A^*, A^T OR $\overline{A}$ [1]

J. VERMEER*

Abstract. A unifying approach is presented between similarity, consimilarity, ^{T}congruence and *congruence of a matrix A to a symmetric, to a Hermitian or to a real matrix. Also studied are similarity consimilarity, ^{T}congruence and *congruence of a matrix A to A^*, A^T, and $\overline{A}$. Attempts are made to find special (con)similarities and congruences, as well as to find connections between these classes of maps. For example, it is shown that if $A\overline{A}$ is nonderogatory and nonsingular, then the consimilarities between A and A^T are precisely the Hermitian similarities between $A\overline{A}$ and $(A\overline{A})^*$. Also, if A is nonsingular, then the coninvolutory ^{T}congruences between A and A^* are in 1-to-1 correspondence with Hermitian similarities between $A(A^{-1})^T$ and $(A(A^{-1})^T)^*$.

Key words. Similarities, Consimilarities, ^{T}congruences, $*$congruences, Canonical forms, Involutions, Coninvolutions.

AMS subject classifications. 15A21, 15A24, 15A48, 15A57, 15A63.

1. Introduction. All matrices are assumed to be complex, unless otherwise restricted. For a given A, p_A denotes its characteristic polynomial and mp_A denotes its minimal polynomial. Matrix A is *nonderogatory* if $p_A = mp_A$.

Matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ are said to be *consimilar* if there exists a nonsingular $S \in \mathcal{M}_n(\mathbb{C})$ with $SA\overline{S^{-1}} = B$. In this case S (and S^{-1}) is said to be a consimilarity transformation between A and B.

Matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ are ^{T}congruent (respectively, *congruent) if there exists a nonsingular $S \in \mathcal{M}_n(\mathbb{C})$ such that $SAS^T = B$ (respectively, $SAS^* = B$).

For a matrix A, we investigate the relation of being similar (respectively, consimilar, ^{T}congruent or *congruent) to a symmetric matrix, to a Hermitian matrix or to a real matrix. We find a kind of unifying approach to this problem, to be called "the standard Proposition". Most of the results in these standard propositions are indeed standard, but some lead to new results, such as:

Proposition 1.1. *Every $A \in \mathcal{M}_n(\mathbb{C})$ is similar to a symmetric matrix via a symmetric matrix.*

We try to find the "natural" similarities (consimilarities, ^{T}congruences, *congruences) between A and A^T, between A and A^*, and between A and $\overline{A}$. In all cases we connect these particular problems to the result in the standard Proposition.

In particular, we are interested in Hermitian similarities between A and A^*. We show:

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* Delft University of Technology, Delft Institute for Applied Mathematics, PO Box 5031, 2600 GA Delft, the Netherlands (J.Vermeer@@tudelft.nl).

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Theorem 1.2. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonderogatory and assume that A and A^* are similar via a Hermitian matrix V. Suppose that a given $S \in \mathcal{M}_n(\mathbb{C})$ satisfies $SA = A^*S$. Then S is Hermitian if and only if there exists a real polynomial p of degree less than n such that $S = Vp(A)$.*

Our second goal is to obtain results corresponding to the following theorem of O. Taussky and H. Zassenhaus.

Theorem 1.3. [12] *Let $\mathbb{F}$ be a field and $A \in \mathcal{M}_n(\mathbb{F})$ then:*
1. There exists a symmetric $X \in \mathcal{GL}_n(\mathbb{F})$ such that $XAX^{-1} = A^T$,
2. Every $X \in \mathcal{M}_n(\mathbb{F})$ such that $XA = A^TX$ is symmetric if and only A is nonderogatory.
3. Every $X \in \mathcal{GL}_n(\mathbb{F})$ such that $XAX^{-1} = A^T$ is symmetric if and only A is nonderogatory.

For consimilarities we obtain:

Theorem 1.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*
1. *A is consimilar to a Hermitian matrix via a symmetric matrix.*
2. *There exists a symmetric consimilarity $V \in \mathcal{GL}_n(\mathbb{C})$ between A and A^*.*
3. *If $A\overline{A}$ is nonderogatory then all $V \in \mathcal{M}_n(\mathbb{C})$ such that $VA = A^*\overline{V}$ are symmetric.*
4. *If A is nonsingular and all $V \in \mathcal{M}_n(\mathbb{C})$ such that $VA = A^*\overline{V}$ are symmetric, then $A\overline{A}$ is nonderogatory.*

Theorem 1.5. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*
1. *There exists a Hermitian $V \in \mathcal{GL}_n(\mathbb{C})$ such that $VA\overline{V^{-1}} = A^T$.*
2. *If A is nonsingular and $A\overline{A}$ is nonderogatory, then all $V \in \mathcal{M}_n(\mathbb{C})$ such that $VA = A^T\overline{V}$ are Hermitian.*

Theorem 1.6. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*
1. *A is consimilar to a real matrix via a coninvolution.*
2. *A is consimilar to $\overline{A}$ via a coninvolution.*

For ^{T}congruences we obtain:

Theorem 1.7. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*
1. ([3],[4] or[14]) *A is ^{T}congruent to A^T via an involution.*
2. *If A is nonsingular and its cosquare $A(A^{-1})^T$ is nonderogatory then all $S \in \mathcal{GL}_n(\mathbb{C})$ such that $SAS^T = A^T$ are involutions.*

Theorem 1.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following assertions are equivalent.*
1. *A is ^{T}congruent to a real matrix,*
2. *A is ^{T}congruent to a real matrix via a coninvolution,*
3. *A and $\overline{A}$ are ^{T}congruent,*
4. *A and $\overline{A}$ are ^{T}congruent via a coninvolution,*
5. *A and A^* are ^{T}congruent,*
6. *A and A^* are ^{T}congruent via a coninvolution,*
7. *A is ^{T}congruent to a Hermitian matrix,*
8. *A is ^{T}congruent to a Hermitian matrix via a coninvolution.*

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For *congruences we obtain:

THEOREM 1.9. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*

1. *A is *congruent to a symmetric matrix via a coninvolution.*
2. ([8]) *A is *congruent to A^T via a coninvolution.*
3. *If A is nonsingular and its *cosquare $A(A^{-1})^*$ is nonderogatory then all $S \in \mathcal{GL}_n(\mathbb{C})$ such that $SAS^* = A^T$ are coninvolutions.*

THEOREM 1.10. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following assertions are equivalent.*

1. *A is *congruent to a real matrix,*
2. *A is *congruent to a real matrix via a coninvolution,*
3. *A and $\overline{A}$ are *congruent,*
4. *A and $\overline{A}$ are *congruent via a coninvolution,*
5. *A and A^* are *congruent,*
6. *A and A^* are *congruent via an involution.*

We do not know whether the results in Theorem 1.5(2), in Theorem 1.7(2) and in Theorem 1.9(3) can be strengthened to "if and only if" (perhaps only in the singular case) as in Theorem 1.3 and Theorem 1.4(3).

2. On similarities between A and A^T, A^* or $\overline{A}$.

2.1. Similarities between A and A or A^T. We need the following fact.

PROPOSITION 2.1. ([6], Theorem 3.2.4.2) *Let $A \in \mathcal{M}_n(\mathbb{C})$ and suppose that A is nonderogatory. Then $BA = AB$ if and only if there exists a polynomial p with $B = p(A)$. The polynomial may be taken to be of degree less than n, in which case p is unique.*

We start with the first "standard Proposition".

PROPOSITION 2.2. *Let $\mathbb{F}$ be a field and let $A \in \mathcal{M}_n(\mathbb{F})$. Assume that each nonsingular symmetric $S \in \mathcal{M}_n(\mathbb{F})$ can be written as $S = U^2$, where $U \in \mathcal{M}_n(\mathbb{F})$ is symmetric. Then the following assertions are equivalent.*

1. *A is similar to a symmetric matrix.*
2. *A is similar to a symmetric matrix via a symmetric matrix.*
3. *A is similar to A^T via a symmetric matrix.*

Proof. (1) → (3) There exists $V \in \mathcal{M}_n(\mathbb{F})$ such that VAV^{-1} is symmetric, i.e., $VAV^{-1} = (VAV^{-1})^T = (V^T)^{-1}A^TV^T$. We obtain:

$$(V^TV)A(V^TV)^{-1} = A^T$$

V^TV is the required symmetric matrix.
(3) → (2) Let S be symmetric such that $SAS^{-1} = A^T$. There exists a symmetric U such that $U^2 = S$. Then: $SAS^{-1} = A^T \Rightarrow U^2A(U^2)^{-1} = A^T \Rightarrow UAU^{-1} = U^{-1}A^TU$. This identity shows UAU^{-1} is symmetric and that A is similar to it via the symmetric matrix U.
(2) → (1) Trivial. ∎

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REMARK 2.3. The proofs of all "standard Proposition" are identical. The main problem is always the implication (3) $\to$ (2), in which a statement of type "each $S \in \mathcal{M}_n(\mathbb{C})$ of type P is the square of some $U \in \mathcal{M}_n(\mathbb{C})$ of type P" is required.

Note that the standard Proposition is only an equivalence of assertions. But since it is known that every $A \in \mathcal{M}_n(\mathbb{C})$ is similar to a symmetric matrix (see [6], Theorem 4.4.9) and since the field $\mathbb{C}$ satisfies the hypothesis in Proposition 2.2 (indeed, if $S \in \mathcal{M}_n(\mathbb{C})$ is nonsingular and symmetric, then there exists $U \in \mathcal{M}_n(\mathbb{C})$, polynomial in S, with $U^2 = S$ (see [7], Theorem 6.4.12.a). U is symmetric, being polynomial in S) we conclude that we have proved Proposition 1.1.

Finally we note that Theorem 1.3 implies that the assertion "A is similar to A^T" could be added to the standard Proposition.

2.2. Similarities between A and $\overline{A}$ or A^*. Note that A and $\overline{A}$ (or A^*) need not be similar at all. $E \in \mathcal{M}_n(\mathbb{C})$ is *involutory* (or "*an involution*") if $E^{-1} = E$ and $J \in \mathcal{M}_n(\mathbb{C})$ is *coninvolutory* (or "*a coninvolution*") if $J^{-1} = \overline{J}$.

LEMMA 2.4.

1. *For each nonsingular $A \in \mathcal{M}_n(\mathbb{C})$, $\overline{A^{-1}}A$ is coninvolutory.*
2. (See [7], Theorem 6.4.22) *If $E \in \mathcal{M}_n(\mathbb{C})$ is coninvolutory, then there exists a coninvolution $X \in \mathcal{M}_n(\mathbb{C})$ such that $X^2 = E$.*

The standard Proposition becomes:

PROPOSITION 2.5. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*

1. *The following assertions are equivalent.*
   (a) *A is similar to a real matrix.*
   (b) *A is similar to a real matrix via a coninvolutory matrix.*
   (c) *A is similar to $\overline{A}$ via a coninvolutory matrix.*
2. *The following assertions are equivalent.*
   (a) *A is similar to a Hermitian matrix.*
   (b) *A is similar to a Hermitian matrix via a Hermitian, positive definite matrix.*
   (c) *A is similar to A^* via a Hermitian, positive definite matrix.*

Proof.

1. $(a) \to (c)$ There exists $S \in \mathcal{M}_n(\mathbb{C})$ such that $SAS^{-1} = R \in \mathcal{M}_n(\mathbb{R})$. So: $SAS^{-1} = R = \overline{R} = \overline{S}\,\overline{A}\,\overline{S^{-1}}$ and we obtain $(\overline{S^{-1}}S)A(\overline{S^{-1}}S)^{-1} = \overline{A}$.
   $E = \overline{S^{-1}}S$ is the required coninvolutory matrix.
   $(c) \to (b)$ Let $U \in \mathcal{M}_n(\mathbb{C})$ be coninvolutory and such that $UAU^{-1} = \overline{A}$. Let $E \in \mathcal{M}_n(\mathbb{C})$ be coninvolutory such that $E^2 = U$. The identity $E^2A(E^2)^{-1} = \overline{A}$ implies $EAE^{-1} = E^{-1}\overline{A}E$, and so $EA\overline{E} = \overline{E}\,\overline{A}\,E$, i.e., EAE^{-1} is real and A is similar to it via the coninvolutory matrix E.
   $(b) \to (a)$ Trivial.
2. $(a) \to (c)$ There exists $V \in \mathcal{M}_n(\mathbb{C})$ such that VAV^{-1} is Hermitian, i.e.,

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$VAV^{-1} = (VAV^{-1})^* = (V^*)^{-1}A^*V^*$. We obtain:

$$(V^*V)A(V^*V)^{-1} = A^*.$$

V^*V is the required Hermitian and positive definite matrix.

$(c) \to (b)$ Let $H \in \mathcal{M}_n(\mathbb{C})$ be Hermitian and positive definite such that $HAH^{-1} = A^*$. There exists a Hermitian and positive definite $U \in \mathcal{M}_n(\mathbb{C})$ such that $U^2 = H$. Then: $HAH^{-1} = A^* \Rightarrow U^2A(U^2)^{-1} = A^* \Rightarrow UAU^{-1} = U^{-1}A^*U$. This identity shows that UAU^{-1} is Hermitian and that A is similar to it via the Hermitian and positive definite matrix U.

$(b) \to (a)$ Trivial. □

The structures of the two proofs of Proposition 2.5 are completely identical to the proof of Proposition 2.2.

It is known from the literature that A and $\overline{A}$ are similar if and only if A is similar to a real matrix, (see [6], Theorem 4.1.7.) and so the statement "A is similar to $\overline{A}$" can be added to Proposition 2.5(1). But the statement "A is similar to A^*" cannot be added to Proposition 2.5(2). Indeed, A and A^* can be similar (via a Hermitian matrix) without A being similar to a Hermitian matrix. For example, let $A \in \mathcal{M}_n(\mathbb{R})$ be any real non-diagonalizable matrix. By Theorem 1.3 there exists a real symmetric (hence Hermitian) S such that $SAS^{-1} = A^T = A^*$. But A is not similar to a Hermitian matrix (as Hermitian matrices are similar to a real diagonal matrix).

For completeness, we mention the following Lemma and Proposition.

Lemma 2.6. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be similar. If A is similar to $\overline{A}$ (via a coninvolution), then B is similar to $\overline{B}$ (via a coninvolution).*

Proof. If $SAS^{-1} = B$ and $JAJ^{-1} = \overline{A}$, then $\overline{B} = (\overline{S}JS^{-1})B(\overline{S}JS^{-1})^{-1}$. Moreover, if J is coninvolutory so is $\overline{S}JS^{-1}$. □

Our proof of the following proposition contains new arguments for the equivalence of conditions (4) and (5).

Proposition 2.7. ([6], page 172) *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following assertions are equivalent.*

1. *A is similar to $\overline{A}$.*
2. *A is similar to $\overline{A}$ via a coninvolutory matrix.*
3. *A is similar to a real matrix.*
4. *A is similar to A^*.*
5. *A is similar to A^* via a Hermitian matrix.*

Proof. (1)→(2) Let $J \in \mathcal{M}_n(\mathbb{C})$ be the Jordan normal form of A. A similar to $\overline{A}$ implies J similar to $\overline{J}$, i.e., if $J_k(\lambda)$ is a Jordan block of A, so is $J_k(\overline{\lambda})$ $(\lambda \notin \mathbb{R})$. So J consists of blocks of type $J_k(\lambda)$ $(\lambda \in \mathbb{R})$, similar to $\overline{J_k(\lambda)}$ via the real involution I_k, and of blocks $J_k(\lambda) \oplus J_k(\overline{\lambda})$ $(\lambda \notin \mathbb{R})$, similar to its conjugate via the reversal matrix, the real involution (here considered consisting of blocks $\begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix}$). So J is similar to $\overline{J}$ by a real involution and Lemma 2.6 implies that A is similar to $\overline{A}$ via a

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coninvolution.
(2)→(1) Trivial.
(2)↔(3) This is Proposition 2.5.
(1)↔(4) This follows from the fact that B and B^T are similar, where $B = \overline{A}$.
(2)→(5) Let $E \in \mathcal{M}_n(\mathbb{C})$ be coninvolutory and such that $EAE^{-1} = \overline{A}$. If $X \in \mathcal{M}_n(\mathbb{C})$ is coninvolutory and $X^2 = E$, then $XAX^{-1} = X^{-1}\overline{A}X$. This identity implies that $X^{-1}\overline{A}X \in \mathcal{M}_n(\mathbb{R})$. By Theorem 1.3 there exits a real symmetric $S \in \mathcal{M}_n(\mathbb{R})$ such that $S(X^{-1}\overline{A}X)S^{-1} = (X^{-1}\overline{A}X)^T = X^TA^*(X^T)^{-1}$. It follows that $((X^T)^{-1}SX)A((X^T)^{-1}SX)^{-1} = A^*$ and as X is coninvolutory we obtain $(X^*SX)A(X^*SX)^{-1} = A^*$. Since S is Hermitian, so is X^*SX.
(5)→(4) Trivial. □

There exists no $A \in \mathcal{M}_n(\mathbb{C})$ for which the assertion "if S is a similarity between A and $\overline{A}$, then S is coninvolutory" is correct. Indeed, if S is a similarity between A and $\overline{A}$, so is αS ($\alpha \in \mathbb{C}$). But αS need not be coninvolutory.

In the same way one notes that not all similarities between A and A^* are Hermitian. (If S is a similarity between A and A^*, so is αS. But if S is Hermitian, αS is not, if $\alpha \notin \mathbb{R}$. Such an S is called "essentially Hermitian" and one might ask whether this is the only possibility.)

For $A \in \mathcal{M}_n(\mathbb{C})$ we define the complex vector space

$$C(A) = \{X \in \mathcal{M}_n(\mathbb{C}) : XA = AX\}.$$

For $A \in \mathcal{M}_n(\mathbb{C})$ which is similar to A^* we define the complex vector spaces:

$$C(A, A^*) = \{S \in \mathcal{M}_n(\mathbb{C}) : SA = A^*S\}$$

and the real vector space:

$$H(A, A^*) = \{H \in \mathcal{M}_n(\mathbb{C}) : H \text{ is Hermitian and } HA = A^*H\} \subset C(A, A^*).$$

$C(A, A^*)$ has the property: "$S \in C(A, A^*)$ implies $S^* \in C(A, A^*)$". Therefore we can define a map $T : C(A, A^*) \to H(A, A^*)$ by $T(S) = \frac{1}{2}S + \frac{1}{2}S^*$. As a map between real vector spaces, T is linear and $\mathrm{Kern}(T) = \{X \in C(A, A^*) : X \text{ is skew Hermitian}\} = iH(A, A^*)$. We have shown the identity $C(A, A^*) = H(A, A^*) \oplus iH(A, A^*)$ (as real vector spaces), and therefore, the complex dimension of $C(A, A^*)$ is equal to the real dimension of $H(A, A^*)$. This also shows that there exist $S \in C(A, A^*)$ which are not essentially Hermitian. For example, if $H_1, H_2 \in H(A, A^*)$ are not multiples of each other, then $S = H_1 + iH_2 \in C(A, A^*)$ is not essentially Hermitian.

Proposition 2.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$ such that A and A^* are similar. The following assertions are equivalent.*

1. *A is nonderogatory,*
2. *$\dim \mathrm{C}(A, A^*) = n$,*
3. *$\dim \mathrm{H}(A, A^*) = n$.*

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Proof. Theorem 4.4.17 of [7] implies that A is nonderogatory if and only if $\dim C(A) = n$. If A and A^* are similar, then $\dim C(A) = \dim C(A, A^*)$. Indeed, if $SAS^{-1} = A^*$, then an isomorphism $R : C(A) \to C(A, A^*)$ can be defined by $R(X) = SX$. ∎

If $A \in \mathcal{M}_n(\mathbb{C})$ is nonderogatory, then Theorem 1.2 represents a characterization of $H(A, A^*) \subset C(A, A^*)$.

Proof of Theorem 1.2

Fix a Hermitian $V \in \mathcal{M}_n(\mathbb{C})$ such that $VAV^{-1} = A^*$. As in the proof of Proposition 2.8, we define an isomorphism $R : C(A, A^*) \to C(A)$ by $R(S) = V^{-1}S$. Let $S \in C(A, A^*)$. Since $V^{-1}S \in C(A)$ and since A is assumed to be nonderogatory, we conclude from Proposition 2.1 that there exists a unique polynomial p_1 of degree less than n such that $V^{-1}S = p_1(A)$.

Note that $S^* \in C(A, A^*)$ and the same argument shows that there is a unique polynomial p_2 of degree less than n such that $V^{-1}S^* = p_2(A)$.

Next we note that

$$V^{-1}S = p_1(A) \Rightarrow S = Vp_1(A) = Vp_1(A)V^{-1}V = p_1(VAV^{-1})V = p_1(A^*)V.$$

Using the identity $[p(A)]^* = \overline{p}(A^*)$ and $V^* = V$ we obtain:

$$S^* = [p_1(A^*)V]^* = V^*[p_1(A^*)]^* = V\overline{p_1}(A),$$

i.e., $V^{-1}S^* = \overline{p_1}(A)$ and the uniqueness imply that $p_2 = \overline{p_1}$.

We conclude: S is Hermitian $\Leftrightarrow S - S^* = 0 \Leftrightarrow V(p_1 - \overline{p_1})(A) = 0 \Leftrightarrow (p_1 - \overline{p_1})(A) = 0$ and since the degree of $p_1 - \overline{p_1}$ is less than the degree of the minimal polynomial of A we see that this is equivalent to $p_1 - \overline{p_1} = 0$, i.e., p_1 is a real polynomial.

We conclude with the observation that the similarities between A and A^* are precisely the nonsingular linear combinations of (the Hermitian matrices) $V, VA, \ldots, VA^{n-1}$. The Hermitian similarities between A and A^* are precisely the nonsingular real linear combinations. ∎

3. On consimilarities between A and A^T, A^*, or $\overline{A}$.

3.1. Consimilarities, general part. Consimilarity between matrices is a well understood phenomenon.

THEOREM 3.1. ([1] or [5])

1. *$A, B \in \mathcal{M}_n(\mathbb{C})$ are consimilar if and only if $A\overline{A}$ is similar to $B\overline{B}$, rank$(A) =$ rank(B), rank$(A\overline{A}) =$ rank$(B\overline{B})$, rank$(A\overline{A}A) =$ rank$(B\overline{B}B)$ etc. until alternating products with n terms.*
2. *Every $A \in \mathcal{M}_n(\mathbb{C})$ is consimilar to some Hermitian matrix and to some real matrix.*

It follows that every $A \in \mathcal{M}_n(\mathbb{C})$ is consimilar to $\overline{A}, A^*$, and A^T (see also [5]).

For $A, B \in \mathcal{M}_n(\mathbb{C})$ we define the *real* vector space:

$$C^{con}(A, B) = \{X \in \mathcal{M}_n(\mathbb{C}) : A\overline{X} = XB\}.$$

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and for $A, B \in \mathcal{M}_n(\mathbb{C})$ we define the complex vector space

$$C(A\overline{A}, B\overline{B}) = \{X \in \mathcal{M}_n(\mathbb{C}) : A\overline{A}X = XB\overline{B}\}.$$

We recall the following results.

LEMMA 3.2. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$.*

1. $S \in \mathrm{C}^{con}(A, B) \;\Rightarrow\; S \in \mathrm{C}_{A\overline{A}}(B\overline{B})$.

If $B \in \mathcal{M}_n(\mathbb{C})$ is nonsingular, then

2. *If* $S, iS \in \mathrm{C}^{con}(A, B)$ *then* $S = 0$.
3. $S \in \mathrm{C}^{con}(A, B) \;\Rightarrow\; A\overline{S}B^{-1} \in \mathrm{C}^{con}(A, B)$.
4. $S \in \mathrm{C}(A\overline{A}, B\overline{B}) \;\Rightarrow\; A\overline{S}B^{-1} \in \mathrm{C}(A\overline{A}, B\overline{B})$.
5. ([1]) $S \in \mathrm{C}(A\overline{A}, B\overline{B}) \;\Rightarrow\; e^{i\theta}A\overline{S}B^{-1} + e^{-i\theta}S \in \mathrm{C}^{con}(A, B)$, *for all* $\theta \in \mathbb{R}$.

Proof. (1) If $A\overline{S} = SB$ then $A\overline{A}S = A\overline{(A\overline{S})} = A\overline{SB} = (A\overline{S})\overline{B} = SB\overline{B}$.
(2) If $S, iS \in C^{con}(A, B)$ then $A\overline{S} = SB$ and $-A\overline{S} = SB$, so $2SB = 0$ and $S = 0$ follows, as B is nonsingular.
(3) If $A\overline{S} = SB$ then $A\overline{A\overline{S}B^{-1}} = A\overline{A}S\overline{B^{-1}} = SB\overline{B}\overline{B} = SB$.
(4) If $A\overline{A}S = SB\overline{B}$ then $A\overline{A}A\overline{S}B^{-1} = A\overline{S}B^{-1}B\overline{B}$.
$\Leftrightarrow \overline{A}A\overline{S}B^{-1} = \overline{S}\overline{B} \Leftrightarrow A\overline{A}S = SB\overline{B}$.
(5) Since $S, A\overline{S}B^{-1} \in C(A\overline{A}, B\overline{B})$, it follows that

$$T = e^{i\theta}A\overline{S}B^{-1} + e^{-i\theta}S \in C(A\overline{A}, B\overline{B}),$$

as $C(A\overline{A}, B\overline{B})$ is a complex vector space. But

$$A\overline{T} = A\overline{(e^{i\theta}A\overline{S}B^{-1} + e^{-i\theta}S)} = e^{-i\theta}A\overline{A}S\overline{B^{-1}} + e^{i\theta}A\overline{S}$$

$$= e^{-i\theta}SB\overline{B}\overline{B^{-1}} + e^{i\theta}A\overline{S} = e^{-i\theta}SB + e^{i\theta}A\overline{S}B^{-1}B = (e^{-i\theta}S + e^{i\theta}A\overline{S}B^{-1})B = TB$$

and the conclusion follows. ☐

PROPOSITION 3.3. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$.*

1. ([5]) *If A and B are consimilar, then $A\overline{A}$ is similar to $B\overline{B}$.*
2. ([1],[5]) *If B is nonsingular and $A\overline{A}$ is similar to $B\overline{B}$, then A and B are consimilar.*

Proof. (1) If $S^{-1}A\overline{S} = B$, then $S^{-1}A\overline{A\overline{S}} = B\overline{B}$.
(2) (From [1]) Assume S is nonsingular and $S^{-1}A\overline{A}S = B\overline{B}$. Note that both A and B are nonsingular. From Lemma 3.2 we conclude that

$$A\overline{(e^{i\theta}A\overline{S}B^{-1} + e^{-i\theta}S)} = (e^{i\theta}A\overline{S}B^{-1} + e^{-i\theta}S)B$$

and so, if θ can be chosen such that $e^{i\theta}A\overline{S}B^{-1} + e^{-i\theta}S$ is nonsingular, the conclusion follows. But $e^{i\theta}A\overline{S}B^{-1} + e^{-i\theta}S$ is nonsingular if and only if

$$(e^{i\theta}A\overline{S}B^{-1} + e^{-i\theta}S)e^{-i\theta}S^{-1} = A\overline{S}B^{-1}S^{-1} + e^{-2i\theta}I$$

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is nonsingular, and this is the case for all but finitely many θ. $\square$

The singular matrices $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are not consimilar, but $A\overline{A}$ and $B\overline{B}$ are similar!

The proof of Lemma 3.2, shows that the real vector space $C^{con}(A,B)$ is a subset of the complex vector space $C(A\overline{A}, B\overline{B})$. They have something in common.

PROPOSITION 3.4. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be consimilar and nonsingular. Then the (real) dimension of $\mathrm{C}^{con}(A,B)$ is equal to the (complex) dimension of $\mathrm{C}(A\overline{A}, B\overline{B})$.*

Proof. We consider the complex vector space $C(A\overline{A}, B\overline{B})$ as a real vector space and we consider the map $T : C(A\overline{A}, B\overline{B}) \to C^{con}(A,B)$ defined by $T(S) = \frac{1}{2}A\overline{S}B^{-1} + \frac{1}{2}S$. (See Lemma 3.2 (5) and put $\theta = 0$ and add $\frac{1}{2}$.) As a map between real vector spaces, the map T is a linear map. Note that $S \in C^{con}(A,B)$ implies $T(S) = S$ and so T is a surjection. We also note $S \in$Kern(T) if and only if $A\overline{S}B^{-1} + S = 0 \Leftrightarrow A\overline{S} = -SB$. We conclude that $S \rightsquigarrow iS$ is an isomorphism from Kern(T) to $C^{con}(A,B)$, so $\dim(\mathrm{Kern}(T)) = \dim(C^{con}(A,B))$. The real dimension of $C(A\overline{A}, B\overline{B})$ is equal to $\dim(\mathrm{Range}(T)) + \dim(\mathrm{Kern}(T)) = 2\dim(C^{con}(A,B))$ and for the complex dimension we obtain $\dim(C(A\overline{A}, B\overline{B})) = \dim(C^{con}(A,B))$. $\square$

COROLLARY 3.5. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be consimilar and nonsingular. Then:*
(a) We have

$$\mathrm{C}^{con}(A,B) \oplus i\mathrm{C}^{con}(A,B) = \mathrm{C}(A\overline{A}, B\overline{B})$$

in the sense that for each $S \in \mathrm{C}(A\overline{A}, B\overline{B})$ there exist unique $S_1, T_1 \in \mathrm{C}^{con}(A,B)$ such that $S = S_1 + iT_1$.
(b) A subset of $T \subset \mathrm{C}^{con}(A,B)$ is real independent in $\mathrm{C}^{con}(A,B)$ (respectively, the (real) span is $\mathrm{C}^{con}(A,B)$) if and only if T is complex independent in $\mathrm{C}(A\overline{A}, B\overline{B})$ (respectively, the complex span is $\mathrm{C}(A\overline{A}, B\overline{B})$).

Proof. (a) This is the essence of the proof of Proposition 3.4.
(b) This is a direct corollary of (a). $\square$

COROLLARY 3.6.
(a) Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular. The following assertions are equivalent.
1. *$A\overline{A}$ is nonderogatory.*
2. $\dim(\mathrm{C}(A\overline{A})) = n$.
3. $\dim(\mathrm{C}^{con}(A,A)) = n$.

If $A\overline{A}$ is not nonderogatory, then $\dim(\mathrm{C}^{con}(A,A)) > n$.
(b) If $A, B \in \mathcal{M}_n(\mathbb{C})$ are nonsingular and consimilar, say $V^{-1}A\overline{V} = B$, then:
4. $\mathrm{C}^{con}(A,B) = \{SV : S \in \mathrm{C}^{con}(A,A)\}$,
5. *If $A\overline{A}$ is not nonderogatory, then $\dim(\mathrm{C}^{con}(A,B)) > n$,*

and the following assertions are equivalent.
6. *$A\overline{A}$ is nonderogatory.*
7. *$B\overline{B}$ is nonderogatory.*

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8. $\dim(\mathrm{C}(A\overline{A}, B\overline{B})) = n$.
9. $\dim(\mathrm{C}^{con}(A, B)) = n$.

Proof. (a) Note that $C(A\overline{A}) = C(A\overline{A}, A\overline{A})$. The equivalence of (1) and (2) is known (see [7], Theorem 4.4.17(d)). The equivalence of (2) and (3) was proved in Proposition 3.4.

Finally, [7], Theorem 4.4.17(d)) implies that if $A\overline{A}$ is not nonderogatory, then $\dim(C(A\overline{A}, A\overline{A})) > n$, and so $\dim(C^{con}(A, A)) > n$.
(b) This follows from part (a) and the observation that $C(A\overline{A}, B\overline{B})$ and $C(A\overline{A}, A\overline{A})$ are isomorphic (see the proof of Theorem 1.2). ◻

Example Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A\overline{A} = A^2 = A$ is nonderogatory and singular. Then: $C^{con}(A, A) = \{\left\{ \begin{bmatrix} r_1 & 0 \\ 0 & r_2 + ir_3 \end{bmatrix} : r_1, r_2, r_3 \in \mathbb{R} \right\}$ and $C(A\overline{A}, A\overline{A}) = \left\{ \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} : s_1, s_2 \in \mathbb{C} \right\}$, i.e., $\dim(C^{con}(A, A)) = 3$ and $\dim(C(A\overline{A}, A\overline{A})) = 2$. So, the nonsingularity of A in Proposition 3.4, Corollary 3.5 and Corollary 3.6 is essential.

If $A\overline{A}$ is nonderogatory, then Proposition 2.1 presents the following description of $C(A\overline{A}, A\overline{A})$:

$$C(A\overline{A}, A\overline{A}) = \{S : \ S \text{ is complex polynomial in } A\overline{A}\}.$$

We obtain the following description of $C^{con}(A, A)$.

Theorem 3.7. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular and suppose that $A\overline{A}$ is nonderogatory. If $S \in \mathcal{M}_n(\mathbb{C})$ then $S \in \mathrm{C}^{con}(A, A)$ if and only if there exists a real polynomial p such that $S = p(A\overline{A})$.*

Proof. Note that $S = A\overline{A} \in C^{con}(A\overline{A}, A\overline{A})$, and so is $(A\overline{A})^k$ (for all k) and therefore $p(A\overline{A}) \in C^{con}(A\overline{A}, A\overline{A})$, for all real polynomial p.

If $A\overline{S} = SA$ then $S \in C(A\overline{A}, A\overline{A})$ and since $A\overline{A}$ is nonderogatory, we know that $S = p(A\overline{A})$ for some polynomial p of degree strictly less than that of the minimal polynomial of $A\overline{A}$.

Write $p(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_k t^k$, where $k <$ degree $mp_{A\overline{A}}$. On the one hand:

$$SA = (\alpha_0 I + \alpha_1(A\overline{A}) + \cdots + \alpha_k(A\overline{A})^k)A = A(\alpha_0 I + \alpha_1(\overline{A}A) + \cdots + \alpha_k(\overline{A}A)^k).$$

On the other hand:

$$SA = A\overline{S} = A(\overline{\alpha_0} I + \overline{\alpha_1}(\overline{A}A) + \cdots + \overline{\alpha_k}(\overline{A}A)^k).$$

Therefore:

$$0 = SA - SA = A[(\alpha_0 - \overline{\alpha_0})I + (\alpha_1 - \overline{\alpha_1})(\overline{A}A) + \cdots + (\alpha_k - \overline{\alpha_k})(\overline{A}A)^k]$$

and as A is nonsingular, we obtain:

$$(\alpha_0 - \overline{\alpha_0})I + (\alpha_1 - \overline{\alpha_1})(\overline{A}A) + \cdots + (\alpha_k - \overline{\alpha_k})(\overline{A}A)^k = 0$$

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But k < degree $mp_{A\overline{A}}$ = degree $mp_{\overline{A}A}$ and so: $\alpha_i - \overline{\alpha_i} = 0$ $(i = 1, \dots, k)$ i.e., p is a real polynomial. □

REMARK 3.8. Consider $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$. Then $S \in C^{con}(A, A)$, but there is no real polynomial such that $S = p(A\overline{A})$. So the nonsingularity of A in Theorem 3.7 is necessary.

COROLLARY 3.9. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular and suppose that $A\overline{A}$ is nonderogatory and $\overline{A}A = A\overline{A}$. If $B \in \mathcal{M}_n(\mathbb{C})$ is consimilar to A via a real consimilarity then all consimilarities between A and B are real (and are therefore similarities).*

Proof. For $U, V \in \mathcal{M}_n(\mathbb{C})$ we obtain: if $UA\overline{U^{-1}} = B = VA\overline{V^{-1}}$ then $(V^{-1}U)A = A(\overline{V^{-1}U})$ and we obtain $V^{-1}U = p(A\overline{A})$, for some real polynomial p. So $U = Vp(A\overline{A})$ and since $A\overline{A} \in \mathcal{M}_n(\mathbb{R})$ we conclude: if $V \in \mathcal{M}_n(\mathbb{R})$ then $U \in \mathcal{M}_n(\mathbb{R})$. □

3.2. Consimilarities between A and A^*. We start with the standard Proposition in this particular case.

PROPOSITION 3.10. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then:*
1. *A is consimilar to a Hermitian matrix.*
2. *A is consimilar to a Hermitian matrix via a symmetric matrix.*
3. *A is consimilar to A^* via a symmetric matrix.*

Proof. The proof of the equivalence (1) ⇒ (3) ⇒ (2) ⇒ (1) is standard, and Theorem 3.1 implies that assertion (1) is correct. □

We conclude that the natural consimilarities between A and A^* are the symmetric ones. We can prove Theorem 1.4. Recall:

THEOREM 1.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*

2. *There exists a symmetric consimilarity $V \in \mathcal{GL}_n(\mathbb{C})$ between A and A^*.*
3. *If $A\overline{A}$ is nonderogatory then all $V \in \mathrm{C}^{con}(A, A^*)$ are symmetric.*
4. *If A is nonsingular and all $V \in \mathrm{C}^{con}(A, A^*)$ are symmetric, then $A\overline{A}$ is nonderogatory.*

Proof. Note that the first statement of Theorem 1.4 as stated in Section 1 is part of Proposition 3.10.

(2) Since every $A \in \mathcal{M}_n(\mathbb{C})$ is consimilar to a Hermitian matrix, Proposition 3.10 implies the existence of a symmetric consimilarity between A and A^*.

(3) Note that $A\overline{V} = VA^*$ implies $A\overline{A}V = VA^*\overline{A^*} = V(A\overline{A})^T$ and Theorem 1.3 implies that such a V is symmetric, provided $A\overline{A}$ is nonderogatory.

(3) If all $V \in C^{con}(A, A^*)$ are symmetric, then Corollary 3.5 implies that all $S \in C(A\overline{A}, (A\overline{A})^T)$ are symmetric. From Theorem 1.3 we can conclude that $A\overline{A}$ is nonderogatory. □

3.3. Consimilarities between A and A^T. We start with the standard Proposition in this particular case.

PROPOSITION 3.11. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following assertions are equivalent.*
1. *A is consimilar to a symmetric matrix.*

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2. *A is consimilar to a symmetric matrix via a Hermitian positive definite matrix.*
3. *A is consimilar to A^T via a Hermitian positive definite matrix.*
4. *A is consimilar to a diagonal matrix.*

Proof. The equivalence (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is standard.
(1) $\rightarrow$ (4) Every complex symmetric matrix is consimilar to a diagonal matrix, even via a unitary matrix (see [6], Theorem 4.4.4).
(4) $\rightarrow$ (1) Trivial. ◻

Not all $A \in \mathcal{M}_n(\mathbb{C})$ are consimilar to a diagonal matrix (see [6], Theorem 4.6.11). And so, not all $A \in \mathcal{M}_n(\mathbb{C})$ are consimilar to a symmetric matrix. But A and A^T are always consimilar and from the previous proposition one might suspect already the type of the natural consimilarities between A and A^T.

We can now prove Theorem 1.5.

Theorem 1.5. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*
1. *There exists a nonsingular Hermitian $V \in \mathrm{C}^{con}(A, A^T)$.*
2. *If A is nonsingular and $A\overline{A}$ is nonderogatory, then all $U \in \mathrm{C}^{con}(A, A^T)$ are Hermitian.*

Proof.
1. We construct a Hermitian $V \in \mathcal{M}_n(\mathbb{C})$ such that $V^{-1}A\overline{V} = A^T$. There exists $S \in \mathcal{M}_n(\mathbb{C})$ with $S^{-1}A\overline{S} = R \in \mathcal{M}_n(\mathbb{R})$, a real matrix. Next, there exists a *real* symmetric $T \in \mathcal{M}_n(\mathbb{R})$ with $T^{-1}RT = R^T$. Thus
$$T^{-1}S^{-1}A\overline{S}T = R^T = (S^{-1}A\overline{S})^T = S^*A^T(S^T)^{-1}.$$
We obtain:
$$((S^*)^{-1}T^{-1}S^{-1})A(\overline{S}TS^T) = A^T$$
and so
$$(STS^*)^{-1}A(\overline{STS^*}) = A^T.$$
Clearly, $V = STS^*$ is the required nonsingular Hermitian matrix.
2. Next we assume that A is nonsingular and $A\overline{A}$ is nonderogatory. Fix the Hermitian V with $V^{-1}A\overline{V} = A^T$. Note that
$$V^{-1}A\overline{A}V = A^T\overline{A}^T = (\overline{A}A)^T.$$
Let $U \in \mathcal{M}_n(\mathbb{C})$ be such that $A\overline{U} = UA^T$. We show that U is Hermitian. Note that $A(\overline{UV^{-1}}) = UA^T\overline{V^{-1}} = (UV^{-1})A$ and so $UV^{-1} = p(A\overline{A})$, for some real polynomial p, i.e., $U = p(A\overline{A})V$. So:
$$U = p(A\overline{A})V = VV^{-1}p(A\overline{A})V = Vp(V^{-1}A\overline{A}V) = Vp((\overline{A}A)^T)$$

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As p is a real polynomial we obtain $\overline{p(A\overline{A})} = p(\overline{A}A)$ and so:

$$U^* = (p(A\overline{A})V)^* = V^*\overline{p(A\overline{A})}^T = Vp((\overline{A}A)^T) = U$$

and we conclude that U is Hermitian. $\square$

REMARK 3.12. Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular. Note: $C(A\overline{A}, A^T\overline{A^T}) = C(A\overline{A}, (A\overline{A})^*)$. If we consider $C(A\overline{A}, (A\overline{A})^*)$ as a real vector space, then we conclude that both the subspace $H(A\overline{A}, (A\overline{A})^*)$, i.e., the subspace of all Hermitian $S \in C(A\overline{A}, (A\overline{A})^*)$ and the subspace $C^{con}(A, A^T)$ have the same dimension. We obtain:

THEOREM 3.13. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular. If $A\overline{A}$ is nonderogatory then:*

$$\mathrm{C}^{con}(A, A^T) = \mathrm{H}(A\overline{A}, (A\overline{A})^*).$$

Proof. Theorem 1.5 implies that $C^{con}(A, A^T) \subset H(A\overline{A}, (A\overline{A})^*)$ and since both vector spaces are n-dimensional, the conclusion follows. $\square$

If $A\overline{A}$ is not nonderogatory the situation is less clear to me. I cannot even answer the following question.

QUESTION. Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular. If $A\overline{A}$ is not nonderogatory, must $C^{con}(A, A^T)$ contain a non-Hermitian matrix?

Nonderogatory real matrices R such that R^2 is not nonderogatory might be especially interesting in connection to this question. Note that if $A\overline{A}$ is not nonderogatory and k is the degree of the minimal polynomial of $A\overline{A}$, then $\{S \in C^{con}(A, A^T) : S \text{ is Hermitian}\}$ has real dimension at least k. The example $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ shows that this particular dimension can be larger then k. For this example it can be checked that:

1. $C^{con}(A, A^T) = \left\{ \begin{bmatrix} a & ib \\ ic & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ has dimension 4,
2. $\{S \in C^{con}(A, A^T) : S \text{ is Hermitian}\} = \left\{ \begin{bmatrix} a & ib \\ -ib & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$ has dimension 3,
3. $C(A\overline{A}, A^T\overline{A^T}) = \mathcal{M}_4(\mathbb{C})$ has real dimension 8,
4. $H(A\overline{A}, A^T\overline{A^T}) = \left\{ \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ has dimension 4.

Another reason why real examples might be interesting is the following:

LEMMA 3.14. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be real and nonsingular and suppose that $A\overline{A} = A^2$ is nonderogatory. Then all consimilarity transformations between A and A^T are real and symmetric.*

Proof. Since $A^T = A^*$ the conditions on A imply by Theorem 1.4 and Theorem 1.5 that any $U \in \mathcal{M}_n(\mathbb{C})$ such that $U^{-1}A\overline{U} = A^T = A^*$ is both symmetric and Hermitian, so it is real symmetric. $\square$

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3.4. Consimilarities between A and $\overline{A}$. Once again we have the standard Proposition. Note that the standard Proposition contains Theorem 1.6.

PROPOSITION 3.15. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then*

1. *A is consimilar to a real matrix.*
2. *A is consimilar to a real matrix via a coninvolutory matrix.*
3. *A is consimilar to $\overline{A}$ via a coninvolutory matrix.*
4. *A is consimilar to $\overline{A}$.*

Proof. The proof of the equivalence (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is standard, and Theorem 3.1 implies that assertion (1) is correct; it also implies that statement (4) can be added to the standard Proposition. ◻

So coninvolutory consimilarities between A and $\overline{A}$ exist for sure. However, there exists no $A \in \mathcal{M}_n(\mathbb{C})$ with the property that all consimilarities between A and $\overline{A}$ are coninvolutory. Indeed, if S is a consimilarity between A and $\overline{A}$, so is αS ($\alpha \in \mathbb{R}$). But S coninvolutory does not imply that αS is coninvolutory.

REMARK 3.16. Here is another proof of Proposition 3.15, (1) $\rightarrow$ (2).
If $UA\overline{U^{-1}} = B \in \mathcal{M}_n(\mathbb{R})$ and $U = RE$ is the RE-decomposition of U (see [7] section 6.4, so $R \in \mathcal{M}_n(\mathbb{R})$ and E is coninvolutory) then $REAER^{-1} = B$. We conclude: $EAE = R^{-1}BR$ is real. So the coninvolutory matrices are a kind of "nucleus" for this type of consimilarities. Note that Proposition 2.7, (3) $\rightarrow$ (2) can be proved in the same way.

Note that $A \in C^{con}(A, \overline{A})$. Therefore we have the following description of $C^{con}(A, \overline{A})$, for nonsingular $A \in \mathcal{M}_n(\mathbb{C})$ such that $A\overline{A}$ is nonderogatory.

PROPOSITION 3.17. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular and such that $A\overline{A}$ is nonderogatory. Then $S \in \mathrm{C}^{con}(A, \overline{A})$ if and only if $S = p(A\overline{A})A$ for some real polynomial p.*

PROPOSITION 3.18. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and suppose that $A\overline{A}$ is nonderogatory.*

1. *If A is symmetric then all $U \in \mathcal{M}_n(\mathbb{C})$ such that $UA\overline{U^{-1}} = \overline{A}(= A^*)$ are symmetric.*
2. *If A is nonsingular and Hermitian then all $U \in \mathcal{M}_n(\mathbb{C})$ such that $UA\overline{U^{-1}} = \overline{A}(= A^T)$ are Hermitian.*

COROLLARY 3.19. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and suppose that $A\overline{A}$ is nonderogatory.*

1. *If A is symmetric then all coninvolutory $U \in \mathcal{M}_n(\mathbb{C})$ such that $UAU = \overline{A}$ are symmetric and unitary.*
2. *If A is nonsingular and Hermitian then all coninvolutory $U \in \mathcal{M}_n(\mathbb{C})$ such that $UAU = \overline{A}$ are Hermitian and orthogonal.*

Proof. This is a corollary of Proposition 3.18 and the fact that a symmetric coninvolutory U is unitary (since $U^{-1} = \overline{U} = U^*$), respectively, a Hermitian coninvolutory U is orthogonal (since $U^{-1} = \overline{U} = U^T$). ◻

3.5. On the history of these results on consimilarity. After we obtained the previous results on consimilarity the referee pointed our attention to [2]. In fact,

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this paper contains Theorem 1.4 and Theorem 1.5 as a corollary of a more general result. Their proof was quite technical and I am convinced the proof presented here is a proof on a more elementary level. Moreover, all the results from section 3 in [2] are in fact corollaries of Theorem 1.4 and Theorem 1.5.

Corollary 3.20. [2] *Let* $A, B, C \in \mathcal{M}_n(\mathbb{C})$.

1. *For the equation* $A\overline{X} - XA^T = C$ *the following are equivalent.*
   (a) $C^T = -C$ *and the equation is consistent.*
   (b) *The equation has a (nonsingular) Hermitian solution.*
2. *If moreover* $A\overline{A}$ *is nonsingular and nonderogatory then* $C^T = -C$ *and* $A\overline{X} - XA^T = C$ *is consistent implies that all solutions of this equation are Hermitian.*

Proof. (1) It is clear that (b) implies (a). To see that (a) implies (b), let X be a solution of the equation $A\overline{X} - XA^T = C$. The identity $C^T = -C$ implies that X^* is also a solution, and so $(X + X^*)/2$ is a Hermitian solution. If V is nonsingular and Hermitian such that $\overline{V^{-1}}AV = A^T$, then one can choose $\alpha \in \mathbb{R}$ such that $(X + X^*)/2 + \alpha V$ becomes a nonsingular Hermitian solution of $A\overline{X} - XA^T = C$.
(2) If X is a solution of the equation $A\overline{X}-XA^T = C$, then $A\overline{X}-XA^T = C = -C^T = -(X^*A^T-AX^T)$. So $A(\overline{X}-X^T)-(X-X^*)A^T = 0 \Rightarrow A(\overline{X}-X^T)-\overline{(\overline{X} - X^T)}A^T = 0$. Theorem 1.5 implies that $\overline{X} - X^T$ is Hermitian as $\overline{X} - X^T$ is skew-Hermitian we obtain that $\overline{X} - X^T = 0$, i.e., X is Hermitian. ◻

Corollary 3.21. [2] *Let* $A, B, C \in \mathcal{M}_n(\mathbb{C})$.

1. *For the equation* $A\overline{X} + XA^T = C$ *the following are equivalent.*
   (a) $C^T = -C$ *and the equation is consistent.*
   (b) *The equation has a (nonsingular) skew-Hermitian solution.*
2. *If moreover* $A\overline{A}$ *is nonsingular and nonderogatory then* $C^T = -C$ *and* $A\overline{X} + XA^T = C$ *is consistent implies that all solutions of this equation are skew-Hermitian.*

Proof. This follows from the observation that $UA = A^T\overline{U}$ if and only if $(iU)A = -A^T\overline{iU}$ and U is Hermitian if and only if iU is skew-Hermitian. The rest is as in the previous corollary. ◻

Likewise, the other statements in section 3 of [2] on the equations of type $A\overline{X} \pm XA^* = C$ are direct corollaries of Theorem 1.4.

4. On ^{T}congruences between A and A^T, A^*, or $\overline{A}$.

4.1. ^{T}congruences, the general part.

In [9] a canonical form for ^{T}congruence was discovered, the so called ^{T}congruence canonical form of a matrix. There are three types of ^{T}congruence canonical matrices: $J_n(\lambda)$ stands for an $n \times n$ Jordan block with eigenvalue λ. $H_{2n}(\mu)$ stands for the $2n \times 2n$ block matrix $H_{2n}(\mu) = \begin{bmatrix} 0 & I_n \\ J_n(\mu) & 0 \end{bmatrix}$

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and Γ_n stands for the $n \times n$ matrix:

$$\Gamma_n = \begin{bmatrix} 0 & & & & & (-1)^{n+1} \\ & & & & \nearrow & (-1)^n \\ & & & -1 & \nearrow & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix}.$$

THEOREM 4.1. [9] *A square complex matrix is ^{T}congruent to a direct sum of canonical matrices, $J_k(0)$, Γ_n and $H_{2n}(\mu)$, where $0 \neq \mu \neq (-1)^{n+1}$ (and μ can be replaced by μ^{-1}). This ^{T}congruence canonical form is uniquely determined up to permutations of summands.*

In [9] the following notation was introduced:

$$(A^{-1})^T = A^{-T},$$

and $A^{-T}A$ (or sometimes AA^{-T}) was called the ^{T}cosquare of A. The question whether nonsingular $A, B \in \mathcal{M}_n(\mathbb{C})$ are ^{T}congruent is in some sense completely determined by the following result from [9].

PROPOSITION 4.2. [9] *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be nonsingular. Then A, B are ^{T}congruent if and only if their ^{T}cosquares $A^{-T}A$ and $B^{-T}B$ are similar.*

Again we start with our standard Proposition.

PROPOSITION 4.3. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*

1. *The following assertions are equivalent.*
   (a) *A is ^{T}congruent to a Hermitian matrix.*
   (b) *A is ^{T}congruent to a Hermitian matrix via a coninvolutory matrix.*
   (c) *A is ^{T}congruent to A^* via a coninvolutory matrix.*
2. *The following assertions are equivalent.*
   (a) *A is ^{T}congruent to a real matrix.*
   (b) *A is ^{T}congruent to a real matrix via a coninvolutory matrix.*
   (c) *A is ^{T}congruent to $\overline{A}$ via a coninvolutory matrix.*
3. *A is ^{T}congruent to a symmetric matrix if and only if A is symmetric.*

Proof. We only prove part (2).

$(a) \to (c)$. If $S \in \mathcal{GL}_n(\mathbb{C})$ such that $SAS^T = R \in \mathcal{M}_n(\mathbb{R})$ then $SAS^T = R = \overline{R} = \overline{S}\,\overline{A}\,\overline{S}^T$. And so: $(\overline{S}^{-1}S)A(S^T(\overline{S}^T)^{-1}) = \overline{A} \Rightarrow (\overline{S}^{-1}S)A(\overline{S}^{-1}S)^T = \overline{A}$, i.e., A and $\overline{A}$ are ^{T}congruent via the the coninvolution $\overline{S}^{-1}S$.

$(c) \to (b)$. Let $J \in \mathcal{M}_n(\mathbb{C})$ be a coninvolution such that $JAJ^T = \overline{A}$. There exists a coninvolution $X \in \mathcal{M}_n(\mathbb{C})$ such that $X^2 = J$. Then: $X^2A(X^2)^T = \overline{A} \Rightarrow XAX^T = X^{-1}\overline{A}(X^T)^{-1} = \overline{X}\,\overline{A}\,\overline{A}^T$. This identity implies that $XAX^T \in \mathcal{M}_n(\mathbb{R})$ and A is ^{T}congruent to it via the coninvolution X.

$(b) \to (a)$ Trivial. $\square$

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Part of the main discussion is whether the statement: "A is ^{T}congruent to A^*" can be added to part (1) of Proposition 4.3, respectively, whether the statement: "A is ^{T}congruent to $\overline{A}$" can be added to part (2) of Proposition 4.3. And what can be said about the ^{T}congruence of A and A^T?

4.2. The ^{T}congruence of A and A^T. Although A is ^{T}congruent to a symmetric matrix if and only if A is symmetric, A and A^T are always ^{T}congruent. In fact:

THEOREM 4.4. (See [4],[14] or [3]) *Let $\mathbb{F}$ be a field and let $A \in \mathcal{M}_n(\mathbb{F})$. Then A and A^T are ^{T}congruent via an involution $J \in \mathcal{M}_n(\mathbb{F})$.*

We can now present a proof of Theorem 1.7.

THEOREM 1.7 Let $A \in \mathcal{M}_n(\mathbb{C})$.

1. ([3],[4] or[14]) A is ^{T}congruent to A^T via an involution.
2. If A is nonsingular and its ^{T}cosquare AA^{-T} is nonderogatory then all $S \in \mathcal{GL}_n(\mathbb{C})$ such that $SAS^T = A^T$ are involutions.

Proof.

(1) Theorem 4.4 implies the existence of an involution J such that $JAJ^T = A^T$.
(2) Assume $SAS^T = A^T$. Then $(S^T)^{-1}(A^{-1})^T S^{-1} = A^{-1}$ and so $SA(A^{-1})^T S^{-1} = A^T A^{-1}$. Since $A^T A^{-1} = AA^{-1}A^T A^{-1} = A[A(A^{-1})^T]^T A^{-1}$, we obtain:

$$(A^{-1}S)[A(A^{-1})^T](A^{-1}S)^{-1} = [A(A^{-1})^T]^T.$$

By Theorem 1.3 we conclude that $A^{-1}S$ is symmetric. But $A^{-1}S$ symmetric implies that $AS^T = SA^T$. The two identities $SAS^T = A^T$ and $S^{-1}AS^T = A^T$ imply, as A and S are nonsingular, that $S = S^{-1}$, i.e., S is an involution. $\square$

4.3. The ^{T}congruence of A and A^*, respectively, A and $\overline{A}$.

The list of ^{T}congruence canonical forms from [9] leads to the conclusion that A and A^*, respectively, A and $\overline{A}$, need not be ^{T}congruent.

EXAMPLE 1. The ^{T}congruence canonical form $H_2(2i) = \begin{bmatrix} 0 & 1 \\ 2i & 0 \end{bmatrix}$ is not ^{T}congruent to $\overline{H_2(2i)} = H_2(-2i)$, since this is a different ^{T}congruence canonical form. The ^{T}cosquare of $H_2(2i)$ is

$$\begin{bmatrix} 2i & 0 \\ 0 & -1/2i \end{bmatrix}$$

and the ^{T}cosquare of $H_2(2i)^*$ is

$$\begin{bmatrix} 1/2i & 0 \\ 0 & -2i \end{bmatrix},$$

which are not similar. So $H_2(2i)$ is not ^{T}congruent to $H_2(2i)^*$. Note that $H_2(2i)$ is ^{T}congruent to its transpose via the involution $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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But we can say the following:

LEMMA 4.5. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be ^{T}congruent.*

1. *If A and $\overline{A}$ are ^{T}congruent, so are B and $\overline{B}$.*
2. *If A and $\overline{A}$ are ^{T}congruent via a coninvolution, so are B and $\overline{B}$.*
3. *If A and A^* are ^{T}congruent, so are B and B^*.*
4. *If A and A^* are ^{T}congruent via a coninvolution, so are B and B^*.*

LEMMA 4.6. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following assertions are equivalent.*

1. *A and $\overline{A}$ are ^{T}congruent.*
2. *The ^{T}congruence canonical form of A is a direct sum of blocks of only the following five types.*
   (a) *$J_k(0)$,*
   (b) *Γ_k,*
   (c) *$H_{2k}(\mu)$ with $\mu \in \mathbb{R}, |u| > 1$,*
   (d) *$H_{2k}(\mu)$ with $\mu \in \mathbb{C}, |\mu| = 1, \mu \neq (-1)^k$ (μ can be replaced by $1/\mu = \overline{\mu}$),*
   (e) *$H_{2k}(\mu) \oplus H_{2k}(\overline{\mu})$ with $\mu \in \mathbb{C} - \mathbb{R}, |\mu| > 1$.*

Proof. (1) $\rightarrow$ (2) According to Lemma 4.5 we have to verify this equivalence only for matrices in ^{T}congruence canonical form. Note that the blocks $J_k(0)$, Γ_k, $H_{2k}(\mu)$ ($\mu \in \mathbb{R}$, $|\mu| > 1$) and $H_{2k}((-1)^{k+1})$ are real. If $H_{2k}(\mu)$ ($\mu \in \mathbb{C}$, $|\mu| > 1$) appears in the ^{T}congruence canonical form, so does $\overline{H_{2k}(\mu)} = H_{2k}(\overline{\mu})$, as these basic canonical forms are of different basic type (and we assume that A and $\overline{A}$ are ^{T}congruent.) However, if $|\mu| = 1$ this argument is incorrect, as $H_{2k}(\mu)$ has the same basic type as $H_{2k}(1/\mu) = H_{2k}(\overline{\mu}) = \overline{H_{2k}(\mu)}$.

(2) $\rightarrow$ (1). Becomes trivial using the previous final observation. □

PROPOSITION 4.7. ([10], Lemma 2.3)([11], Theorem 7) *Consider the Jordan block $J_n(\mu)$, $|\mu| = 1$ and $\mu \neq \pm 1$. Then there exists a Toeplitz matrix $C \in \mathcal{M}_n(\mathbb{C})$ such that $(C^*)^{-1}C = J_n(\mu)$.*

We can present a proof of Theorem 1.8. Recall:

THEOREM 1.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following assertions are equivalent.*

3. *A and $\overline{A}$ are ^{T}congruent,*
4. *A and $\overline{A}$ are ^{T}congruent via a coninvolution,*
5. *A and A^* are ^{T}congruent,*
6. *A and A^* are ^{T}congruent via a coninvolution.*

Proof. Note that once we have established the equivalence of (3), (4), (5) and (6), the equivalence of the other assertions in Theorem 1.8 of section 1 follows from the standard Proposition 4.3.

(3) $\leftrightarrow$ (5) This follows from Theorem 4.4 and Lemma 4.5, since $\overline{A}$ is ^{T}congruent to A^*.

(3) $\rightarrow$ (4) From Lemma 4.5 we conclude that we only have to prove this for matrices in ^{T}congruence canonical form. We may assume that A is a direct sum of the described ^{T}canonical blocks (a),...,(e) in Lemma 4.6.

Note that the ^{T}canonical blocks of type (a), (b) or (c) and $H_{2k}((-1)^{k+1})$ of type (d) are real, i.e., the real involution I has the property $IBI^T = \overline{B}$, for

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all these blocks.

Let $H_{2k}(\mu)$, $|\mu| = 1$ and $\mu \neq \pm 1$ be a canonical form of type (d). Use Proposition 4.7 to find a Toeplitz matrix $C \in \mathcal{M}_k(\mathbb{C})$ such that $(C^*)^{-1}C = J_n(\mu)$ and consider:

$$S = \begin{bmatrix} 0 & C^* \\ C^{-T} & 0 \end{bmatrix}.$$

S is a coninvolution, since $S\overline{S} = \begin{bmatrix} 0 & C^* \\ C^{-T} & 0 \end{bmatrix} \begin{bmatrix} 0 & C^T \\ C^{*-1} & 0 \end{bmatrix} = I$.

Moreover, $SH_{2k}(\mu)S^T = \begin{bmatrix} 0 & C^* \\ C^{-T} & 0 \end{bmatrix} \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix} \begin{bmatrix} 0 & C^{-1} \\ \overline{C} & 0 \end{bmatrix} =$ $\begin{bmatrix} 0 & C^* J_k(\mu) C^{-1} \\ C^{-T}\overline{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & C^* J_k(\mu) C^{-1} \\ \overline{(C^*)^{-1}C} & 0 \end{bmatrix} =$ $\begin{bmatrix} 0 & I \\ \overline{J_k(\mu)} & 0 \end{bmatrix} = \overline{H_{2k}(\mu)}$.

If a ^{T}canonical block of type (e) (i.e., $H_{2k}(\mu) \oplus H_{2k}(\overline{\mu})$ with $\mu \in \mathbb{C} - \mathbb{R}$, $|\mu| > 1$) appears in the direct sum, then

$$S = \begin{pmatrix} 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_k \\ I_k & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \end{pmatrix}$$

is a real involution (and so a coninvolution) such that

$$S(H_{2k}(\mu) \oplus H_{2k}(\overline{\mu}))S^T = H_{2k}(\overline{\mu}) \oplus H_{2k}(\mu) = \overline{H_{2k}(\mu) \oplus H_{2k}(\overline{\mu})}.$$

(4) → (3) Trivial, so we have obtained the equivalence of (1), (2) and (3).

(5) → (6) Assume A is ^{T}congruent to A^*. By the equivalence of (1), (2) and (3) and the standard Proposition 4.3 part (2) we conclude that $JAJ^T = R \in \mathcal{M}_n(\mathbb{R})$, for some coninvolution $J \in \mathcal{M}_n(\mathbb{C})$. By Theorem 4.4 there exists a real involution $S \in \mathcal{M}_n(\mathbb{R})$ such that:

$$SRS^T = R^T \Rightarrow SJAJ^TS^T = R^T = R^* = \overline{J}A^*J^*.$$

Thus,

$$(\overline{J}^{\,-1}SJ)A(J^TS^T(J^*)^{-1}) = A^* \Rightarrow (\overline{J}^{\,-1}SJ)A(\overline{J}^{\,-1}SJ)^T = A^*.$$

Finally we observe that $S \in \mathcal{M}_n(\mathbb{R})$ and $S = S^{-1}$ implies that $\overline{J}^{\,-1}SJ$ is coninvolutory.

(6) → (5) Trivial. □

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The rest of this subsection is devoted to the question: "Assume that A and A^* are ^{T}congruent. Under what conditions are all ^{T}congruences between A and A^* coninvolutory?"

One might hope that this condition is: "if AA^{-T} is nonderogatory." This is not the case, as we will see.

LEMMA 4.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular.*

1. *If p is a polynomial of degree less than n and we define G by $G = p(AA^{-T})$, then $GAG^T = A$ is equivalent to $p(AA^{-T})p((AA^{-T})^{-1}) = I_n$.*

Moreover, if AA^{-T} is nonderogatory then:

2. *If $G \in \mathcal{M}_n(\mathbb{C})$ is nonsingular and $GAG^T = A$, then there exists a unique polynomial of degree less than n, such that $G = p(AA^{-T})$.*

Proof.

1. Next we note that $GAG^T = A$ is equivalent to:
$p(AA^{-T})A(p(AA^{-T}))^T = A \Leftrightarrow p(AA^{-T})Ap((AA^{-T})^T) = A \Leftrightarrow$
$p(AA^{-T})Ap(A^{-1}A^T) = A \Leftrightarrow p(AA^{-T})Ap(A^{-1}A^T)A^{-1}A = A \Leftrightarrow$
$p(AA^{-T})p(AA^{-1}A^TA^{-1})A = A \Leftrightarrow p(AA^{-T})p(A^TA^{-1}) = I_n \Leftrightarrow$
$p(AA^{-T})p((AA^{-T})^{-1}) = I_n$.
2. $GAG^T = A$ implies $GAA^{-T}G^{-1} = AA^{-T}$ and as AA^{-T} is nonderogatory, we conclude that $G = p(AA^{-T})$ for some unique polynomial of degree less than n. $\square$

Lemma 4.8 provides us with many ^{T}congruences between A and A, for example $G = \pm(AA^{-T})^k$ $(k \in \mathbb{Z})$.

Next we observe the following. If $SAS^T = A^*$ then $SAA^{-T}S^{-1} = A^*(A^*)^{-T} = A^*(\overline{A})^{-1} = \overline{A}(\overline{A})^{-1}A^*(\overline{A})^{-1} = \overline{A}[AA^{-T}]^*(\overline{A})^{-1}$. And so:

$$((\overline{A})^{-1}S)[AA^{-T}]((\overline{A})^{-1}S)^{-1} = [AA^{-T}]^*.$$

LEMMA 4.9. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be nonsingular and assume $SAS^T = A^*$. Then S is coninvolutory if and only if $(\overline{A})^{-1}S$ is Hermitian.*

Proof. Note that $((\overline{A})^{-1}S)$ is the corresponding similarity between AA^{-T} and $[AA^{-T}]^*$.

Well: $(\overline{A})^{-1}S$ is Hermitian $\Leftrightarrow$ $((\overline{A})^{-1}S)^* = (\overline{A})^{-1}S \Leftrightarrow S^*A^{-T} = (\overline{A})^{-1}S$ $\Leftrightarrow$ $S^T(A^*)^{-1} = (\overline{A})^{-1}\overline{S} \Leftrightarrow$ $(\overline{S})^{-1}AS^T = A^*$. Together with the equation $SAS^T = A^*$ (and A, S nonsingular) this is equivalent to $S = (\overline{S})^{-1}$, i.e., S is coninvolutory. $\square$

PROPOSITION 4.10. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and assume that AA^{-T} is nonderogatory. Let J be a coninvolutory ^{T}congruence between A and A^*. The following assertions are equivalent.*

1. *All ^{T}congruences between A and A^* are coninvolutory.*
2. *Any $G \in \mathcal{M}_n(\mathbb{C})$ such that $GAG^T = A$ can be written as $G = p(AA^{-T})$, where p is a real polynomial of degree less than n.*

Proof. Note first that the coninvolutory ^{T}congruence J induces the similarity

$$((\overline{A})^{-1}J)[AA^{-T}]((\overline{A})^{-1}J)^{-1} = [AA^{-T}]^*,$$

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which is Hermitian.
(1) $\to$ (2) Fix $G \in \mathcal{M}_n(\mathbb{C})$ such that $GAG^T = A$. Then $S = JG$ has the property $SAS^T = A^*$, and is therefore, by assumption, coninvolutory. S induces the Hermitian similarity

$$((\overline{A})^{-1}S)[AA^{-T}]((\overline{A})^{-1}S)^{-1} = [AA^{-T}]^*.$$

By Theorem 1.2 we conclude: $(\overline{A})^{-1}S = (\overline{A})^{-1}Jp(AA^{-T})$ for some real polynomial p of degree less than n, and so $G = p(AA^{-T})$, as required.
(2) $\to$ (1) Fix $S \in \mathcal{M}_n(\mathbb{C})$ such that $SAS^T = A^*$. Find $G \in \mathcal{M}_n(\mathbb{C})$ with the property $GAG^T = A$ such that $S = JG$. (Of course, $G = J^{-1}S$.) By assumption $G = p(AA^{-T})$ for some real polynomial of degree less than n. For the induced similarities on AA^{-T} we obtain: $(\overline{A})^{-1}S = (\overline{A})^{-1}Jp(AA^{-T})$ for a real polynomial. By Theorem 1.2, $(\overline{A})^{-1}S$ is Hermitian, and so S is a coninvolution. □

Proposition 4.10 allows us to construct an example $A \in \mathcal{M}_n(\mathbb{C})$ such that A and A^* are ^{T}-congruent, AA^{-T} is nonderogatory, but A and A^* admit ^{T}congruences that are not coninvolutory.

Example 2. Define $A_1 = H_4(2i)$ and $A_2 = H_4(-2i)$ and consider $A = A_1 \oplus A_2 = H_4(2i) \oplus H_4(-2i) \in \mathcal{M}_8(\mathbb{C})$. The matrix AA^{-T} has four eigenvalues $\pm 2i, \pm 1/2i$, and has four 2×2 Jordan blocks with *different* eigenvalues. This implies that AA^{-T} is nonderogatory. Consider the 8×8 reversal matrix J; J is a real coninvolution and $JAJ^T = A^*$, as one can check.

Next define $H = A_1A_1^{-T} \oplus (A_2A_2^{-T})^2$. (Note that the different exponents imply that if you write $H = p(AA^{-T})$ then $\overline{p(2i)} \neq p(-2i)$, and so p cannot be a real polynomial.) One checks for $S = JH$ that $SAS^T = A^*$, but S is not coninvolutory.

In the same way one can show:

Corollary 4.11. *If $A = H_{2k}(\mu) \oplus H_{2k}(\overline{\mu})$ with $\mu \in \mathbb{C} - \mathbb{R}, |\mu| > 1$, then AA^{-T} is nonderogatory, but there exist ^{T}congruences between A and A^* that are not coninvolutory.*

Which $A \in \mathcal{M}_n(\mathbb{C})$ has the property that AA^{-T} is nonderogatory? Recall that B is nonderogatory if and only if different Jordan blocks in the Jordan normal form of B belong to different eigenvalues of B. Note that

1. $J_k(0)$ is not invertible,
2. the ^{T}cosquare of Γ_ℓ is nonderogatory, since $\Gamma_\ell\Gamma_\ell^{-T}$ is similar to $J_\ell((-1)^{\ell+1})$ (see [9]),
3. the ^{T}cosquare of $H_{2k}(\mu)$ with $\mu \in \mathbb{R}, |u| \geq 1, \mu \neq (-1)^{k+1}$ is similar to $J_k(\mu) \oplus J_k(1/\mu)$ (see [9]), so it is nonderogatory if and only if $\mu \neq (-1)^k$.

We conclude that AA^{-T} is nonderogatory if and only if the ^{T}congruence canonical blocks contains

1. at most one block of type Γ_ℓ and ℓ is odd,
2. at most one block of type Γ_ℓ and ℓ is even,

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3. for each μ such that $|\mu| > 1$ at most one block of type $H_{2k}(\mu)$,
4. for each μ such that $|\mu| = 1, \mu \notin \mathbb{R}$ at most one block of type $H_{2k}(\mu)$ (or $H_{2k}(\overline{\mu})$) (and so: no block of type $H_{2k}((-1)^k)$).

5. On *congruences between A and A^T, A^*, or $\overline{A}$.

5.1. *congruences, the general part. In [9] a canonical form for *congruence was discovered, the so called *congruence canonical form of a matrix. There are three types of *congruence canonical matrices: again $J_n(\lambda)$ stands for an $n \times n$ Jordan block with eigenvalue λ, $H_{2n}(\mu)$ stands for the $2n \times 2n$ block matrix $\begin{bmatrix} 0 & I_n \\ J_n(\mu) & 0 \end{bmatrix}$ and Δ_n stands for the $n \times n$ matrix:

$$\Delta_n = \begin{bmatrix} 0 & & & & 1 \\ & & & 1 & i \\ & & \nearrow & \nearrow & \\ & 1 & i & & \\ 1 & i & & & 0 \end{bmatrix}.$$

THEOREM 5.1. [9] *A square complex matrix is *congruent to a direct sum of canonical matrices, $J_k(0)$, $\lambda\Delta_n$ ($|\lambda| = 1$) and $H_{2n}(\mu)$ ($|\mu| > 1$). This *congruence canonical form is uniquely determined up to permutations of summands.*

The question whether $A, B \in \mathcal{M}_n(\mathbb{C})$ are *congruent can be handled in the same way as the ^{T}congruence. In [9] the notation $(A^{-1})^* = A^{-*}$ was introduced and $A^{-*}A$ (or sometimes AA^{-*}) was called the *cosquare of A. The question whether nonsingular $A, B \in \mathcal{M}_n(\mathbb{C})$ are ^{T}congruent is partially determined by the following result.

PROPOSITION 5.2. [9] *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be nonsingular. If A, B are *congruent then their *cosquares $A^{-*}A$ and $B^{-*}B$ are similar (but not conversely).*

The standard Proposition becomes:

PROPOSITION 5.3. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*
1. *The following assertions are equivalent.*
   (a) *A is *congruent to a symmetric matrix.*
   (b) *A is *congruent to a symmetric matrix via a coninvolutory matrix.*
   (c) *A is *congruent to A^T via a coninvolutory matrix.*
2. *The following assertions are equivalent.*
   (a) *A is *congruent to a real matrix.*
   (b) *A is *congruent to a real matrix via a coninvolutory matrix.*
   (c) *A is *congruent to $\overline{A}$ via a coninvolutory matrix.*
3. *A is *congruent to a Hermitian matrix if and only if A is Hermitian.*

5.2. The *congruence of A and A^T. We recall from the literature:

THEOREM 5.4. (See [8]) *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A and A^T are *congruent via a coninvolutory $J \in \mathcal{M}_n(\mathbb{C})$.*

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This result implies that the statement "A is *congruent to A^T" could be added to the list in part (1) of the standard Proposition 5.3, and it also implies that the statements in this list are not only equivalent. Each of the assertions is correct. Note that we have obtained the first two statements in Theorem 1.9. We still have to prove statement (3) of Theorem 1.9. Recall:

THEOREM 1.9. *Let $A \in \mathcal{M}_n(\mathbb{C})$.*

*3. If A is nonsingular and its *cosquare AA^{-*} is nonderogatory then all $S \in \mathcal{GL}_n(\mathbb{C})$ such that $SAS^* = A^T$ are coninvolutions.*

Proof. Assume $SAS^* = A^T$. Then

$$(S^*)^{-1}(A^{-1})^*S^{-1} = \overline{A^{-1}}$$

and so $SA(A^{-1})^*S^{-1} = A^T\overline{A^{-1}}$. Since

$$A^T\overline{A^{-1}} = \overline{A}\,\overline{A^{-1}}A^T\overline{A^{-1}} = \overline{A}[A(A^{-1})^*]^T\overline{A^{-1}},$$

we obtain:

$$(\overline{A^{-1}}S)[AA^{-*}](\overline{A^{-1}}S)^{-1} = [AA^{-*}]^T.$$

By Theorem 1.3 we conclude that $\overline{A^{-1}}S$ is symmetric. But $\overline{A^{-1}}S$ symmetric implies $\overline{A}S^T = SA^*$ and so $AS^* = \overline{S}A^T$. The two identities $SAS^* = A^T$ and $\overline{S^{-1}}AS^* = A^T$ imply, as A and S are nonsingular, that $S = \overline{S^{-1}}$, i.e., S is a coninvolution. ∎

5.3. The *congruence of A and A^*, respectively, A and $\overline{A}$. Theorem 5.4 implies the following lemma.

LEMMA 5.5. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A is *congruent to $\overline{A}$ if and only if A is *congruent to A^*.*

However, A and A^*, respectively, A and $\overline{A}$, need not be *congruent, as we can conclude from Proposition 5.2.

EXAMPLE 1. (Continuation 1). Again consider $H_2(2i) = \begin{bmatrix} 0 & 1 \\ 2i & 0 \end{bmatrix}$. $H_2(2i)$ is not *congruent to $\overline{H_2(2i)}$, since the *cosquare of $H_2(2i))$ is $\begin{bmatrix} 2i & 0 \\ 0 & 1/2i \end{bmatrix}$, which is not similar to $\begin{bmatrix} -2i & 0 \\ 0 & -1/2i \end{bmatrix}$, the *cosquare of $\overline{H_2(2i)}$.

The *cosquare of $H_2(2i)^*$ is

$$\begin{bmatrix} -1/2i & 0 \\ 0 & -2i \end{bmatrix},$$

which is not similar to the *cosquare of $H_2(2i)$, i.e., $H_2(2i)$ is not *congruent to $H_2(2i)^*$. Again we note that $H_2(2i)$ is *congruent to its transpose via the reversal matrix $S_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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The next question we discuss is whether the statement "A is *congruent to $\overline{A}$" could be added to the equivalence in the standard Proposition 5.3, part (2). First we note the following.

LEMMA 5.6. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ and assume that A, B are *congruent.*

1. *If A is *congruent to $\overline{A}$, then B is *congruent to $\overline{B}$.*
2. *If A is *congruent to $\overline{A}$ via a coninvolutory matrix, then B is *congruent to $\overline{B}$ via a coninvolutory matrix.*
3. *If A is *congruent to A^*, then B is *congruent to B^*.*
4. *If A is *congruent to A^* via an involution, then B is *congruent to B^* via an involution.*

Proof. (1) and (2). If $SAS^* = B$ and $JAJ^* = \overline{A}$, then $\overline{B} = (\overline{S}JS^{-1})B(\overline{S}JS^{-1})^*$. Moreover, if J is coninvolutory, so is $\overline{S}JS^{-1}$.
(3) and (4). If $SAS^* = B$ and $JAJ^* = A^*$, then $B^* = (SJS^{-1})B(SJS^{-1})^*$. Moreover, if J is an involution, so is SJS^{-1}. ☐

What does it mean for the *congruence canonical form that A and $\overline{A}$ (or A^*) are *congruent?

1. The matrices $J_k(0)$ and $H_{2m}(\mu)$ ($|\mu| > 1$ $\mu \in \mathbb{R}$) are real, so they are *congruent via the identity to their conjugates and ^{T}congruent to their transposes via a real involution, i.e., *congruent to their conjugate transpose via a real involution.
2. $\pm\Delta_\ell$ is *congruent to $\pm\overline{\Delta_\ell} = \pm{\Delta_\ell}^*$ via S =diag$(1, -1, 1, -1, \ldots)$, a real involution.
3. If $\lambda \notin \mathbb{R}$ and $|\lambda| = 1$ then $\lambda\Delta_\ell$ and $\overline{\lambda\Delta_\ell}$ are not *congruent. This follows from the observation that $\overline{\lambda\Delta_\ell} = \overline{\lambda}\,\overline{\Delta_\ell}$ is *congruent to $\overline{\lambda}\Delta_\ell$ by (2), which is not *congruent to $\lambda\Delta_\ell$, being of different *congruence canonical form type. And so: $\lambda\Delta_\ell$ and $(\lambda\Delta_\ell)^*$ are not *congruent.
4. $H_{2m}(\mu)$ ($|\mu| > 1$ $\mu \notin \mathbb{R}$) is not *congruent to its conjugate, since $\overline{H_{2m}(\mu)} = H_{2m}(\overline{\mu})$ is of a different *congruence canonical form type. And so: $H_{2m}(\mu)$ and $H_{2m}(\mu)^*$ are not *congruent.

We conclude:

COROLLARY 5.7. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following assertions are equivalent.*

1. *A is *congruent to $\overline{A}$ (or A^*).*
2. *A is *congruent to a direct sum of matrices of the following type*
   (a) $J_k(0)$,
   (b) Δ_ℓ *or* $-\Delta_\ell$,
   (c) $H_{2m}(\mu)$ ($|\mu| > 1$, $\mu \in \mathbb{R}$),
   (d) $H_{2m}(\mu) \oplus H_{2m}(\overline{\mu})$ ($|\mu| > 1$, $\mu \notin \mathbb{R}$),
   (e) $\lambda\Delta_\ell \oplus \overline{\lambda}\Delta_\ell$, ($|\lambda| = 1$, $\lambda \notin \mathbb{R}$).

(Note that type (e) uses the *congruence of $\overline{\lambda\Delta_\ell}$ and $\overline{\lambda}\Delta_\ell$.)

We prove Theorem 1.10. Recall:

THEOREM 1.10. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following assertions are equivalent.*

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*3. A and $\overline{A}$ are *congruent,*

*4. A and $\overline{A}$ are *congruent via a coninvolution,*

*5. A and A^* are *congruent,*

*6. A and A^* are *congruent via an involution.*

Proof. Note that once we have established the equivalence of $(3), (4), (5)$ and (6), the standard Proposition 5.3 implies the equivalence of the rest of the assertions of Theorem 1.10 of section 1.
$(3) \leftrightarrow (5)$. This is Lemma 5.5.
$(3) \rightarrow (4)$ and $(3) \rightarrow (5)$ By Corollary 5.7 we obtain that if A and $\overline{A}$ (or A^*) are *congruent, a *congruence canonical form of A becomes a direct sum of matrices of the 5 types described in Corollary 5.7. By Lemma 5.6 it suffices to show that each of these basic types is *congruent to its conjugate via a coninvolution, respectively, is *congruent to its conjugate transposes via an involution.

We already have seen the first 3 types to be *congruent with their conjugate via a coninvolution, respectively, to be *congruent with their conjugate transpose via a real involution.

In the proof of Theorem 1.8, part (2), we have shown that type 4, $H_{2m}(\mu) \oplus H_{2m}(\overline{\mu})$, is ^{T}congruent to its conjugate (respectively, conjugate transpose) via a real involution, and so these are *congruent via a real involution.

Finally, type 5: $\lambda\Delta_\ell \oplus \overline{\lambda}\Delta_\ell$, $(|\lambda| = 1,\, \lambda \notin \mathbb{R})$ is *congruent to its conjugate (= its conjugate transpose) via the real involution $J = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix}$
$(4) \rightarrow (3)$ and $(5) \rightarrow (4)$ are trivial.
The following proof of $(5) \rightarrow (6)$ might be interesting. If A is *congruent to A^* then the equivalence of $(3), (4)$ and (5) and the standard Proposition 5.3, part (2) imply that $JAJ^* = R \in \mathcal{M}_n(\mathbb{R})$, for some coninvolution J. And so, Theorem 4.4 implies that there exists a real involution S such that $SRS^T = R^T$. Therefore,

$$SJAJ^*S^T = R^T = R^* = JA^*J^* \Rightarrow (J^{-1}SJ)A(J^*S^T(J^*)^{-1}) = A^*.$$

Thus $(J^{-1}SJ)A(J^{-1}SJ)^* = A^*$, since $S \in \mathcal{M}_n(\mathbb{R})$. We observe that $J^{-1}SJ$ is an involution. ◻

In the same way as in section 4.3 on ^{T}congruences, one can show the following: Every *congruence $SAS^* = A^*$ induces a similarity

$$(A^{-*}S)AA^{-*}(A^{-*}S)^{-1} = (AA^{-*})^*.$$

Lemma 5.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and assume that A and A^* are *congruent. The following assertions are equivalent for any nonsingular S such that $SAS^* = A^*$.*

*1. S is an involutory *congruence between A and A^*.*

2. $A^{-*}S$ is a Hermitian similarity between AA^{-*} and $(AA^{-*})^*$.

Therefore, we obtain the following equivalence.

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LEMMA 5.9. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and assume that AA^{-*} is nonderogatory. Assume J is an involutory *congruence between A and A^*. For any $*$congruence S such that $SAS^* = A^*$ the following assertions are equivalent.*

1. *S is involutory.*
2. *$S = Jp(AA^{-*})$, where p is a real polynomial.*

Finally we mention that there exists no $A \in \mathcal{M}_n(\mathbb{C})$ with the property that all *congruences between A and A^* are involutions. Indeed, if $JAJ^* = A^*$ is a *congruence between A and A^*, so is αJ, provided $|\alpha| = 1$. But αJ need not be an involution.

Dedication. This article is dedicated to the three brothers, Hans (Sr), Jan and Robert Vermeer, and is written in the hope that they too will meet again.

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