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MAXIMAL ORIENTATIONS OF GRAPHS*

STEPHEN W. DRURY†

Abstract. Orientations of connected graphs that maximize the spectral norm of the adjacency matrix are studied, and a conjecture of Hoppen, Monsalve and Trevisan is solved.

Key words. Spectral norm, Oriented graph, Digraph.

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1. Introduction.

Let G be a simple graph without loops. An orientation of G is a digraph with the same vertex set as G but in which each edge ab of G is replaced by either by the arc $\overrightarrow{ab}$ or the arc $\overrightarrow{ba}$. The set of all orientations of G is denoted $\mathcal{O}(G)$. An oriented graph is sink-source if every vertex is either a sink or a source.

We denote by $A(H)$ the adjacency matrix of a graph or digraph H. The spectral norm of a matrix C is given by $\sup |y^*Cx|$ where the supremum is taken over all unit vectors x and y. It is also the largest singular value $\sigma_1(C)$ of C. We denote by C' the transpose of C. For nonnegative $n \times n$ matrices C (such as those considered in this article), the spectral norm is $\sup y'Cx$ where the supremum is taken over all nonnegative unit vectors x and y and the singular values $\sigma_j(C) = \sqrt{\mu_j}$, $(j = 1, \dots, n)$ where the μ_j are the necessarily nonnegative eigenvalues of $C'C$ or of CC' taken in decreasing order.

We denote e_a the vector with 1 in the a^{th} place and zeros elsewhere. We denote by C_p the graph that is a p-cycle. For X a subset of $\{1, 2, \dots, n\}$ and S a symmetric $n \times n$ matrix, we denote by $S_{[X]}$ the principal submatrix of S corresponding to the set X. For more information on graphs and digraphs see [1], and on linear algebra see [3]. The following conjecture is proposed in [2, Conjecture 4.2].

Conjecture 1.1. *Let G be a graph and let H be an orientation of G such that*

$$\sigma_1(A(H)) = \max_{K \in \mathcal{O}(G)} \sigma_1(A(K)). \tag{1.1}$$

Then H is an acyclic digraph.

As it stands, Conjecture 1.1 is trivially false. Let $G = C_3 \cup C_4$, then the right hand side of (1.1) is 2 and it is attained for any orientation that reduces to a sink-source orientation on the C_4 irrespective of whether the C_3 is oriented cyclically or not. We settle the issue by establishing the following theorem.

Theorem 1.2. *Let G be a connected graph and let H be an orientation of G such that*

$$\sigma_1(A(H)) = \max_{K \in \mathcal{O}(G)} \sigma_1(A(K)).$$

Then H is an acyclic digraph.

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†Department of Mathematics and Statistics, McGill University, Montreal, H3A 0B9, Canada (stephen.drury@mcgill.ca).

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2. Development of results. Let H be an orientation of G. We will say that H is maximal (as an orientation of G) if (1.1) holds. We see that $A(H)'A(H)$ is a nonnegative symmetric matrix which therefore can be viewed as a block diagonal matrix with p irreducible blocks. We view zero blocks as being 1×1. Each block has a unit Perron vector, strictly positive on the block and extended to be zero outside it. We can proceed similarly with $A(H)A(H)'$ which has q irreducible blocks. A block X of $A(H)'A(H)$ is said to be maximal if $\sigma_1(A(H)'A(H)_{[X]}) = \sigma_1(A(H))^2$ and similarly for $A(H)A(H)'$.

The following example may help to make these ideas clearer.

$$A(H) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{spectral norm } \sigma_1(A(H)) = \sqrt{2}.$$

giving

$$A(H)'A(H) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(H)A(H)' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

with $p = 5$ and $q = 3$. The blocks and Perron vectors are $X_j = \{j\}$, $\xi_j = e_j$ for $j = 1, \dots, 5$, $Y_1 = \{1\}$, $\eta_1 = e_1$ and

$$Y_2 = \{2, 4\}, \quad \eta_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad Y_3 = \{3, 5\}, \quad \eta_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Thus, in the example, the maximal blocks are X_1, X_3, Y_2 and Y_3. The following lemma is well-known.

Lemma 2.1. *The maximal blocks of $A(H)'A(H)$ and $A(H)A(H)'$ are in one-to-one correspondence.*

Proof. If X_j is a maximal block of $A(H)'A(H)$, then

$$A(H)A(H)'(\sigma_1^{-1}A(H)\xi_j) = \sigma_1^{-1}A(H)(A(H)'A(H)\xi_j) = \sigma_1^2(\sigma_1^{-1}A(H)\xi_j).$$

So that $\sigma_1^{-1}A(H)\xi_j$ is an eigenvector of $A(H)A(H)'$ for the eigenvalue σ_1^2. If now X_k is a different maximal block, then

$$(\sigma_1^{-1}A(H)\xi_k)'(\sigma_1^{-1}A(H)\xi_j) = \sigma_1^{-2}\xi_k'A(H)'A(H)\xi_j = \sigma_1^{-2}\xi_k'\sigma_1^2\xi_j = 0$$

since ξ_j and ξ_k have disjoint supports. The positive unit eigenvectors $\sigma_1^{-1}A(H)\xi_j$ and $\sigma_1^{-1}A(H)\xi_k$ of $A(H)A(H)'$ are orthogonal and therefore equal η_ℓ and η_m respectively for different blocks Y_ℓ and Y_m. It is also clear that $\sigma_1^{-1}A(H)'\eta_\ell = \xi_j$ so that the maximal blocks of $A(H)'A(H)$ and the maximal blocks of $A(H)A(H)'$ are in one-to-one correspondence. ∎

Proposition 2.2. *Let G be a graph and H a maximal orientation of G. Choose a corresponding pair of maximal blocks as in Lemma 2.1 and let x and y be the corresponding unit Perron vectors, so that $y'A(H)x = \sigma_1(A(H))$. Then, for all arcs $\overrightarrow{ab}$ of H, we have $x_a y_b \le x_b y_a$. Furthermore, if $x_a y_b = x_b y_a$, then necessarily $x_a = x_b = y_a = y_b = 0$.*

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Proof. Since $\sigma_1 = \sum_{a,b} y_a A(H)_{a,b} x_b$, we see that if $x_a y_b > x_b y_a$ and $A(H)_{a,b} = 1$ (i.e., $\overrightarrow{ab} \in H$), then we would do better to switch the arc $\overrightarrow{ab}$ to $\overrightarrow{ba}$. Explicity this means that if K is the oriented graph identical to H except that the arc $\overrightarrow{ab}$ is replaced by the arc $\overrightarrow{ba}$ then we would have $\sigma_1(A(K)) \geq y'A(K)x > y'A(H)x = \sigma_1(A(H))$ contradicting the maximality of H. This establishes the first assertion.

For the second assertion, and defining K as above, we have $\sigma_1(A(K)) \geq y'A(K)x = y'A(H)x = \sigma_1(A(H))$. By the maximality of H we get $\sigma_1(A(K)) = y'A(K)x$ and this forces both $A(K)x = A(H)x$ and $A(K)'y = A(H)'y$ or equivalently $x_a e_b = x_b e_a$ and $y_a e_b = y_b e_a$. Since $\{e_a, e_b\}$ is linearly independent, we have the required conclusion. ∎

Proposition 2.3. *Let G be a graph and H a maximal orientation of G. Choose a corresponding pair of maximal blocks as in Lemma 2.1 and let x and y be the corresponding unit Perron vectors, so that $y'A(H)x = \sigma_1(A(H))$. Suppose that $x_a = y_a = 0$ and let ac be an edge of G. Then $x_c = y_c = 0$.*

Proof. We have that either $\overrightarrow{ac}$ or $\overrightarrow{ca}$ is an arc in H. In the first case we apply Proposition 2.2 with b replaced by c and in the second we apply Proposition 2.2 with a replaced by c and b replaced by a. ∎

Corollary 2.4. *Let G be a connected graph and H a maximal orientation of G. Choose a corresponding pair of maximal blocks as in Lemma 2.1 and let x and y be the corresponding unit Perron vectors, so that $y'A(H)x = \sigma_1(A(H))$. Then, for all arcs $\overrightarrow{ab}$ of H, we have $x_a y_b < x_b y_a$.*

Proof. By Proposition 2.2 we may conclude unless there is a vertex a of G with $x_a = y_a = 0$. But, in that case we may find for every vertex c a path $c_1, c_2, \ldots, c_r$ in G with $c_1 = a$ and $c_r = c$. Then applying Proposition 2.3 successively along the path we find $x_c = y_c = 0$ for all vertices c. But this contradicts the fact that x and y are unit vectors. ∎

Proof. Proof of Theorem 1.2. Suppose that H has a directed cycle of length ℓ indexed over $\mathbb{Z}(\ell)$ as $\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \ldots, \overrightarrow{v_{\ell-2}v_{\ell-1}}, \overrightarrow{v_{\ell-1}v_0}$. Then, by Corollary 2.4, we have $0 \leq x_{v_j} y_{v_{j+1}} < x_{v_{j+1}} y_{v_j}$ for all $j \in \mathbb{Z}(\ell)$. This forces $x_{v_{j+1}} > 0$ and $y_{v_j} > 0$ for all $j \in \mathbb{Z}(\ell)$. Hence, $0 < x_{v_j} y_{v_{j+1}} < x_{v_{j+1}} y_{v_j}$ for all $j \in \mathbb{Z}(\ell)$. Taking the product we find

$$\prod_{j\in\mathbb{Z}(\ell)} x_{v_j} y_{v_j} = \prod_{j\in\mathbb{Z}(\ell)} x_{v_j} y_{v_{j+1}} < \prod_{j\in\mathbb{Z}(\ell)} x_{v_{j+1}} y_{v_j} = \prod_{j\in\mathbb{Z}(\ell)} x_{v_j} y_{v_j},$$

a contradiction. ∎

We now have the following corollary of Theorem 1.2

Corollary 2.5. *Let G be a connected graph and H a maximal orientation of G. Then, after reordering the vertices of H, $A(H)$ is a strictly triangular matrix.*

Proof. We define a partial order on the vertices of H by $a \geq b$ if and only if there is a directed walk from vertex a to vertex b. To verify that this is indeed a partial order, we note that $a \geq a$ by considering the empty directed walk from a to a. The transitivity of the order follows by adjoining walks. If $a \geq b$ and $b \geq a$, then adjoining the corresponding walks gives a directed walk which implies the existence of a directed cycle unless $a = b$.

It is well known that every partial order can be extended to a total order and it is clear that in this total order, the adjacency matrix is strictly triangular. ∎

Proposition 2.6. *Let G be a connected graph and H a maximal orientation of G. Then $A(H)$ cannot have repeated maximal singular values. In particular, there is only one pair of maximal corresponding blocks. Furthermore, all the non maximal blocks are zero 1×1 blocks.*

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Proof. Suppose that there are two singular values equal to the spectral norm σ_1. Then, following the proof of Lemma 2.1, we have two pairs of maximal blocks with associated vector pairs (x, y) and (u, v). Note that the supports of x and u are disjoint and the supports of y and v are disjoint. Since $\sigma_1 v = Au$ there exists an arc $\overrightarrow{ab}$ of H such that $v_a > 0$ and $u_b > 0$. It follows that $x_b = 0$ and $y_a = 0$. But according to Corollary 2.4 we have $0 \le x_a y_b < x_b y_a = 0$ giving a contradiction.

So, there is a unique corresponding pair X, Y of maximal blocks and associated Perron vectors x and y strictly positive on X and Y respectively. Now let $\overrightarrow{ab}$ be any arc of H then again $0 \le x_a y_b < x_b y_a$. So $x_b = 0$ is impossible, as is $y_a = 0$. Thus, all arcs go from a vertex in Y to a vertex in X. This implies that all the nonmaximal blocks are zero, and hence, 1×1. ∎

3. Questions.

There are some natural questions.

QUESTION 3.1. *Let G be a connected graph and suppose that H and K are maximal orientations of G. Is it necessarily true that either H and K are isomorphic as digraphs or that H and the reversal of K are isomorphic as digraphs?*

The graph $G = C_3 \cup C_4$ shows that the connectedness hypothesis is needed here. Also, an oriented graph and its reversal need not be isomorphic, but they do have the same adjacency singular values.

QUESTION 3.2. *Given a connected graph G and a maximal orientation H does there necessarily exist a spanning tree T of G which becomes a sink-source digraph when oriented with the orientation inherited from H?*

Consider the oriented graph with adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is not a maximal orientation. It is acyclic and has only one corresponding pair of maximal blocks, all other blocks being zero. Nevertheless, the associated graph has no spanning tree which inherits a sink-source orientation. This shows that some additional features of maximal orientations are needed to solve Question 3.2.

The author has affirmed Questions 3.1 and 3.2 for all graphs of order ≤ 8.

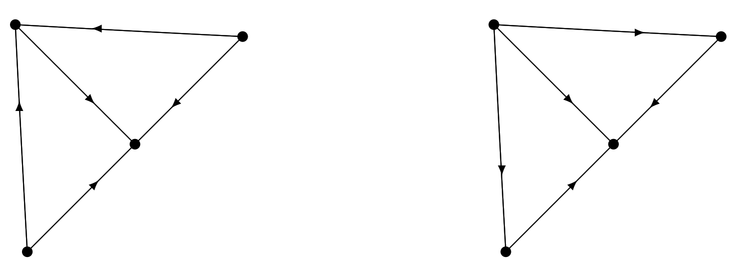

FIGURE 1. *Two orientations H and K of the same graph.*

QUESTION 3.3. *Given a graph G how might one efficiently find a maximal orientation of G?*

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Obviously the method of working through all orientations is very time consuming. In the proof of Proposition 2.2, we have used the idea of switching arcs. However, we may consider the example depicted in Figure 1.

The orientation H is a maximal orientation of the underlying graph. We have $\sigma_1(A(H)) > 2$. On the other hand $\sigma_1(A(K)) = 2$. For each of the 5 orientations L obtained by reversing a single arc of K, we have $\sigma_1(A(L)) < 2$. In fact, H is a global maximum and we see that K is a local maximum, the bane of optimization. One cannot get from K to H by a series of steps switching just one arc at a time while steadily increasing the spectral norm.

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