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ON THE PERRON-FROBENIUS THEORY OF M_V-MATRICES AND EQUIVALENT PROPERTIES TO EVENTUALLY EXPONENTIALLY NONNEGATIVE MATRICES*

THANIPORN CHAYSRI† AND DIMITRIOS NOUTSOS†

Abstract. M_v-matrix is a matrix of the form $A = sI - B$, where $0 \le \rho(B) \le s$ and B is an eventually nonnegative matrix. In this paper, M_v-matrices concerning the Perron-Frobenius theory are studied. Specifically, sufficient and necessary conditions for an M_v-matrix to have positive left and right eigenvectors corresponding to its eigenvalue with smallest real part without considering or not if $index_0 B \le 1$ are stated and proven. Moreover, analogous conditions for eventually nonnegative matrices or M_v-matrices to have all the non Perron eigenvectors or generalized eigenvectors not being nonnegative are studied. Then, equivalent properties of eventually exponentially nonnegative matrices and M_v-matrices are presented. Various numerical examples are given to support our theoretical findings.

Key words. $M-$matrices, M_v-matrices, Eventually exponentially nonnegative matrices, Perron-Frobenius theory.

AMS subject classifications. 65F10, 15A48.

1. Introduction. The Perron-Frobenius theory was established by Perron [17] in 1907, who proved that the dominant eigenvalue of an entry-wise positive matrix is positive and its corresponding eigenvector is positive, and later by Frobenius [7] in 1912, who extended it to irreducible nonnegative matrices. Since then the well-known Perron-Frobenius theory was studied by many researchers. Extensions and generalizations to the Perron-Frobenius theory were given by Friedland [6], Eschenbach and Johnson [5], Tarazaga et al. [18], Naqvi and McDonald [12], Maroulas et al. [11], Johnson and Tarazaga [9], Le and McDonald [10], Elhashash and Szyld [4], Gao [8], etc.

In 2006, Noutsos [13] extended the Perron-Frobenius theory by introducing the definitions of the Perron-Frobenius property and the strong Perron-Frobenius property and connected matrices having these properties with eventually positive and eventually nonnegative matrices. Later in 2012, this theory was extended into complex matrices by Noutsos and Varga [15].

The class of $M-$matrices is that of matrices of the form $A = sI - B$, where B is entrywise nonnegative ($B \ge 0$) and $0 \le \rho(B) \le s$. Pseudo $M-$matrices are of the form $A = sI - B$, where $0 < \rho(B) < s$ and B being an eventually positive matrix. They were introduced by Johnson and Tarazaga [9] in 2004. Next, the term M_v-matrix was introduced and studied by Olesky et al.[16] in 2006 for matrices of the form $A = sI - B$, where $0 \le \rho(B) \le s$ and B is an eventually nonnegative matrix. Finally, the class of generalized $M-$matrices or $GM-$matrices was studied by Elhashash and Szyld [3] in 2008 and contains matrices of the form $A = sI - B$, where $0 < \rho(B) \le s$ and both B, B^T possess the Perron-Frobenius property. From the definitions above, the class of $M-$matrices is a subclass of M_v-matrices and the class of pseudo $M-$matrices is also a subclass of M_v-matrices; however, an $M-$matrix may not be a pseudo $M-$matrix. The class of M_v-matrices is also a subclass of $GM-$matrices because, for every B eventually nonnegative, both B and B^T possess the Perron-Frobenius property (see [13, Theorem 2.3]).

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†Department of Mathematics, University of Ioannina, Ioannina 45110, Greece (tchaysri@cc.uoi.gr, dnoutsos@uoi.gr).

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Extensions of $M-$matrices are applied in many fields such as in mathematics (iterative methods, discretizations of differential operators), economics (gross substitutability, stability of a general equilibrium and Leontief's input-output analysis in economic systems), optimization, Markov chains in the field of probability theory and operation research like queuing theory, engineering (control theory) and also biology (population dynamics).

Many equivalent properties that characterize M-matrices were stated and proven by many researchers. In the book of Berman and Plemmons [1], over than 70 such properties are presented not all of which are valid for M_v-matrices. In this paper, we study the M_v-matrices in connection with the Perron-Frobenius theory. Specifically, sufficient conditions for an M_v-matrix with $index_0 B \leq 1$ to have positive left and right eigenvectors corresponding to its eigenvalue with smallest real part are studied. Also, sufficient and necessary conditions are proven without considering that $index_0 B \leq 1$. Then analogous properties of such class of matrices having all non Perron eigenvectors and generalized eigenvectors not being nonnegative are presented and proven. Finally, we give equivalent properties of eventually exponentially nonnegative matrices and M_v-matrices.

This work is organized as follows: In Section 2, we present the main notation that is used in this paper as well as some definitions and preliminary results. In Section 3, we present the main result of the Perron-Frobenius theory for M_v-matrices. In Section 4, we apply the result from the previous section to prove the equivalent properties of eventually exponentially nonnegative matrices and M_v-matrices. Finally, in Section 5, we summarize all our results. We also give various numerical examples to confirm our theoretical findings.

2. Notation, definitions and preliminaries. Let $A \in \mathbb{R}^{n,n}$ be a square matrix and let $\lambda_i \in \mathbb{C}$ be the eigenvalues of A. Then,

- $\sigma(A) := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is called the *spectrum* of the matrix A;
- $\rho(A) := \max\limits_{i=1(1)n} |\lambda_i|$ is called the *spectral radius* of the matrix A;
- λ is called a *dominant eigenvalue* of the matrix A if $|\lambda| = \rho(A)$;
- $\lambda \in \sigma(A)$ is called the *strictly dominant eigenvalue* of the matrix A if $|\lambda| > |\mu|$, $\forall \mu \in \sigma(A), \mu \neq \lambda$;
- $index_\lambda(A)$ denotes the degree of λ as a root of the minimal polynomial of the matrix A.

Let $A \in \mathbb{R}^{n,n}$ be a square matrix partitioned as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{2.1}$$

Then, by

$$\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}$$

we denote a block matrix of A^k partitioned conformably to (2.1).

Definition 2.1. A matrix $A \in \mathbb{C}^{n,n}$ is called *reducible* matrix if there exists a permutation matrix $P \in \mathbb{R}^{n,n}$ such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{2.2}$$

where $A_{11} \in \mathbb{C}^{r,r}$, $A_{22} \in \mathbb{C}^{n-r,n-r}$ and $A_{12} \in \mathbb{C}^{r,n-r}$, $0 < r < n$. Otherwise, A is called *irreducible*.

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Definition 2.2. A matrix $A \in \mathbb{R}^{n,n}$ is called

- *positive*, denoted by $A > 0$, if A is entrywise positive;
- *nonnegative*, denoted by $A \geq 0$, if A is entrywise nonnegative;
- *primitive* if $A \geq 0$ and there exists a positive integer k such that $A^k > 0$;
- *cyclic of index* $k > 1$ if $A \geq 0$ and there exists a permutation matrix $P \in \mathbb{R}^{n,n}$ such that PAP^T is partitioned in the form:

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{11} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.3}$$

  where all the diagonal blocks are square zero matrices;
- *weakly cyclic of index* $k > 1$ if there exists a permutation matrix $P \in \mathbb{R}^{n,n}$ such that PAP^T is partitioned as in (2.3);
- *eventually nonnegative (positive)*, denoted by $A \overset{\text{v}}{\geq} 0$ ($A \overset{\text{v}}{>} 0$), if there exists an integer $k_0 > 0$ such that $A^k \geq 0$ ($A^k > 0$) for all $k \geq k_0$; the smallest such positive integer is called the *power index* of A;
- *exponentially nonnegative (positive)* if for all $t > 0$, $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \geq 0$ ($e^{tA} > 0$);
- *eventually exponentially nonnegative (positive)* if there exists $t_0 \in [0, \infty)$ such that for all $t > t_0$, $e^{tA} \geq 0$ ($e^{tA} > 0$). The smallest such nonnegative number is called the *exponential index* of A.

Definition 2.3 ([13]). A matrix $A \in \mathbb{R}^{n,n}$ possesses

- the *Perron-Frobenius property* if it has a positive dominant eigenvalue $\lambda_1 > 0$ and the corresponding eigenvector $x^{(1)} \geq 0$;
- the *strong Perron-Frobenius property* if it has a positive strictly dominant eigenvalue $\lambda_1 > 0$, $\lambda_1 > |\lambda_i|$, $i = 2, 3, \ldots, n$, and the corresponding eigenvector $x^{(1)} > 0$.

Theorem 2.4 (Perron-Frobenius). *Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,*

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;
(ii) to $\rho(A)$ there corresponds an eigenvector $x > 0$;
(iii) $\rho(A)$ increases when any entry of A increases;
(iv) $\rho(A)$ is a simple eigenvalue of A;
(v) all nonnegative eigenvectors of A are multiples of x.

Theorem 2.5 ([13], Theorem 2.3). *Let $A \in \mathbb{R}^{n,n}$ be an eventually nonnegative matrix which is not nilpotent. Then, both matrices A and A^T possess the Perron-Frobenius property.*

3. Eigenvectors of M_v-matrices.

We will study the eigenvalues and eigenvectors of M_v-matrices, e.g., matrices that are based on eventually nonnegative matrices.

Theorem 3.1. *Let A be an irreducible M_v-matrix, written in the form $sI - B$ with $B \overset{\text{v}}{\geq} 0$, $0 \leq \rho(B) \leq s$ and $\text{index}_0 B \leq 1$. Then, to the smallest real eigenvalue $\lambda_1 \geq 0$ of A there correspond positive right and left eigenvectors. Moreover, $\lambda_1 < \operatorname{Re} \lambda_i$, $i = 2, 3, \ldots, n$.*

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Proof. Suppose that $\mu_1 = \rho(B) \geq |\mu_2| \geq \cdots \geq |\mu_n|$ are the eigenvalues of B and let k_0 is the power index of B. Since B is irreducible and $index_0 B \leq 1$, from [12, Theorem 3.4] we obtain that there exist integers $k \geq k_0$ such that B^k is irreducible and nonnegative. This means, see also [2, Proposition 2.1], either:

1. B^k is a primitive matrix, for all $k \geq k_0$, which means that $\rho(B^k)$ is a simple eigenvalue of B^k, for all $k \geq k_0$, implying that $\rho(B)$ is a simple one of B.
2. B^k is a nonnegative cyclic matrix of index r, for $k \geq k_0$, $k \neq mr$, m integer, implying that $\rho(B^k)$ is a simple eigenvalue of B^k and therefore $\rho(B)$ is a simple one of B.

In both cases, the right and left Perron eigenvectors of B^k are positive and so are the ones of B. For the other eigenvalues of B there hold $\operatorname{Re}\mu_i < \rho(B) = \mu_1$, $i = 2, 3, \ldots, n$. Thus, for the eigenvalues of $A = sI - B$ there hold $\lambda_1 = s - \rho(B) < \operatorname{Re}\lambda_i = s - \operatorname{Re}\mu_i$, $i = 2, 3, \ldots, n$. Obviously, to λ_1 there correspond the same right and left eigenvectors. ∎

EXAMPLE 3.2. (See [14, Example 3.11]) We consider the matrix

$$A = 3I - B, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \overset{\text{v}}{\geq} 0.$$

B is an irreducible matrix with $index_0 B = 2$. All powers $B^k, k \geq 2$ become reducible and Theorem 3.1 does not hold: $\rho(B) = 2$ is a double eigenvalue and the right and left eigenvectors are both nonnegative and not positive.

COROLLARY 3.3. *Let A be an irreducible symmetric M_v-matrix. Then, its smallest real eigenvalue $\lambda_1 \geq 0$ is a simple one, and the corresponding eigenvector is positive.*

Proof. In view of the symmetry, we get that $index_0 B \leq 1$ and the assumptions of Theorem 3.1 hold true. ∎

We have to remark that the assumption $index_0 B \leq 1$ is sufficient and not necessary. This is shown in the following example.

EXAMPLE 3.4. Consider

$$A = 3I - B, \quad B = \begin{bmatrix} 0.0163 & -0.2113 & 0.6667 & 0.2887 & 0.5163 \\ 0.183 & 0.5 & 0 & 1 & 0.6830 \\ 0.6667 & 0.5774 & 0.3335 & 0.5774 & 0.6667 \\ 0.3943 & 0.5 & 1.1547 & 0 & -0.1057 \\ -0.3497 & 0.7887 & 0.6667 & 0.2887 & -0.8497 \end{bmatrix} \overset{\text{v}}{\geq} 0.$$

B is irreducible with $index_0 B = 2$. However, $\rho(B) = 2$ is a simple eigenvalue, both right and left eigenvectors of B are positive: $x = y = (0.2887 \ \ 0.5 \ \ 0.5774 \ \ 0.5 \ \ 0.2887)^T$. B^k is a positive matrix for $k \geq 4$,

$$B^4 = \begin{bmatrix} 0.7622 & 2.56 & 3.4887 & 2.06 & 0.2622 \\ 1.4047 & 4.2504 & 5.7741 & 3.7504 & 0.9047 \\ 2.3339 & 4.6201 & 5.6683 & 4.6201 & 2.3339 \\ 3.2144 & 3.7506 & 3.465 & 4.2506 & 3.7144 \\ 2.5717 & 2.0598 & 1.1792 & 2.5598 & 3.0717 \end{bmatrix}.$$

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The following theorem shows the equivalent conditions for irreducible M_v-matrices which may have positive right and left eigenvectors corresponding to the smallest eigenvalue.

THEOREM 3.5. *Let A be an irreducible M_v-matrix, written in the form $sI-B$ with $B \overset{\vee}{\geq} 0$, $0 \leq \rho(B) \leq s$. Then, the following statements are equivalent:*

(i) There exists $\alpha > 0$ such that $B + \alpha I \overset{\vee}{\geq} 0$.
(ii) $B + \alpha I \overset{\vee}{>} 0$ for all $\alpha > 0$.
(iii) The smallest real eigenvalue $\lambda_1 \geq 0$ of A is simple, to λ_1 correspond positive right and left eigenvectors and $\lambda_1 < \operatorname{Re}\lambda_i$, $i = 2, 3, \ldots, n$.

Proof. $(ii) \Rightarrow (i)$: Holds trivially.

$(ii) \Rightarrow (iii)$: Since $B + \alpha I \overset{\vee}{>} 0$ for all $\alpha > 0$, it has positive right and left eigenvectors and obviously statement (iii) holds true.

$(iii) \Rightarrow (ii)$: Since the right and left eigenvectors of λ_1 are positive, so are the Perron eigenvectors of B and therefore of $B + \alpha I$ for all $\alpha > 0$. Since $B \overset{\vee}{\geq} 0$ and λ_1 is simple, for the eigenvalues of B there hold $\rho(B) = \mu_1 \geq |\mu_2| \geq |\mu_3| \geq \cdots \geq |\mu_n|$ and $\mu_1 > \operatorname{Re}\mu_2$. For any $\alpha > 0$, the eigenvalues of $B + \alpha I$ are $\mu_i + \alpha$, $i = 1, 2, 3, \ldots, n$. Then, $|\mu_2 + \alpha|^2 = (\operatorname{Re}\mu_2 + \alpha)^2 + (\operatorname{Im}\mu_2)^2 = (\operatorname{Re}\mu_2)^2 + 2\alpha\operatorname{Re}\mu_2 + \alpha^2 + (\operatorname{Im}\mu_2)^2 = |\mu_2|^2 + 2\alpha\operatorname{Re}\mu_2 + \alpha^2 < \mu_1^2 + 2\alpha\mu_1 + \alpha^2 = (\mu_1 + \alpha)^2$. Thus, $\mu_1 + \alpha > |\mu_2 + \alpha|$ which means that $B + \alpha I$ and $B^T + \alpha I$ have the strong Perron-Frobenius property, implying that, [13, Theorem 2.2], $B + \alpha I \overset{\vee}{>} 0$.

To complete the proof, we only need to show (i) implies (iii). Suppose (i) holds. Then we distinguish two cases.

Case 1: $index_0 B \leq 1$ or $index_0(B + \alpha I) \leq 1$.

Then, $B^k \geq 0$ or $(B + \alpha I)^k \geq 0$ remains irreducible for all $k = rm + 1 \geq k_0$, where r is the index of cyclicity of B ($r = 1$ if B is primitive) and $k_0 = \max\{k_0(B),\ k_0(B + \alpha I)\}$. This means that the right and left Perron eigenvectors of B^k or $(B + \alpha I)^k$ are positive. But these eigenvectors are also the Perron eigenvectors of B. Therefore, statement (iii) holds true.

Case 2: $index_0 B \geq 2$ and $index_0(B + \alpha I) \geq 2$.

Let $r_0 = index_0(B)$, $r_\alpha = index_0(B+\alpha I)$, k_0 is the power index of B and k_α the power index of $B+\alpha I$. If B^k is an irreducible matrix for some $k \geq k_0$ or $(B + \alpha I)^k$ is irreducible for some $k \geq k_\alpha$, then, obviously B has positive right and left Perron vectors and statement (iii) holds true. Thus, we suppose that B^k and $(B + \alpha I)^k$ are reducible matrices for all $k \geq k_0$ and $k \geq k_\alpha$, respectively. First, we will prove that B^k and $(B + \alpha I)^k$ do not have the same Frobenius normal form. Looking for a contradiction, we suppose these two matrices have the same Frobenius normal form. For simplicity, we assume that B^k and $(B+\alpha I)^k$ are in their reducible form: $B^k = \begin{bmatrix} B_{11}^{(k)} & B_{12}^{(k)} \\ 0 & B_{22}^{(k)} \end{bmatrix}$ and $(B+\alpha I)^k = \begin{bmatrix} (B_\alpha)_{11}^{(k)} & (B_\alpha)_{12}^{(k)} \\ 0 & (B_\alpha)_{22}^{(k)} \end{bmatrix}$, where $B_{11}^{(k)}, (B_\alpha)_{11}^{(k)} \in \mathbb{R}^{m,m}$ and $B_{22}^{(k)}, (B_\alpha)_{22}^{(k)} \in \mathbb{R}^{n-m,n-m}$. On the other hand, we have that

$$(B + \alpha I)^k = \alpha^k I + \binom{k}{1}\alpha^{k-1}B + \binom{k}{2}\alpha^{k-2}B^2 + \cdots + B^k. \qquad (3.4)$$

For each row index $i = m+1, m+2, \ldots, n$ and column index $j = 1, 2, \ldots, m$ that correspond to zero entries

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of both matrices, relation (3.4) takes the form

$$((B+\alpha I)^k)_{ij} = \binom{k}{1}\alpha^{k-1}b_{ij} + \binom{k}{2}\alpha^{k-2}(B^2)_{ij} + \cdots + \binom{k}{r_0-1}\alpha^{k-r_0+1}(B^{r_0-1})_{ij} = 0, \tag{3.5}$$

for all $k \geq \max\{r_0, k_\alpha\}$.

Taking $r_0 - 1$ successive values of k; i.e., $k, k+1, \ldots, k+r_0-2$, we get the linear system

$$\begin{array}{ccccccc} \binom{k}{1}\alpha^{k-1}b_{ij} & + & \binom{k}{2}\alpha^{k-2}(B^2)_{ij} & + \cdots + & \binom{k}{r_0-1}\alpha^{k-r_0+1}(B^{r_0-1})_{ij} & = & 0, \\ \binom{k+1}{1}\alpha^{k}b_{ij} & + & \binom{k+1}{2}\alpha^{k-1}(B^2)_{ij} & + \cdots + & \binom{k+1}{r_0-1}\alpha^{k-r_0+2}(B^{r_0-1})_{ij} & = & 0, \\ \vdots & & \vdots & & \vdots & & \vdots \\ \binom{k+r_0-2}{1}\alpha^{k+r_0-3}b_{ij} & + & \binom{k+r_0-2}{2}\alpha^{k+r_0-4}(B^2)_{ij} & + \cdots + & \binom{k+r_0-2}{r_0-1}\alpha^{k-1}(B^{r_0-1})_{ij} & = & 0, \end{array} \tag{3.6}$$

considering as unknown vector: $\begin{pmatrix} b_{ij} & (B^2)_{ij} & \cdots & (B^{r_0-1})_{ij} \end{pmatrix}^T$. The coefficient matrix is a Vandermonde type matrix, and thus, it is a nonsingular one. Obviously, this system has the unique solution of zeros. This means that $b_{ij} = 0$, and this happens for all $i = m+1, m+2, \ldots, n$ and $j = 1, 2, 3, \ldots, m$. Thus, the matrix B is a reducible matrix which constitutes a contradiction.

Now we consider the matrix

$$C^{(k)} = B^k + (B+\alpha I)^k \tag{3.7}$$

for some $k \geq \max\{r_0, r_\alpha, k_0, k_\alpha\}$. This matrix is a nonnegative irreducible one, since otherwise B^k and $(B+\alpha I)^k$ should have the same reducible form and we arrive at the same contradiction. Since $C^{(k)}$ is a polynomial of B, it has the same eigenvectors of B, Thus, the Perron right and left eigenvectors of B are those of $C^{(k)}$, which are positive vectors, proving the validity of statement (iii), and the proof is complete. □

The above theorem does not hold for $GM-$matrices. From the definition of $GM-$matrices and [13] we obtain that any $GM-$matrix (which is not an M_v-matrix) may have nonnegative eigenvector corresponding to its spectral radius (see [3], Example 2.2).

The following examples show that if a matrix has $index_0 B \geq 2$ but there exists $\alpha > 0$ such that $B + \alpha I \overset{v}{\geq} 0$, Theorem 3.5 is valid.

Example 3.6. Consider

$$A = 14I - B, \quad B = \begin{bmatrix} 4 & -3 & 15 & 1 & 2 & 4 \\ 1 & -1 & 7 & 1 & 1 & 1 \\ 1.5 & 1 & 3 & 1.5 & 1.5 & 1.5 \\ 2 & 1 & 16 & 2 & 2 & 1 \\ 1.5 & -1 & 1 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 \end{bmatrix} \overset{v}{\geq} 0.$$

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B is irreducible with $index_0 B = 2$. B^k is a positive matrix for $k \geq 3$,

$$B^3 = \begin{bmatrix} 554 & 21.75 & 1757.25 & 414.5 & 461 & 519.5 \\ 233.75 & 10.75 & 756.25 & 178.25 & 196.75 & 218.25 \\ 254.375 & 34.25 & 756.25 & 197.375 & 216.375 & 239.875 \\ 543.75 & 42.75 & 1722.25 & 419.25 & 460.75 & 508.25 \\ 179.375 & 21.75 & 528.75 & 137.375 & 151.375 & 169.875 \\ 235.125 & 37.875 & 680.625 & 183.375 & 200.625 & 222.375 \end{bmatrix}.$$

$\rho(B) = 12.8955$ is a simple eigenvalue, both right and left eigenvectors of B: $(0.6137\ \ 0.2621\ \ 0.2807\ \ 0.6087\ \ 0.1965\ \ 0.2582)^T$ and $(0.2825\ \ 0.0282\ \ 0.8631\ \ 0.2168\ \ 0.2387\ \ 0.2657)^T$ are positive since there exist $\alpha = 4$, such that $B + 4I$ is an eventually nonnegative matrix with power index 4 ($(B + \alpha I)^k > 0$, $\forall k \geq 4$).

Example 3.7. Consider the matrix

$$B = \frac{1}{155}\begin{bmatrix} 2021 & 4346 & 3318 & -8517 & 9414 & -5835 \\ -2810 & -3895 & -2225 & 9325 & -9060 & 6120 \\ 2402 & 3642 & 2591 & -7699 & 8438 & -5055 \\ -318 & -628 & -394 & 1591 & -1392 & 1270 \\ -877 & -2272 & -1536 & 5289 & -5193 & 3405 \\ -1224 & -2464 & -1227 & 6583 & -5896 & 3815 \end{bmatrix} \overset{\text{v}}{\geq} 0.$$

B is irreducible, $\sigma(B) = \{7,\ 5,\ -3,\ -3,\ 0,\ 0\}$ with $index_0 B = 2$ and $index_{-3} B = 2$. B^k is a positive matrix for $k \geq 15$. $\rho(B) = 7$ is a simple eigenvalue, both right and left eigenvectors of B: $(0.3123\ \ 0.4685\ \ 0.3123\ \ 0.3123\ \ 0.3123\ \ 0.6247)^T$ and $(0.3041\ \ 0.3041\ \ 0.5744\ \ 0.4054\ \ 0.5406\ \ 0.1689)^T$ are positive since there exists $\alpha = 3$, such that $B + 3I$ is an eventually nonnegative matrix with power index 23 ($(B + \alpha I)^k > 0$, $\forall k \geq 23$).

Example 3.8. Consider the matrix

$$B = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \overset{\text{v}}{\geq} 0.$$

B is irreducible, $\sigma(B) = \{2,\ 1,\ 0,\ 0\}$ with $index_0 B = 2$. The right and left eigenvectors of B corresponding to $\rho(B)$ are $(1\ 1\ 1\ 1)^T > 0$ and $(0\ 0\ 1\ 1)^T \geq 0$, respectively. However, the left eigenvector $(0\ 0\ 1\ 1)^T$ is a nonnegative vector. B^k is a reducible matrix for $k \geq 2$,

$$B^k = \begin{bmatrix} 0.5 & 0.5 & 2^{k-1} - 0.5 & 2^{k-1} - 0.5 \\ 0.5 & 0.5 & 2^{k-1} - 0.5 & 2^{k-1} - 0.5 \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix}.$$

If $\alpha > 0$, then $B + \alpha I$ is not an eventually nonnegative matrix because $(B + \alpha I)^k$ has the submatrix

$$(B + \alpha I)^{(k)}_{21} = \begin{bmatrix} k\alpha^{k-1} & -k\alpha^{k-1} \\ -k\alpha^{k-1} & k\alpha^{k-1} \end{bmatrix}$$

that always has negative entries ($-k\alpha^{k-1}$).

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There is no $\alpha > 0$ such that $B + \alpha I \overset{v}{\geq} 0$. Properties (i) and (ii) of Theorem 3.5 do not hold true and the left Perron vector of B is nonnegative and not positive.

THEOREM 3.9. *The eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ of an irreducible eventually nonnegative matrix B with $index_0 B \leq 1$ are not nonnegative vectors.*

Proof. Let $\lambda \neq \rho(B)$ and y be a nonnegative right eigenvector corresponding to λ. Let also $w > 0$ be the left eigenvector corresponding to $\rho(B)$. Then

$$w^T By = \rho(B)w^T y > 0 \quad \text{and} \quad w^T By = \lambda w^T y > 0,$$

which means that $\lambda = \rho(B)$ and constitutes a contradiction.

Let y_s be a nonnegative generalized eigenvector corresponding to $\lambda \neq \rho(B)$ of order s. $[y_1\, y_2\, \cdots\, y_s\, \cdots\, y_q]$ is the chain of the generalized eigenspace of λ. Then, y_s is an eigenvector of $(B - \lambda I)^s$ and w is a left eigenvector of $B - \lambda I$ corresponding to the eigenvalue $(\rho(B) - \lambda)^s$. Hence,

$$w^T (B - \lambda I)^s y_s = 0 \quad \text{and} \quad w^T (B - \lambda I)^s y_s = (\rho(B) - \lambda)^s w^T y_s > 0,$$

which constitutes a contradiction.

The proof for the left eigenvector is analogous. □

REMARK 3.10. Theorem 3.9 could be stated equivalently as:

The eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \lambda_{\min}(A)$ of an irreducible M_v-matrix $A = sI - B$ with $index_0 B \leq 1$ are not nonnegative vectors.

The assumption $index_0 B \leq 1$ is sufficient since otherwise the matrix B^k may be reducible and the Perron eigenvectors should be nonnegative and not positive. But this assumption is not necessary. We now state and prove sufficient and necessary conditions in the following theorem.

THEOREM 3.11. *The right and left eigenvectors and generalized eigenvectors corresponding to the eigenvalue $\lambda \neq \rho(B)$ of an irreducible eventually nonnegative matrix B are not nonnegative vectors iff there exists $\alpha > 0$ such that $B + \alpha I \overset{v}{\geq} 0$.*

Proof. In the proof of Theorem 3.5, we have proven that the Perron eigenvector of B is positive iff there exists $\alpha > 0$ such that $B + \alpha I \overset{v}{\geq} 0$, even if $index_0 B \geq 2$ and $index_0(B + \alpha I) \geq 2$. Thus, from Theorem 3.9, we get our result. □

REMARK 3.12. As in Remark 3.10, Theorem 3.11 could be stated in an analogous way for M_v-matrices.

We now give examples which support the result of Theorem 3.11. The following example shows that if there is no $\alpha > 0$ such that $B + \alpha I \overset{v}{\geq} 0$, the right and left eigenvectors and generalized eigenvectors corresponding to the eigenvalue $\lambda \neq \rho(B)$ may be nonnegative.

EXAMPLE 3.13. Consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 0.5 & 0.5 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & -1 & 0.5 & 0.5 \\ -1 & 1 & 0.5 & 0.5 \end{bmatrix} \overset{v}{\geq} 0.$$

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B is an irreducible matrix, $\sigma(B) = \{2,\ 1,\ 0,\ 0\}$ with $index_0 B = 2$. The right and left eigenvectors of B corresponding to $\rho(B)$ are $(1\ 1\ 0\ 0)^T \geq 0$ and $(1\ 1\ 1\ 1)^T > 0$, respectively. The right and left eigenvectors of B corresponding to 1 are $(-1\ \ -1\ 1\ 1)^T$ and $(0\ 0\ 1\ 1)^T \geq 0$, respectively. The right and left eigenvectors of B corresponding to 0 are $(0\ \ 0\ \ 1\ \ -1)^T$ and $(-1\ \ 1\ \ 0\ \ 0)^T$, respectively.

However, the left eigenvector corresponding to $\lambda = 1:\ (0\ 0\ 1\ 1)^T$ is a nonnegative vector. The power of B:

$$B^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & 2^{k-1} - 0.5 & 2^{k-1} - 0.5 \\ 2^{k-1} & 2^{k-1} & 2^{k-1} - 0.5 & 2^{k-1} - 0.5 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

is a reducible nonnegative matrix for $k \geq 2$.

If $\alpha > 0$, then $B + \alpha I$ is not an eventually nonnegative matrix because $(B + \alpha I)^k$ has a submatrix

$$(B + \alpha I)^{(k)}_{21} = \begin{bmatrix} k\alpha^{k-1} & -k\alpha^{k-1} \\ -k\alpha^{k-1} & k\alpha^{k-1} \end{bmatrix}$$

that always has negative entries ($-k\alpha^{k-1}$).

There is no $\alpha > 0$ such that $B + \alpha I \overset{\mathrm{v}}{\geq} 0$. The assumption of Theorem 3.11 does not hold and the left eigenvector corresponding to the eigenvalue $\lambda = 1 \neq 2 = \rho(B)$ of B is a nonnegative vector.

The following example shows that the assumption $index_0 B \leq 1$ in Theorem 3.9 is sufficient and not necessary.

Example 3.14. Consider the matrix

$$B = \begin{bmatrix} 2 & 4 & 2 & 5 & -1 \\ 3 & 2 & 4 & 4 & 3 \\ 1 & 5 & -2 & 4 & -5 \\ 1 & -2 & 2 & -1 & 4 \\ 2 & -4 & 4 & -2 & 8 \end{bmatrix} \overset{\mathrm{v}}{\geq} 0.$$

B is an irreducible matrix, $\sigma(B) = \{7.7572,\ 2.8635,\ 0,\ 0,\ -1.6207\}$ with $index_0 B = 2$. The right and left eigenvectors of B corresponding to $\rho(B)$ are $(0.6250\ \ 0.6735\ \ 0.3794\ \ 0.0485\ \ 0.0970)^T > 0$ and $(0.4692\ \ 0.2632\ \ 0.4945\ \ 0.5105\ \ 0.4532)^T > 0$, respectively. The eigenvectors or generalized right and left eigenvectors corresponding to $\lambda \neq \rho(B)$ of B are, respectively, as follows: the eigenvectors $(0.3886\ \ 0.0581\ \ 0.5473\ \ -0.3305\ \ -0.6610)^T$ and $(-0.2212\ \ 0.4338\ \ -0.2465\ \ 0.3106\ \ -0.7782)^T$ corresponding to 2.8635; the eigenvectors $(-0.4584\ \ 0.6845\ \ -0.1432\ \ -0.1991\ \ 0.4854)^T$ and $(-0.3756\ \ 0.3756\ \ 0\ \ 0.6676\ \ -0.5216)^T$ corresponding to 0; the generalized eigenvectors $(0.5416\ \ -0.1301\ \ 0.4765\ \ -0.4115\ \ -0.5416)^T$ and $(0.4973\ \ -0.4973\ \ 0\ \ -0.5044\ \ 0.5009)^T$ corresponding to 0; the eigenvectors $(-0.0107\ \ 0.2317\ \ -0.8077\ \ 0.2424\ \ -0.4848)^T$ and $(-0.2566\ \ -0.2869\ \ 0.8064\ \ 0.1163\ \ 0.4336)^T$ corresponding to -1.6207 which are not nonnegative vectors.

If $\alpha = 2$, then $B + 2I$ is an eventually nonnegative matrix with power index 6 ($(B + \alpha I)^k \geq 0,\ \forall k \geq 6$). From Theorem 3.11, the eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ of B are not nonnegative vectors because there exists $\alpha = 2$ such that $B + \alpha I \overset{\mathrm{v}}{\geq} 0$.

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4. Equivalence of eventually exponentially nonnegative and M_v-matrices. Properties connecting eventually nonnegative matrices and eventually exponentially nonnegative matrices have been proven in [14, Theorem 3.7] when $index_0 B \leq 1$. In this section, we give results connecting M_v-matrices and eventually exponentially nonnegative matrices for every case of $index_0 B$.

First, we prove a lemma for series of the inverse of a matrix $A = sI - B$.

LEMMA 4.1. *Let $A \in \mathbb{R}^{n,n}$ be an invertible matrix, written in the form $A = sI - B$. Then, $(A^{-1})^k$ is given in the following series form:*

$$(A^{-1})^k = \frac{1}{s^k}I + \frac{1}{s^{k+1}}\binom{k}{1}B + \frac{1}{s^{k+2}}\binom{k+1}{2}B^2 + \frac{1}{s^{k+3}}\binom{k+2}{3}B^3 + \cdots + \frac{1}{s^{k+m}}\binom{k+m-1}{m}B^m + \cdots . \tag{4.8}$$

Proof. By induction, for $k = 1$, the statement is true because

$$\begin{aligned}(A^{-1})^1 &= \frac{1}{s}(I - \frac{1}{s}B)^{-1} = \frac{1}{s}I + \frac{1}{s^2}B + \frac{1}{s^3}B^2 + \frac{1}{s^4}B^3 + \cdots + \frac{1}{s^{m+1}}B^m + \cdots \\ &= \frac{1}{s}I + \frac{1}{s^2}\binom{1}{1}B + \frac{1}{s^3}\binom{2}{2}B^2 + \frac{1}{s^4}\binom{3}{3}B^3 + \cdots + \frac{1}{s^{m+1}}\binom{m}{m}B^m + \cdots\end{aligned}$$

Assume (4.8) holds. Then,

$$\begin{aligned}(A^{-1})^{k+1} &= (A^{-1})^k (A^{-1})^1 \\ &= \left(\frac{1}{s^k}I + \frac{1}{s^{k+1}}\binom{k}{1}B + \frac{1}{s^{k+2}}\binom{k+1}{2}B^2 + \frac{1}{s^{k+3}}\binom{k+2}{3}B^3 + \cdots + \frac{1}{s^{k+m}}\binom{k+m-1}{m}B^m + \cdots\right) \\ &\quad \times \left(\frac{1}{s}I + \frac{1}{s^2}B + \frac{1}{s^3}B^2 + \cdots + \frac{1}{s^{m+1}}B^m + \cdots\right) \\ &= \frac{1}{s^{k+1}}I + \frac{1}{s^{k+2}}\left(\binom{k}{1} + \binom{k-1}{0}\right)B + \frac{1}{s^{k+3}}\left(\binom{k+1}{2} + \binom{k}{1} + \binom{k-1}{0}\right)B^2 \\ &\quad + \cdots + \frac{1}{s^{k+m+1}}\left(\binom{k+m-1}{m} + \binom{k+m-2}{m-1} + \cdots + \binom{k}{1} + \binom{k-1}{0}\right)B^m + \cdots\end{aligned}$$

We have to prove that

$$\binom{k+i-1}{i} + \binom{k+i-2}{i-1} + \cdots + \binom{k-1}{0} = \binom{k+i}{i}, \quad i = 1, 2, \ldots, m. \tag{4.9}$$

For this, we use induction:

For $i = 1$, we have $\binom{k}{1} + \binom{k-1}{0} = k + 1 = \binom{k+1}{1}$, thus (4.9) holds true.

Suppose that (4.9) holds true for $i = j$, we prove it for $i = j + 1$:

$$\begin{aligned}&\binom{k+j}{j+1} + \binom{k+j-1}{j} + \binom{k+j-2}{j-1} + \cdots + \binom{k}{1} + \binom{k-1}{0} \\ &= \binom{k+j}{j+1} + \binom{k+j}{j} \\ &= \frac{(k+j)(k+j-1)\cdots k}{(j+1)!} + \frac{(k+j)(k+j-1)\cdots(k+1)}{j!} \\ &= \frac{(k+j)(k+j-1)\cdots(k+1)}{j!}\left(\frac{k}{j+1} + 1\right) \\ &= \frac{(k+j)(k+j-1)\cdots(k+1)}{j!}\frac{k+j+1}{j+1} = \binom{k+j+1}{j+1},\end{aligned}$$

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and the proof is complete. □

THEOREM 4.2. *Let $B \in \mathbb{R}^{n,n}$ be an eventually nonnegative matrix. Let $A \in \mathbb{R}^{n,n}$, of the form $A = sI - B$, is the associated M_v−matrix and $0 \leq \rho(B) < s$. Then, the following statements are equivalent:*

(i) There exists $\alpha > 0$ such that $(B + \alpha I) \overset{\mathrm{v}}{\geq} 0$.
(ii) B is an eventually exponentially nonnegative matrix.
(iii) $A^{-1} \overset{\mathrm{v}}{\geq} 0$.

Proof. Statement (i) means that there exists $k_\alpha > 0$ such that $(B + \alpha I)^k \geq 0$ for all $k \geq k_\alpha$. Let B has power index k_0 and we choose $k > \max\{k_0, k_\alpha\}$. Then,

$$(B+\alpha I)^k = \alpha^k \left(I + \binom{k}{1}\left(\frac{B}{\alpha}\right) + \binom{k}{2}\left(\frac{B}{\alpha}\right)^2 + \cdots + \binom{k}{k_0-1}\left(\frac{B}{\alpha}\right)^{k_0-1} + \binom{k}{k_0}\left(\frac{B}{\alpha}\right)^{k_0} \right.$$
$$\left. + \cdots + \binom{k}{k}\left(\frac{B}{\alpha}\right)^k \right) \geq 0. \tag{4.10}$$

Statement (ii) means that there exists $t_0 > 0$ such that $e^{tB} \geq 0$ for all $t > t_0$. Thus,

$$e^{tB} = I + tB + \frac{t^2}{2!}B^2 + \frac{t^3}{3!}B^3 + \cdots + \frac{t^{k_0-1}}{(k_0-1)!}B^{k_0-1} + \frac{t^{k_0}}{k_0!}B^{k_0} + \cdots \geq 0. \tag{4.11}$$

Statement (iii) means that there exists $m_0 > 0$ such that $(A^{-1})^m \geq 0$ for all $m > m_0$. Taking into account the expansion proven in Lemma 4.1 we get that

$$(A^{-1})^m = \frac{1}{s^m}\left(I + \binom{m}{1}\left(\frac{B}{s}\right) + \binom{m+1}{2}\left(\frac{B}{s}\right)^2 \right. \tag{4.12}$$
$$\left. + \cdots + \binom{m+k_0-2}{k_0-1}\left(\frac{B}{s}\right)^{k_0-1} + \binom{m+k_0-1}{k_0}\left(\frac{B}{s}\right)^{k_0} + \cdots \right) \geq 0.$$

We observe that in (4.10), we have a polynomial in B which should be nonnegative, while in (4.11) and (4.12) we have series expansions in B to be nonnegative. Since $B \overset{\mathrm{v}}{\geq} 0$, the first k_0 terms may have negative entries in all cases. These entries should be the same for the three cases, because all the coefficients in the powers of B are positive.

Case 1: B is irreducible and $index_0 B \leq 1$.

Suppose first that B is not a weakly cyclic matrix. Then, both right and left Perron vectors of B are positive, and thus, B should be eventually positive and the validity of (i) is guaranteed from Theorem 3.5. This means that the last $k - k_0 + 1$ terms dominate the first k_0 ones in order to eliminate the negative entries. We observe that in statement (ii) we can choose a large enough t such that the $(k_0 + 1)$st term (monomial in t of degree k_0) should dominate all the previous sum (polynomial in t of degree $k_0 - 1$). Thus, $(i) \Rightarrow (ii)$ is proven. We observe also that in statement (iii) we can choose large enough m such that the $(k_0 + 1)$st term should dominate all the previous sum, since the coefficient of this term is a polynomial in m of degree k_0 while the coefficients of the previous terms, are polynomials in m of smaller degrees. Thus, $(i) \Rightarrow (iii)$ is proven.

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The proof in the opposite directions is exactly the same. Indeed, the validity of (ii) means that the series of (k_0+1)st term and thereafter, dominates the first sum. Then we can choose a large enough k such that the (k_0+1)st term of polynomial (4.10) should dominate all the previous sum, proving that $(ii) \Rightarrow (i)$. Similarly, we prove that $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.

In the case where B is a weakly cyclic matrix of index r, we consider r sums of (4.10), taking in each sum the terms of modulus r, i.e.,

$$\alpha^k \left(\binom{k}{i} \left(\frac{B}{\alpha}\right)^i + \binom{k}{r+i} \left(\frac{B}{\alpha}\right)^{r+i} + \binom{k}{2r+i} \left(\frac{B}{\alpha}\right)^{2r+i} + \cdots \right), \quad i = 0, 1, \ldots, r-1.$$

Each term in this sum has the same cyclic structure. Analogously, we consider r subseries of (4.11) and (4.12) taking in each subseries the powers of modulus r, as in (4.10). Then, the proof follows the same steps as before, connecting each polynomial of (4.10) with each subseries of (4.11) and (4.12) having the same cyclic structure.

Case 2: B is irreducible and $index_0 B \geq 2$.

Suppose first that (i) holds true. Then, from Theorem 3.5 we obtain that both the right and left Perron vectors of B are positive. Thus, there exists k_0 such that B^k is irreducible and $B^k \geq 0$ for all $k \geq k_0$. Then, to prove $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ we follow the same arguments as in case 1.

Now suppose that (ii) holds true. Then, from (4.11), since B is irreducible, e^{tB} is irreducible even if we consider that B^k maybe reducible for all $k \geq r_0$ ($r_0 = index_0 B$). Otherwise, supposing e^{tB} is reducible, we arrive at the same contradiction following the proof of case 2 in Theorem 3.5, where in (3.5) we consider the associated terms of $\left(e^{tB}\right)_{ij}$ instead of $\left((B+\alpha I)^k\right)_{ij}$, and system (3.6) is taken by choosing different values of t. Thus, e^{tB}, and therefore B, has positive right and left Perron vectors. Now, following the same steps as in the proof of case 1, we prove that $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$.

Finally, we suppose that (iii) holds true. Then, from (4.12), following the same steps previously, we obtain that $\left(A^{-1}\right)^m$ is irreducible and thus, $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.

Case 3: B is reducible.

For simplicity, and without loss of generality, suppose that B is in its Frobenius normal form

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1q} \\ & B_{22} & \cdots & B_{2q} \\ & & \ddots & \vdots \\ & & & B_{qq} \end{bmatrix}, \tag{4.13}$$

where B_{ii}, $i = 1, 2, \ldots, q$ are square irreducible matrices or 1×1 zero ones. Since $B \overset{\mathrm{v}}{\geq} 0$, we have that

$$B^k = \begin{bmatrix} B_{11}^k & B_{12}^{(k)} & \cdots & B_{1q}^{(k)} \\ & B_{22}^k & \cdots & B_{2q}^{(k)} \\ & & \ddots & \vdots \\ & & & B_{qq}^k \end{bmatrix} \geq 0, \tag{4.14}$$

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for all $k \geq k_0$. If B_{ii} is 1×1 zero matrix then so is B_{ii}^k, $\forall k \geq 0$. Thus, $((B+\alpha I)^k)_{ii} = \alpha^k > 0$ for relation (4.10). For relation (4.11), we have $e^{tB} = I + \sum_{j=1}^{\infty} \frac{(tB)^j}{j!}$, thus $(e^{tB})_{ii} = 1 > 0$, and for relation (4.12), $(A^{-1})^m = \frac{1}{s^m}\left(I + \sum_{j=1}^{\infty} \binom{m+j-1}{j}\left(\frac{B}{s}\right)^j\right)$, we get $((A^{-1})^m)_{ii} = \frac{1}{s^m} > 0$.

If B_{ii} is irreducible, $B_{ii} \overset{\text{v}}{\geq} 0$, then we follow the proof of case 1, if $index_0 B_{ii} \leq 1$ or of case 2, if $index_0 B_{ii} \geq 2$, where we consider the matrix B_{ii} in the places of B. Thus, the proof of theorem concerning the diagonal blocks is complete.

We consider the (i,j) off diagonal block, $i < j$. Then, (4.10) gives us

$$((B+\alpha I)^k)_{ij} = \alpha^k \left(\binom{k}{1}\left(\frac{B_{ij}}{\alpha}\right) + \binom{k}{2}\left(\frac{B_{ij}^{(2)}}{\alpha^2}\right) + \cdots + \binom{k}{k}\left(\frac{B_{ij}^{(k)}}{\alpha^k}\right)\right),$$

(4.11) presents

$$(e^{tB})_{ij} = tB_{ij} + \frac{t^2}{2!}B_{ij}^{(2)} + \cdots + \frac{t^{k_0}}{k_0!}B_{ij}^{(k_0)} + \cdots,$$

while (4.12)

$$((A^{-1})^m)_{ij} = \frac{1}{s^m}\left(\binom{m}{1}\left(\frac{B_{ij}}{s}\right) + \binom{m+1}{2}\left(\frac{B_{ij}^{(2)}}{s^2}\right) + \cdots + \binom{m+k_0-1}{k_0}\left(\frac{B_{ij}^{(k_0)}}{s^{k_0}}\right) + \cdots\right).$$

Let $B_{ij} \in \mathbb{R}^{i_n, j_n}$. We consider the (μ,ν) entry of B_{ij}, $1 \leq \mu \leq i_n$, $1 \leq \nu \leq j_n$. Then, the sequence of matrices $\left\{B_{ij}^{(k)}\right\}_{k=1}^{\infty}$ defines a sequence of real number for the associated (μ,ν) entries: $\left\{\left(B_{ij}^{(k)}\right)_{\mu,\nu}\right\}_{k=1}^{\infty}$. For simplicity, we symbolize this sequence by $\{b_k\}_{k=1}^{\infty}$. From the fact that $B \overset{\text{v}}{\geq} 0$, we have that $b_k \geq 0$, $\forall k \geq k_0$. Relations (4.10), (4.11) and (4.12) for this entry, are given us

$$\left(((B+\alpha I)^k)_{ij}\right)_{\mu,\nu} = \alpha^{k-1}\binom{k}{1}b_1 + \alpha^{k-2}\binom{k}{2}b_2 + \cdots + \binom{k}{k}b_k,$$

$$\left((e^{tB})_{ij}\right)_{\mu,\nu} = tb_1 + \frac{t^2}{2!}b_2 + \cdots + \frac{t^{k_0}}{k_0!}b_{k_0} + \cdots,$$

and

$$\left(((A^{-1})^m)_{ij}\right)_{\mu,\nu} = \frac{1}{s^{m+1}}\binom{m}{1}b_1 + \frac{1}{s^{m+2}}\binom{m+1}{2}b_2 + \cdots + \frac{1}{s^{m+k_0}}\binom{m+k_0-1}{k_0}b_{k_0} + \cdots.$$

Suppose first that $\{b_k\}_{k=1}^{\infty}$ is the zero sequence: $b_k = 0$, $\forall k \geq 1$, then the relations above give all zeros. Thus, the equivalence of the three statements of the Theorem, concerning this entry, is trivially proven.

Let $b_k = 0$ for all $k > k_1 > 0$. The validity of statement (i) means that $b_{k_1} > 0$ and k is chosen large enough, such that the last nonzero term $\alpha^{k-k_1}\binom{k}{k_1}b_{k_1}$ dominates all the previous sum. Then, the same hold

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true for the terms $\frac{t^{k_1}}{k_1!}b_{k_1}$ and $\frac{1}{s^{m+k_1}}\binom{m+k_1-1}{k_1}b_{k_1}$ for large enough t and m, respectively. This is because the k_1 terms are polynomials in k, in t or in m, respectively, of degree k_1, while all the previous sums are polynomials of smaller degree.

Finally, suppose that b_k has nonzero entries as k tends to infinity. Then, we choose $k_1 > k_0$ such that $b_{k_1} > 0$. We follow the same argument of the previous case, for such k_1, to prove the equivalence of $(i), (ii)$ and (iii), for the associated entry.

Applying the same argument for any entry of B_{ij}, and every off-diagonal block, the theorem is proven. ∎

The following examples show the validity of Theorem 4.2 for all cases.

EXAMPLE 4.3. Consider the M_v-matrix

$$A = 7I - B, \quad B = \begin{bmatrix} 3 & 2 & -1 & -2 \\ 1 & 2 & 5 & -1 \\ 1 & 3 & 1 & 3 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

$\rho(B) = 5.9389$ and B is an irreducible eventually nonnegative matrix with power index 8 and $index_0 B = 0$. The right and left Perron eigenvectors are $(0.3251 \;\; 0.7714 \;\; 0.5467 \;\; 0.0203)^T$ and $(0.4452 \;\; 0.6676 \;\; 0.5950 \;\; 0.0459)^T$, respectively.

If $\alpha = 2$, then $B+2I$ is an eventually nonnegative matrix with power index 11 ($(B+\alpha I)^k \geq 0, \forall k \geq 11$). From Theorem 4.2, the matrix

$$A^{-1} = \begin{bmatrix} 0.3042 & 0.1993 & 0.1014 & -0.0839 \\ 0.2483 & 0.5420 & 0.4161 & 0.0350 \\ 0.1958 & 0.3007 & 0.3986 & 0.0839 \\ 0.0420 & -0.0070 & 0.0140 & 0.1608 \end{bmatrix}$$

is an eventually nonnegative matrix, $\left(A^{-1}\right)^k \geq 0, \forall k \geq 3$.

Also, the matrix B is an eventually exponentially nonnegative matrix. Choosing $t = 1.8275$, we get

$$e^{tB} = \begin{bmatrix} 7536.4060 & 11383.9671 & 10170.9115 & 796.2579 \\ 18034.4030 & 26999.1535 & 24064.8506 & 1877.0609 \\ 12781.4358 & 19145.2405 & 17045.1966 & 1290.6572 \\ 468.1169 & 723.8724 & 623.3201 & 0.0124 \end{bmatrix} > 0.$$

EXAMPLE 4.4. Consider the M_v-matrix

$$A = 8I - B, \quad B = \begin{bmatrix} 6 & 2 & -2 & 5 & -1 \\ 2 & 4 & 1 & 1 & 2 \\ 2 & 4 & 1 & -1 & 3 \\ -3 & 1 & 2 & -4 & 1 \\ -3 & 1 & 2 & -4 & 1 \end{bmatrix}.$$

B is an irreducible eventually nonnegative matrix with power index 8 and $\sigma(B) = \{6.6286, \; 3.4354, \; 0, \; 0, \; -2.064\}$ with $index_0 B = 2$. The right and left Perron eigenvectors are $(0.4311 \;\; 0.6396 \;\; 0.6301 \;\; 0.0630 \;\; 0.0630)^T$ and $(0.5254 \;\; 0.7679 \;\; 0.1213 \;\; 0.2026 \;\; 0.2802)^T$, respectively, even if $index_0 B = 2$.

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If $\alpha = 1$, then $B + I$ is an eventually nonnegative matrix with power index 8 ($(B+\alpha I)^k \geq 0$, $\forall k \geq 8$). From Theorem 4.2, the matrix

$$A^{-1} = \begin{bmatrix} 0.3988 & 0.2262 & -0.0238 & 0.1793 & 0.0231 \\ 0.2202 & 0.4881 & 0.0714 & 0.0766 & 0.1496 \\ 0.2262 & 0.3571 & 0.1905 & 0.0551 & 0.1592 \\ -0.0476 & 0.0476 & 0.0476 & 0.0476 & 0.0476 \\ -0.0476 & 0.0476 & 0.0476 & -0.0774 & 0.1726 \end{bmatrix}$$

is an eventually nonnegative matrix, $\left(A^{-1}\right)^k \geq 0$, $\forall k \geq 4$.

The matrix B is also an eventually exponentially nonnegative matrix. Choosing $t = 1.1925$, we get

$$e^{tB} = \begin{bmatrix} 809.6378 & 1078.1401 & 134.7511 & 359.9745 & 348.2426 \\ 1084.3121 & 1617.5316 & 266.1097 & 403.9465 & 603.3538 \\ 1078.1401 & 1591.0872 & 257.4735 & 407.0276 & 588.6072 \\ 72.1968 & 164.3321 & 46.0677 & 1.0062 & 85.2909 \\ 72.1968 & 164.3321 & 46.0677 & 1.0062 & 85.2909 \end{bmatrix} > 0.$$

EXAMPLE 4.5. Consider the reducible M_v-matrix

$$A = 7I - B,\ B = \left[\begin{array}{cc|cccc} 4 & 3 & 1 & -1 & -2 & 1 \\ 5 & -2 & 4 & 4 & 2 & -4 \\ \hline 0 & 0 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 & -2 & -1 \\ 0 & 0 & 1 & -1 & -2 & -2 \end{array}\right].$$

$\rho(B) = 5.8990$ and B is an eventually nonnegative matrix with power index 18 and the right and left Perron eigenvectors are $(0.8499\ \ 0.5348\ \ 0\ \ 0\ \ 0\ \ 0)^T$ and $(0.5270\ \ 0.2002\ \ 0.5559\ \ 0.5801\ \ 0.0920\ \ 0.1679)^T$, respectively. The block matrix $B_{11} = \begin{bmatrix} 4 & 3 \\ 5 & -2 \end{bmatrix}$ is an irreducible eventually nonnegative matrix with power index 4 and $B_{22} = \begin{bmatrix} 2 & 3 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & -1 & -2 & -1 \\ 1 & -1 & -2 & -2 \end{bmatrix}$ is also an irreducible eventually nonnegative matrix with power index 18 and $\sigma(B_{22}) = \{4.4051, -3.4051,\ 0,\ 0\}$, with $index_0 B_{22} = 2$.

If $\alpha = 5$, then $B + 5I$ is an eventually nonnegative matrix with power index 8 ($(B+\alpha I)^k \geq 0$, $\forall k \geq 8$). From Theorem 4.2, the matrix

$$A^{-1} = \left[\begin{array}{cc|cccc} 0.75 & 0.25 & 0.4456 & 0.3696 & -0.0045 & 0.1128 \\ 0.4167 & 0.25 & 0.3927 & 0.4049 & 0.0836 & 0.0581 \\ \hline 0 & 0 & 0.2562 & 0.1625 & 0.0491 & 0.0695 \\ 0 & 0 & 0.0771 & 0.2819 & 0.062 & 0.0416 \\ 0 & 0 & 0.0181 & -0.0121 & 0.1126 & -0.0098 \\ 0 & 0 & 0.0159 & -0.0106 & -0.0265 & 0.1164 \end{array}\right]$$

is an eventually nonnegative matrix, $\left(A^{-1}\right)^k \geq 0$, $\forall k \geq 5$.

The matrix B is also an eventually exponentially nonnegative matrix. Choosing $t = 1.0584$, we get

$$e^{tB} = \left[\begin{array}{rr|rrrr} 414.9039 & 157.5727 & 349.3079 & 327.0651 & 30.9242 & 90.9654 \\ 262.6211 & 99.7586 & 261.8324 & 267.2673 & 39.594 & 76.0199 \\ \hline 0 & 0 & 49.6756 & 70.932 & 22.2564 & 23.3148 \\ 0 & 0 & 37.1674 & 56.5734 & 18.406 & 17.3476 \\ 0 & 0 & 1.989 & 1.6553 & 0.6663 & 0.7247 \\ 0 & 0 & 1.5059 & 1.5061 & 0.0002 & 1.0002 \end{array}\right] \geq 0.$$

5. Summary. To study the eigenvectors of an M_v-matrix, we categorize into M_v-matrices with $index_0 B \leq 1$ and M_v-matrices with $index_0 B > 1$ and we obtain results as follows:

1. For an irreducible M_v-matrix with $index_0 B \leq 1$, to the smallest real eigenvalue $\lambda_1 \geq 0$ of A there correspond positive right and left eigenvectors.
2. We gave equivalent statements for M_v-matrices with $index_0 B > 1$ to have positive right and left Perron eigenvectors.
3. For an irreducible eventually nonnegative matrix B with $index_0 B \leq 1$, its eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ are not nonnegative vectors.
4. For an irreducible eventually nonnegative matrix B (with $index_0 B \leq 1$ or $index_0 B > 1$), its right and left eigenvectors and generalized eigenvectors corresponding to $\lambda \neq \rho(B)$ are not nonnegative vectors iff there exists $\alpha > 0$ such that $B + \alpha I \overset{\mathrm{v}}{\geq} 0$.

Finally, we gave and proved equivalent properties of eventually exponentially nonnegative and M_v-matrices.

It is trivial from the definition of $GM-$matrices and [13] that any $GM-$matrix (which is not an M_v-matrix) may have nonnegative eigenvector corresponding to its spectral radius. Hence, Theorems 3.1 and 3.5 do not hold for $GM-$matrices.

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