

## COMPLETE CHARACTERISATION OF KRONECKER INVARIANTS OF A MATRIX PENCIL WITH A PRESCRIBED QUASI-REGULAR SUBPENCIL\*

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**Abstract.** In this paper, the possible Kronecker invariants of a matrix pencil with a prescribed quasi-regular subpencil are determined.

**Key words.** Completion of matrix pencils, Quasi-regular pencils, Polynomial chains.

**AMS subject classifications.** 15A83, 05A17.

**1. Introduction.** Matrix and matrix pencils completion problems have been thoroughly studied in the past. The General Matrix Pencil Completion Problem (GMPCP) consists of determining the possible Kronecker invariants of a matrix pencil with a prescribed subpencil. For the motivation and most important classical results, see e.g. [1, 2, 4, 21, 24, 25, 27, 29–32]. There are also new insights to matrix pencils completion problems coming from Representation Theory (see [22, 28]), but still far away from the compact and explicit form of solutions. However, these problem have been put away from the main stream of Modern Linear Algebra, despite its connection to Control Theory and Linear Systems (both Singular and Regular), and possible applications in engineering, see e.g. [3, 7, 20, 23, 26]. In the last decade, we have done significant effort in bringing these problems back to light. Many new combinatorial structures and results were obtained while trying to find the best path toward solution of GMPCP, see [5, 8–10, 13–16, 19]. The results obtained in [8, 11, 12] have made possible to obtain a solution to many important cases of GMPCP.

Overall, there are two different approaches in studying GMPCP. The *minimal* one is a novel approach that deals with GMPCP by restricting the dimension of the completion. By following this approach, we have used up all its force and solved minimal case completion of GMPCP in [8]. The other, *classical* approach consists in restricting the number of invariants, i.e., by restricting the structure of the involved matrix pencils. In this paper, we follow the classical approach and come close to GMPCP as possible, by studying and solving the following problem:

**PROBLEM 1.** *Characterise the possible Kronecker invariants of a matrix pencil with a prescribed quasi-regular subpencil.*

In Theorem 3.1, we give a complete, explicit and constructive solution to Problem 1. In this way, we generalize results in [1, 2, 4, 6, 9, 10, 13, 16, 22, 24, 27–31]. Indeed, for example in [9] and [13] we have solved Problem 1 with additional restrictions on the resulting pencil – in [9], we have required that the resulting pencil is without nontrivial homogeneous invariant factors, and in [13], we have considered it to be quasi-

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\*Received by the editors on February 10, 2020. Accepted for publication on May 7, 2020. Handling Editor: Froilán M. Dopico.

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regular. In [10], we have studied and solved the minimal case of Problem 1, with restrictions on the size of a completion. Also, in [6] and [16] Problem 1 with additional assumption on the prescribed quasi-regular subpencil was solved – in [16], we assumed that it was without nontrivial homogeneous invariant factors, while in [6], it was assumed to be regular. In this paper, by using powerful recent result on polynomial chains from [12], we give a solution to Problem 1 in full generality. This solution naturally generalizes [6, 9, 10, 13, 16] as well as many related classical results like [1, 2, 4, 24, 27, 29–31].

The paper is organised in four sections and one appendix. In Section 2, we give definitions and auxiliary results that will be used in the proof of the main result. In Section 3, we write Problem 1 as a combinatorial problem, and we give the main result of the paper (Theorem 3.1). In Section 4, we prove the main result. Also, there is an Appendix in which we prove that partitions  $\mathbf{f}$  and  $\mathbf{w}$ , appearing in the main result, are well defined.

**2. Auxiliary results.** Let  $\mathbb{F}$  be an algebraically closed field. All polynomials throughout the paper are assumed to be homogeneous polynomials from  $\mathbb{F}[\lambda, \mu]$ , and monic with respect to  $\lambda$ . For details on homogeneous polynomials see e.g. [5]. Let  $\alpha_1 | \dots | \alpha_n$  be any chain of homogeneous polynomials, we assume  $\alpha_i = 1$ , for all  $i < 1$  and  $\alpha_i = 0$ , for all  $i > n$ . By  $d(\alpha_i)$  we denote the degree of a polynomial  $\alpha_i$ .

By a partition we mean a non increasing sequence of integers. For any integers  $a_1 \geq \dots \geq a_s$  we define a corresponding partition by  $\mathbf{a} = (a_1, \dots, a_s)$ . Moreover, for any such sequence, we assume  $a_i = +\infty$ , for  $i \leq 0$ , and  $a_i = -\infty$ , for  $i > s$ . Also, for any integers  $c_1, \dots, c_r$ , we put  $\sum_{i=a}^b c_i = 0$  whenever  $a > b$ .

In [16], we have introduced the concept of generalized majorization:

DEFINITION 2.1. Let  $d_1 \geq \dots \geq d_{m+k-s}$ ,  $g_1 \geq \dots \geq g_{m+k}$ ,  $a_1 \geq \dots \geq a_s$  be integers. Consider partitions  $\mathbf{d} = (d_1, \dots, d_{m+k-s})$ ,  $\mathbf{g} = (g_1, \dots, g_{m+k})$  and  $\mathbf{a} = (a_1, \dots, a_s)$ . If

$$(2.1) \quad d_i \geq g_{i+s}, \quad i = 1, \dots, m+k-s,$$

$$(2.2) \quad \sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j-j} d_i \leq \sum_{i=1}^j a_i, \quad j = 1, \dots, s$$

$$(2.3) \quad \sum_{i=1}^{m+k} g_i = \sum_{i=1}^{m+k-s} d_i + \sum_{i=1}^s a_i,$$

where

$$h_j := \min\{i | d_{i-j+1} < g_i\}, \quad j = 1, \dots, s,$$

then we say that  $\mathbf{g}$  is *majorized* by  $\mathbf{d}$  and  $\mathbf{a}$ . This type of majorization we call *the generalized majorization*, and we write

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}).$$

Notice that, if (2.3) is satisfied, then (2.2) is equivalent to the following:

$$(2.4) \quad \sum_{i=h_j+1}^{m+k} g_i \geq \sum_{i=h_j-j+1}^{m+k-s} d_i + \sum_{i=j+1}^s a_i, \quad j = 1, \dots, s.$$

DEFINITION 2.2. If partitions  $\mathbf{a}$ ,  $\mathbf{d}$  and  $\mathbf{g}$  in Definition 2.1 satisfy (2.1), (2.4) and

$$(2.5) \quad \sum_{i=1}^{m+k} g_i \geq \sum_{i=1}^{m+k-s} d_i + \sum_{i=1}^s a_i,$$

then we say that  $\mathbf{g}$  is *weakly majorized* by  $\mathbf{d}$  and  $\mathbf{a}$ , and we write

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}).$$

REMARK 2.3. We note that (2.4) for  $j = s$ , and (2.1) together imply:

$$\sum_{i=h_s+1}^{m+k} g_i \geq \sum_{i=h_s-s+1}^{m+k-s} d_i \geq \sum_{i=h_s-s+1}^{m+k-s} g_{i+s},$$

i.e.,

$$d_i = g_{i+s}, \quad \text{for all } i = h_s - s + 1, \dots, m + k - s.$$

Let  $h_0 := 0$ , then (2.2) is trivially satisfied for  $j = 0$ . In addition, if (2.5) is valid, then (2.4) is also trivially satisfied for  $j = 0$ .

Let  $\mathbf{c} = (c_1, \dots, c_m)$  and  $\mathbf{d} = (d_1, \dots, d_{m+x-s})$  be two partitions of nonnegative integers, and let  $\mathbf{a} = (a_1, \dots, a_s)$  and  $\mathbf{b} = (b_1, \dots, b_x)$  be partitions of integers.

Let  $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$  be the following condition: there exists a partition  $\mathbf{g} = (g_1, \dots, g_{m+x})$  of nonnegative integers such that

$$(2.6) \quad \mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

We shall use this condition in our main result.

In [13], we obtained the following result:

THEOREM 2.4. Let  $A(\lambda) \in \mathbb{F}[\lambda]^{n \times (n+m)}$  be a quasi-regular matrix pencil. Denote by  $\alpha_1 | \dots | \alpha_n$  and  $c_1 \geq \dots \geq c_c > c_{c+1} = \dots = c_m = 0$  its homogeneous invariant factors and column minimal indices, respectively. Let  $Q(\lambda) \in \mathbb{F}[\lambda]^{(n+s) \times (n+m+x)}$  be a quasi-regular matrix pencil,  $s \leq m + x$ . Denote by  $\beta_1 | \dots | \beta_{n+s}$  and  $d_1 \geq \dots \geq d_d > d_{d+1} = \dots = d_{m+x-s} = 0$  its homogeneous invariant factors and column minimal indices, respectively.

There exist matrix pencils  $X(\lambda) \in \mathbb{F}[\lambda]^{n \times x}$ ,  $Y(\lambda) \in \mathbb{F}[\lambda]^{s \times (n+m)}$ , and  $Z(\lambda) \in \mathbb{F}[\lambda]^{s \times x}$ , such that the pencil

$$\left[ \begin{array}{c|c} A(\lambda) & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array} \right]$$

is strictly equivalent to  $Q(\lambda)$  if and only if

$$(2.7) \quad c \leq s + d,$$

$$(2.8) \quad \beta_i | \alpha_i | \beta_{i+x+s}, \quad i = 1, \dots, n,$$

and there exists a partition of nonnegative integers  $\mathbf{g} = (g_1, \dots, g_{m+x})$  satisfying:

$$(2.9) \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}),$$

$$(2.10) \quad \mathbf{g} \prec' (\mathbf{d}, \mathbf{a}),$$

where  $\mathbf{a} = (a_1, \dots, a_s)$  and  $\mathbf{b} = (b_1, \dots, b_x)$  are given by

$$a_j = d(\Psi_{s-j+1}) - d(\Psi_{s-j}) - 1, \quad j = 1, \dots, s,$$

$$b_j = d(\Phi_{j-1}) - d(\Phi_j), \quad j = 1, \dots, x,$$

with  $\Psi_j = \prod_{i=1}^{n+j} \text{lcm}(\psi_{i-j}, \beta_i)$ ,  $j = 0, \dots, s$ , and  $\Phi_j = \prod_{i=1}^n \text{lcm}(\psi_i, \alpha_{i-j})$ ,  $j = 0, \dots, x$ , where

$$\psi_i := \text{gcd}(\alpha_i, \beta_{i+s}), \quad i = 1, \dots, n.$$

Also, as a direct corollary of [18, Lemma 2.4], we have the following:

**THEOREM 2.5.** *Let  $\mathbf{c} = (c_1, \dots, c_m)$ ,  $\mathbf{d} = (d_1, \dots, d_{m+x-s})$ ,  $\mathbf{a} = (a_1, \dots, a_s)$ , and  $\mathbf{b} = (b_1, \dots, b_x)$ , be partitions such that*

$$\sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^s a_i = \sum_{i=1}^m c_i + \sum_{i=1}^x b_i.$$

*If there exists a partition  $\mathbf{g}' = (g'_1, \dots, g'_{m+x})$  such that*

$$\mathbf{g}' \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g}' \prec'' (\mathbf{c}, \mathbf{b}),$$

*then there exists a partition  $\mathbf{g} = (g_1, \dots, g_{m+x})$  such that*

$$(2.11) \quad \mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

*Moreover, if  $\mathbf{g}'$  is a partition of nonnegative integers and  $\sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^s a_i \geq 0$ , then there exists a partition  $\mathbf{g}$  of nonnegative integers satisfying (2.11).*

In [12], we have studied polynomial chains. We cite here a version of Theorem 1 from [12] as will be used in the proof of the main result:

**THEOREM 2.6.** *Let  $x, y$  and  $s$  be nonnegative integers such that  $y \geq s$ . Let  $\alpha : \alpha_1 | \dots | \alpha_n$  and  $\gamma : \gamma_1 | \dots | \gamma_{n+s}$  be polynomial chains. Let  $X_1, \dots, X_{x+s-1}$ ,  $Y_1, \dots, Y_{y-s-1}$  and  $Z$  be nonnegative integers.*

*There exists a polynomial chain  $\beta : \beta_1 | \dots | \beta_{n+s}$  satisfying*

$$(2.12) \quad \beta_i | \alpha_i | \beta_{i+x+s}, \quad i = 1, \dots, n,$$

$$(2.13) \quad \gamma_i | \beta_i | \gamma_{i+y-s}, \quad i = 1, \dots, n+s,$$

$$(2.14) \quad \sum_{i=1}^{n+s} d(\beta_i) = Z,$$

$$(2.15) \quad \sum_{i=1}^{n+\min(x+s-j,s)} d(\text{lcm}(\alpha_{i-x-s+j}, \beta_i)) \leq X_j, \quad j = 1, \dots, x+s-1,$$

$$(2.16) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\beta_{i-k}, \gamma_i)) \leq Y_k, \quad k = 1, \dots, y-s-1$$

*if and only if*

$$(2.17) \quad \gamma_i | \alpha_i | \gamma_{i+x+y}, \quad i = 1, \dots, n,$$

$$(2.18) \quad \sum_{i=1}^{n+\min(s,x+s-j+k)} d(\text{lcm}(\alpha_{i-x-s+j-k}, \gamma_i)) \leq X_j + Y_k - Z,$$

*for  $j = 0, \dots, x+s$ ,  $k = 0, \dots, y-s$ , such that  $j \leq x$  or  $k = 0$ .*

*Here  $X_{x+s} = \sum_{i=1}^n d(\alpha_i)$ ,  $X_0 = Z$ ,  $Y_{y-s} = \sum_{i=1}^{n+s} d(\gamma_i)$  and  $Y_0 = Z$ .*

Also, we shall use the following result from [19, Lemma 10]:

LEMMA 2.7. *Let  $A(\lambda) \in \mathbb{F}[\lambda]^{(n+p) \times (n+m+l)}$  and  $B(\lambda) \in \mathbb{F}[\lambda]^{l \times (n+m+l)}$  be matrix pencils,  $n = \text{rank } A(\lambda)$ . Let  $M(\lambda) = \begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix}$  and let  $s = \text{rank } M(\lambda) - \text{rank } A(\lambda)$ . There exist matrix pencils  $X(\lambda) \in \mathbb{F}[\lambda]^{s \times (n+m+l)}$*

*and  $Y(\lambda) \in \mathbb{F}[\lambda]^{(l-s) \times (n+m+l)}$  such that  $\begin{bmatrix} A(\lambda) \\ X(\lambda) \\ Y(\lambda) \end{bmatrix}$  is strictly equivalent to  $M(\lambda)$ , and such that  $\begin{bmatrix} A(\lambda) \\ X(\lambda) \end{bmatrix}$  has rank equal to  $n + s$ , and has the same row minimal indices as  $A(\lambda)$ , and the same column minimal indices as  $M(\lambda)$ .*

**3. Main result.** Let  $A(\lambda) \in \mathbb{F}[\lambda]^{n \times (n+m)}$  be a quasi-regular matrix pencil, hence  $\text{rank } A(\lambda) = n$ . Denote by  $\alpha_1 | \dots | \alpha_n$  and  $c_1 \geq \dots \geq c_m$  its homogeneous invariant factors and column minimal indices, respectively. Let us denote by  $c$  the number of nonzero column minimal indices of  $A(\lambda)$ .

Let  $M(\lambda) \in \mathbb{F}[\lambda]^{(n+y) \times (n+m+x)}$  be a matrix pencil, with  $\text{rank } M(\lambda) = n + s$ . Denote by  $\gamma_1 | \dots | \gamma_{n+s}$ ,  $d_1 \geq \dots \geq d_{m+x-s}$  and  $\bar{r}_1 \geq \dots \geq \bar{r}_{y-s}$  its homogeneous invariant factors, column and row minimal indices, respectively. Let us denote by  $d$  the number of nonzero column minimal indices of  $M(\lambda)$ .

By the ranks of the pencils  $A(\lambda)$  and  $M(\lambda)$ , we have

$$(3.19) \quad \sum_{i=1}^n d(\alpha_i) + \sum_{i=1}^m c_i + s = \sum_{i=1}^{n+s} d(\gamma_i) + \sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^{y-s} \bar{r}_i.$$

Let us define integers  $f_1, \dots, f_s$  by

$$(3.20) \quad \sum_{i=j+1}^s f_i = \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \gamma_i)) - \sum_{i=1}^n d(\alpha_i) - s + j, \quad j = 0, \dots, s-1,$$

and integers  $w_1, \dots, w_x$  by

$$(3.21) \quad \sum_{i=j+1}^x w_i = \max_{k=0, \dots, y-s} \left[ \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-k}, \gamma_i)) - \sum_{i=k+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i) \right], \quad j = 0, \dots, x-1.$$

In Appendix, we prove that if  $\gamma_i | \alpha_i$ ,  $i = 1, \dots, n$ , then  $f_1 \geq \dots \geq f_s \geq -1$ . Also, we prove that if the condition

$$(3.22) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-x-k}, \gamma_i)) \leq \sum_{i=k+1}^{y-s} \bar{r}_i + \sum_{i=1}^{n+s} d(\gamma_i), \quad k = 0, \dots, y-s$$

is valid, then  $w_1 \geq \dots \geq w_x \geq 0$ . Thus, if  $\gamma_i | \alpha_i$ ,  $i = 1, \dots, n$ , and (3.22) holds we have that both partitions  $\mathbf{f} = (f_1, \dots, f_s)$  and  $\mathbf{w} = (w_1, \dots, w_x)$  are well defined.

Now we can give the main result:

THEOREM 3.1. *Let  $A(\lambda) \in \mathbb{F}[\lambda]^{n \times (n+m)}$  be a quasi-regular matrix pencil and  $M(\lambda) \in \mathbb{F}[\lambda]^{(n+y) \times (n+m+x)}$  be a matrix pencil, as above. There exist matrix pencils  $X(\lambda) \in \mathbb{F}[\lambda]^{n \times x}$ ,  $Y(\lambda) \in \mathbb{F}[\lambda]^{y \times (n+m)}$ , and  $Z(\lambda) \in \mathbb{F}[\lambda]^{y \times x}$ , such that the pencil*

$$(3.23) \quad \left[ \begin{array}{c|c} A(\lambda) & X(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array} \right]$$

is strictly equivalent to  $M(\lambda)$  if and only if

- (i)  $c \leq d + s$ ,
- (ii)  $\gamma_i | \alpha_i | \gamma_{i+x+y}, \quad i = 1, \dots, n$ ,
- (iii)  $\sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-x-k}, \gamma_i)) \leq \sum_{i=k+1}^{y-s} \bar{r}_i + \sum_{i=1}^{n+s} d(\gamma_i), \quad k = 0, \dots, y - s$ ,
- (iv)  $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{f}, \mathbf{w})$  holds.

REMARK 3.2. Once [11] is published, the condition  $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{f}, \mathbf{w})$  in Theorem 3.1 can be replaced with its explicit form  $\Omega(\mathbf{c}, \mathbf{d}, \mathbf{f}, \mathbf{w})$  given in [11].

**3.1. Properties of the pencil (3.23) and its subpencils.** Since  $A(\lambda)$  is a quasi-regular matrix pencil, so is the pencil

$$(3.24) \quad \left[ \begin{array}{c|c} A(\lambda) & X(\lambda) \end{array} \right] \in \mathbb{F}[\lambda]^{n \times (n+m+x)}$$

and its rank equals  $n$ . Moreover, since the rank of (3.23) is  $n + s$ , by Lemma 2.7 we have that the pencil (3.23) is strictly equivalent to the pencil

$$\left[ \begin{array}{c|c} A(\lambda) & X(\lambda) \\ \hline Y_1(\lambda) & Z_1(\lambda) \\ \hline Y_2(\lambda) & Z_2(\lambda) \end{array} \right],$$

where the subpencil

$$(3.25) \quad \left[ \begin{array}{cc} A(\lambda) & X(\lambda) \\ Y_1(\lambda) & Z_1(\lambda) \end{array} \right] \in \mathbb{F}[\lambda]^{(n+s) \times (n+m+x)}$$

is such that its rank is exactly  $n + s$ , i.e., it is quasi-regular and its column minimal indices coincide with the column minimal indices of (3.23) ( $d_1 \geq \dots \geq d_{m+x-s}$ ).

Let us denote the homogeneous invariant factors of (3.25) by  $\beta_1 | \dots | \beta_{n+s}$ .

Next, let us consider the completion of  $A(\lambda)$  up to (3.25). It is a completion from a quasi-regular up to a quasi-regular matrix pencil, so we can apply Theorem 2.4:

A completion from  $A(\lambda)$  up to (3.25) is possible if and only if there exists a partition of nonnegative integers  $\mathbf{g} = (g_1, \dots, g_{m+x})$  satisfying conditions (2.7)-(2.10). We note that from the ranks of the involved matrix pencils we also have

$$(3.26) \quad \sum_{i=1}^n d(\alpha_i) + \sum_{i=1}^m c_i + s = \sum_{i=1}^{n+s} d(\beta_i) + \sum_{i=1}^{m+x-s} d_i.$$

By the definition of the general majorization, (2.9) is equivalent to

$$(3.27) \quad c_i \geq g_{i+x}, \quad i = 1, \dots, m,$$

$$(3.28) \quad \sum_{i=h_j+1}^{m+x} g_i - \sum_{i=h_j-j+1}^m c_i \geq \sum_{i=j+1}^x b_i, \quad j = 0, \dots, x,$$

and

$$(3.29) \quad \sum_{i=1}^{m+x} g_i = \sum_{i=1}^m c_i + \sum_{i=1}^x b_i,$$

where  $h_j := \min\{i | c_{i-j+1} < g_i\}$ ,  $j = 1, \dots, x$ , and  $h_0 := 0$ .

By the definition of  $b_i$ 's, if (2.8) is satisfied, then we have that

$$\sum_{i=j+1}^x b_i = \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-j-s}, \beta_i)) - \sum_{i=1}^{n+s} d(\beta_i), \quad j = 0, \dots, x-1.$$

Hence, (3.28) becomes

$$(3.30) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-j-s}, \beta_i)) \leq \sum_{i=h_j+1}^{m+x} g_i - \sum_{i=h_j-j+1}^m c_i + \sum_{i=1}^{n+s} d(\beta_i)$$

for all  $j = 0, \dots, x$ . Also, (3.29) becomes

$$(3.31) \quad \sum_{i=1}^{m+x} g_i = \sum_{i=1}^m c_i + \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s}, \beta_i)) - \sum_{i=1}^{n+s} d(\beta_i).$$

Analogously, by the definition of the general majorization, (2.10) is equivalent to

$$(3.32) \quad d_i \geq g_{i+s}, \quad i = 1, \dots, m+x-s,$$

$$(3.33) \quad \sum_{i=v_j+1}^{m+x} g_i - \sum_{i=v_j-j+1}^{m+x-s} d_i \geq \sum_{i=j+1}^s a_i, \quad j = 0, \dots, s,$$

and

$$(3.34) \quad \sum_{i=1}^{m+x} g_i = \sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^s a_i,$$

where  $v_j := \min\{i | d_{i-j+1} < g_i\}$ ,  $j = 1, \dots, s$ , and  $v_0 := 0$ .

By the definition of  $a_i$ 's, if (2.8) is satisfied, then we have that

$$\sum_{i=j+1}^s a_i = \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \beta_i)) - \sum_{i=1}^n d(\alpha_i) - s + j, \quad j = 0, \dots, s-1.$$

Hence, (3.33) becomes

$$(3.35) \quad \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \beta_i)) \leq \sum_{i=v_j+1}^{m+x} g_i - \sum_{i=v_j-j+1}^{m+x-s} d_i + \sum_{i=1}^n d(\alpha_i) + s - j, \quad j = 0, \dots, s,$$

and (3.34) becomes

$$(3.36) \quad \sum_{i=1}^{m+x} g_i = \sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s}, \beta_i)) - \sum_{i=1}^n d(\alpha_i) - s.$$

We note that if (3.26) is valid, then the righthand sides of (3.31) and (3.36) coincide, as needed.

Finally, let us consider the row completion from (3.25) up to (3.23). By Theorem 2 in [15] (see also [14, 19]), we have that this completion is possible if and only if the following conditions are valid:

$$(3.37) \quad \gamma_i | \beta_i | \gamma_{i+y-s}, \quad i = 1, \dots, n+s,$$

$$(3.38) \quad \sum_{i=1}^{n+s} d(\beta_i) = \sum_{i=1}^{n+s} d(\gamma_i) + \sum_{i=1}^{y-s} \bar{r}_i,$$

$$(3.39) \quad \sum_{i=1}^k \bar{r}_i \leq \sum_{i=1}^k \hat{b}_i, \quad k = 1, \dots, y-s,$$

where

$$\sum_{i=1}^k \hat{b}_i = \sum_{i=1}^{y-s} \bar{r}_i + \sum_{i=1}^{n+s} d(\gamma_i) - \sum_{i=1}^{n+s} d(\text{lcm}(\beta_{i-k}, \gamma_i)), \quad k = 1, \dots, y-s.$$

Hence, (3.39) is equivalent to

$$(3.40) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\beta_{i-k}, \gamma_i)) \leq \sum_{i=k+1}^{y-s} \bar{r}_i + \sum_{i=1}^{n+s} d(\gamma_i), \quad k = 1, \dots, y-s.$$

REMARK 3.3. We note that if (3.37) and (3.38) hold, the inequality (3.40) is also trivially satisfied for  $k = 0$ .

Finally, as the result of the above analysis we conclude that our starting completion problem given in Theorem 3.1 is equivalent to the existence of homogeneous polynomials  $\beta_1 | \dots | \beta_{n+s}$  and a partition  $\mathbf{g} = (g_1, \dots, g_{m+x})$  of nonnegative integers, which satisfy conditions (2.7), (2.8), (3.26), (3.27), (3.30), (3.31), (3.32), (3.35), (3.36), (3.37), (3.38) and (3.40).

Now we pass to the proof of Theorem 3.1 given in the following section.

#### 4. Proof of Theorem 3.1.

**4.1. Necessity of the conditions (i) – (iv).** Let us suppose that there exists the wanted completion. As we have shown in Section 3.1 that is equivalent to the existence of homogeneous polynomials  $\beta_1 | \dots | \beta_{n+s}$  and a partition  $\mathbf{g} = (g_1, \dots, g_{m+x})$  of nonnegative integers, which satisfy conditions (2.7), (2.8), (3.26), (3.27), (3.30), (3.31), (3.32), (3.35), (3.36), (3.37), (3.38) and (3.40).

Condition (2.7) is equal to condition (i). Conditions (2.8) and (3.37) give (ii). Conditions (2.8), (3.38) and (3.40) give (iii).

Before proceeding with the proof of (iv), we shall write conditions (3.30), (3.35) and (3.40) in slightly different equivalent forms, according to the notation from Theorem 2.6. Let

$$(4.41) \quad Z := \sum_{i=1}^{y-s} \bar{r}_i + \sum_{i=1}^{n+s} d(\gamma_i).$$



Consider condition (3.30). By (3.38), it is equal to

$$(4.42) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-x+j-s}, \beta_i)) \leq \sum_{i=h_{x-j}+1}^{m+x} g_i - \sum_{i=h_{x-j-x+j+1}}^m c_i + Z, \quad j = 0, \dots, x.$$

Let us denote by

$$(4.43) \quad X_j := \sum_{i=h_{x-j}+1}^{m+x} g_i - \sum_{i=h_{x-j-x+j+1}}^m c_i + Z, \quad j = 0, \dots, x.$$

Then (4.42) becomes

$$(4.44) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-x-s+j}, \beta_i)) \leq X_j, \quad j = 0, \dots, x.$$

Next consider condition (3.35). It is equal to

$$(4.45) \quad \sum_{i=1}^{n+x+s-j} d(\text{lcm}(\alpha_{i-x-s+j}, \beta_i)) \leq \sum_{i=v_{j-x}+1}^{m+x} g_i - \sum_{i=v_{j-x-j+x+1}}^{m+x-s} d_i + \sum_{i=1}^n d(\alpha_i) + s - j + x$$

for all  $j = x, \dots, x + s$ . Let us denote by

$$(4.46) \quad X_j := \sum_{i=v_{j-x}+1}^{m+x} g_i - \sum_{i=v_{j-x-j+x+1}}^{m+x-s} d_i + \sum_{i=1}^n d(\alpha_i) + s - j + x, \quad j = x, \dots, x + s.$$

Now (4.45) becomes

$$(4.47) \quad \sum_{i=1}^{n+x+s-j} d(\text{lcm}(\alpha_{i-x-s+j}, \beta_i)) \leq X_j, \quad j = x, \dots, x + s.$$

Note that for  $j = x$  the definitions of  $X_x$  from (4.43) and (4.46) coincide. Also, we note that  $X_0 = Z$ , and  $X_{x+s} = \sum_{i=1}^n d(\alpha_i)$ .

Finally, let us denote by

$$(4.48) \quad Y_k := \sum_{i=k+1}^{y-s} \bar{r}_i + \sum_{i=1}^{n+s} d(\gamma_i), \quad k = 0, \dots, y - s.$$

By Remark 3.3, (3.40) becomes

$$(4.49) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\beta_{i-k}, \gamma_i)) \leq Y_k, \quad k = 0, \dots, y - s.$$

Note that  $Y_0 = Z$ , and  $Y_{y-s} = \sum_{i=1}^{n+s} d(\gamma_i)$ .

Hence, (4.44), (4.47) and (4.49), together with (2.8), (3.37), (3.38) and (4.41), by Theorem 2.6 give that the following is valid:

$$(4.50) \quad \gamma_i |\alpha_i| \gamma_{i+x+y}, \quad i = 1, \dots, n,$$

$$(4.51) \quad \sum_{i=1}^{n+\min(s, x+s-j+k)} d(\text{lcm}(\alpha_{i-x-s+j-k}, \gamma_i)) \leq X_j + Y_k - Z,$$

for  $j = 0, \dots, x + s$ ,  $k = 0, \dots, y - s$ , such that  $j \leq x$  or  $k = 0$ .

Due to restriction on the indices  $j$  and  $k$  ( $j \leq x$  or  $k = 0$ ), condition (4.51) can be split into

$$(4.52) \quad \sum_{i=1}^{n+x+s-j} d(\text{lcm}(\alpha_{i-x-s+j}, \gamma_i)) \leq X_j, \quad j = x, \dots, x + s,$$

and

$$(4.53) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-x-s+j-k}, \gamma_i)) \leq X_j + Y_k - Z, \quad j = 0, \dots, x, \quad k = 0, \dots, y - s.$$

Before proceeding we note that (4.52) for  $j = x$  coincide with (4.53) for  $j = x$  and  $k = 0$ .

From the definition of  $X_j$ ,  $j = x, \dots, x + s$ , condition (4.52) is equal to

$$(4.54) \quad \sum_{i=v_j-x+1}^{m+x} g_i - \sum_{i=v_j-x-j+1}^{m+x-s} d_i \geq \sum_{i=1}^{n+x+s-j} d(\text{lcm}(\alpha_{i-x-s+j}, \gamma_i)) - \sum_{i=1}^n d(\alpha_i) - s + j - x,$$

for  $j = x, \dots, x + s$ . The last is equal to

$$(4.55) \quad \sum_{i=v_j+1}^{m+x} g_i - \sum_{i=v_j-j+1}^{m+x-s} d_i \geq \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \gamma_i)) - \sum_{i=1}^n d(\alpha_i) - s + j, \quad j = 0, \dots, s.$$

Since the condition (ii) is valid, from the definition of the weak generalized majorization, and by the definition (3.20) of  $\mathbf{f} = (f_1, \dots, f_s)$ , we have that (3.32) and (4.55) are equivalent to

$$(4.56) \quad \mathbf{g} \prec'' (\mathbf{d}, \mathbf{f}).$$

As for condition (4.53), by the definition of  $X_j$ ,  $j = 0, \dots, x$  and  $Y_k$ ,  $k = 0, \dots, y - s$ , it is equal to

$$(4.57) \quad \sum_{i=h_x-j+1}^{m+x} g_i - \sum_{i=h_x-j-x+1}^m c_i \geq \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-x-s+j-k}, \gamma_i)) - \sum_{i=k+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i)$$

for all  $j = 0, \dots, x$ ,  $k = 0, \dots, y - s$ .

The last is equal to

$$(4.58) \quad \sum_{i=h_j+1}^{m+x} g_i - \sum_{i=h_j-j+1}^m c_i \geq \max_{k=0, \dots, y-s} \left[ \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-k}, \gamma_i)) - \sum_{i=k+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i) \right]$$

for all  $j = 0, \dots, x$ .

Since the condition (iii) is valid, from the definition of the weak generalized majorization, and by the definition (3.21) of  $\mathbf{w} = (w_1, \dots, w_x)$ , we have that (3.27) and (4.58) are equivalent to

$$(4.59) \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{w}).$$

Finally, (4.56) and (4.59) give (iv) as wanted.

**4.2. Sufficiency of the conditions (i) – (iv).** Let us suppose that conditions (i) – (iv) are valid. We are left with proving the existence of homogeneous polynomials  $\beta_1 | \dots | \beta_{n+s}$  and a partition of nonnegative integers  $\mathbf{g} = (g_1, \dots, g_{m+x})$  which satisfy conditions (2.7), (2.8), (3.26), (3.27), (3.30), (3.31), (3.32), (3.35), (3.36), (3.37), (3.38) and (3.40).

First note that (i) coincide with (2.7). Next, we have that condition (iv) implies the existence of a partition  $\mathbf{g}' = (g'_1, \dots, g'_{m+x})$  of nonnegative integers such that

$$(4.60) \quad \mathbf{g}' \prec'' (\mathbf{c}, \mathbf{w})$$

and

$$(4.61) \quad \mathbf{g}' \prec'' (\mathbf{d}, \mathbf{f}).$$

By the definition of the weak generalized majorization, together with the definition of  $\mathbf{w}$  from (3.21), we have that (4.60) is equivalent to

$$(4.62) \quad c_i \geq g'_{i+x}, \quad i = 1, \dots, m,$$

and

$$(4.63) \quad \sum_{i=h'_j+1}^{m+x} g'_i - \sum_{i=h'_j-j+1}^m c_i \geq \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-k}, \gamma_i)) - \sum_{i=k+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i)$$

for all  $j = 0, \dots, x$ ,  $k = 0, \dots, y - s$ , where  $h'_j := \min\{i | c_{i-j+1} < g'_j\}$ ,  $j = 1, \dots, x$ ,  $h'_0 := 0$ .

Also, by the definition of the weak generalized majorization, together with the definition of  $\mathbf{f}$  from (3.20), condition (4.61) is equivalent to

$$(4.64) \quad d_i \geq g'_{i+s}, \quad i = 1, \dots, m + x - s,$$

and

$$(4.65) \quad \sum_{i=v'_j+1}^{m+x} g'_i - \sum_{i=v'_j-j+1}^{m+x-s} d_i \geq \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \gamma_i)) - \sum_{i=1}^n d(\alpha_i) - s + j, \quad j = 0, \dots, s,$$

where  $v'_j := \min\{i | d_{i-j+1} < g'_j\}$ ,  $j = 1, \dots, s$ ,  $v'_0 := 0$ .

Equation (4.63) is equal to

$$(4.66) \quad \sum_{i=h'_{x-j}+1}^{m+x} g'_i - \sum_{i=h'_{x-j-x+j+1}}^m c_i \geq \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-x-s+j-k}, \gamma_i)) - \sum_{i=k+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i)$$

for all  $j = 0, \dots, x, k = 0, \dots, y - s$ .

Equation (4.65) is equal to

$$(4.67) \quad \sum_{i=v'_j-x+1}^{m+x} g'_i - \sum_{i=v'_j-x-j+x+1}^{m+x-s} d_i \geq \sum_{i=1}^{n+s-j+x} d(\text{lcm}(\alpha_{i-s+j-x}, \gamma_i)) - \sum_{i=1}^n d(\alpha_i) - s + j - x$$

for all  $j = x, \dots, x + s$ .

Let  $Y_k$  be as in (4.48), and let  $Z$  be as in (4.41).

Let

$$(4.68) \quad X'_j := \sum_{i=h'_{x-j}+1}^{m+x} g'_i - \sum_{i=h'_{x-j-x+j+1}}^m c_i + Z, \quad j = 0, \dots, x,$$

$$(4.69) \quad X'_j := \sum_{i=v'_j-x+1}^{m+x} g'_i - \sum_{i=v'_j-x-j+x+1}^{m+x-s} d_i + \sum_{i=1}^n d(\alpha_i) + s - j + x, \quad j = x, \dots, x + s.$$

We note that  $X'_0 (= Y_0) = Z$ , and  $X'_{s+x} = \sum_{i=1}^n d(\alpha_i)$ .

Then we write (4.66) and (4.67) together as

$$(4.70) \quad \sum_{i=1}^{n+\min(s, x+s-j+k)} d(\text{lcm}(\alpha_{i-x-s+j-k}, \gamma_i)) \leq X'_j + Y_k - Z,$$

for  $j = 0, \dots, x + s, k = 0, \dots, y - s$ , such that  $j \leq x$  or  $k = 0$ .

By Theorem 2.6, conditions (ii) and (4.70) give the existence of homogeneous polynomials  $\beta_1 | \dots | \beta_{n+s}$  which satisfy

$$(4.71) \quad \sum_{i=1}^{n+s} d(\beta_i) = Z,$$

conditions (2.8), (3.37), (3.40),

$$(4.72) \quad \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j}, \beta_i)) \leq \sum_{i=h'_j+1}^{m+x} g'_i - \sum_{i=h'_j-j+1}^m c_i + \sum_{i=1}^{n+s} d(\beta_i), \quad j = 0, \dots, x,$$

and

$$(4.73) \quad \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \beta_i)) \leq \sum_{i=v'_j+1}^{m+x} g'_i - \sum_{i=v'_j-j+1}^{m+x-s} d_i + \sum_{i=1}^n d(\alpha_i) + s - j, \quad j = 0, \dots, s - 1.$$

We note that (4.73) is also trivially satisfied for  $j = s$ .

Condition (4.71) is equal to (3.38), and (4.71) together with (3.19) gives (3.26).

If denote by

$$(4.74) \quad \sum_{i=j+1}^x b'_i := \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j}, \beta_i)) - \sum_{i=1}^{n+s} d(\beta_i), \quad j = 0, \dots, x-1,$$

and by

$$(4.75) \quad \sum_{i=j+1}^s a'_i := \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \beta_i)) - \sum_{i=1}^n d(\alpha_i) - s + j, \quad j = 0, \dots, s-1,$$

conditions (4.62) and (4.72), as well as (4.64) and (4.73), give

$$(4.76) \quad \mathbf{g}' \prec'' (\mathbf{c}, \mathbf{b}') \quad \text{and} \quad \mathbf{g}' \prec'' (\mathbf{d}, \mathbf{a}').$$

Here  $\mathbf{a}' = (a'_1, \dots, a'_s)$  and  $\mathbf{b}' = (b'_1, \dots, b'_x)$ .

By (3.26), we have  $\sum_{i=1}^m c_i + \sum_{i=1}^x b'_i = \sum_{i=1}^{m+x-s} d_i + \sum_{i=1}^s a'_i \geq 0$ . By Theorem 2.5, we have that (4.76) implies the existence of a partition  $\mathbf{g} = (g_1, \dots, g_{m+x})$  of nonnegative integers such that

$$(4.77) \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}') \quad \text{and} \quad \mathbf{g} \prec' (\mathbf{d}, \mathbf{a}').$$

By the definition of the generalized majorization condition (4.77) give conditions (3.27), (3.30), (3.31), (3.32), (3.35) and (3.36), as wanted.

This finishes our proof.

**Appendix.** The following lemma is a simple generalization of [17, Lemma 2]:

LEMMA 4.1. *Let  $\bar{\alpha} : \bar{\alpha}_1 | \dots | \bar{\alpha}_n$  and  $\bar{\gamma} : \bar{\gamma}_1 | \dots | \bar{\gamma}_m$  be two chains of monic polynomials. Let*

$$(4.78) \quad \pi_j(\bar{\alpha}, \bar{\gamma}) = \prod_{i=\min(1, j+1)}^{n+j} \text{lcm}(\bar{\alpha}_{i-j}, \bar{\gamma}_i), \quad j \leq m-n$$

and let

$$(4.79) \quad \sigma_j(\bar{\alpha}, \bar{\gamma}) = \frac{\pi_j(\bar{\alpha}, \bar{\gamma})}{\pi_{j-1}(\bar{\alpha}, \bar{\gamma})}, \quad j \leq m-n.$$

Then

$$\sigma_j(\bar{\alpha}, \bar{\gamma}) | \sigma_{j+1}(\bar{\alpha}, \bar{\gamma}), \quad j \leq m-n-1.$$

In this section, we shall prove that the integers  $f_1, \dots, f_s$  defined by (3.20) as well as the integers  $w_1, \dots, w_x$  defined by (3.21) are both non-increasing, thus defining partitions  $\mathbf{f} = (f_1, \dots, f_s)$  and  $\mathbf{w} = (w_1, \dots, w_x)$ .

Let  $\gamma_i | \alpha_i$ ,  $i = 1, \dots, n$ . By using definitions (4.78) and (4.79), from (3.20) we have

$$f_j = \sum_{i=1}^{n+s-j+1} d(\text{lcm}(\alpha_{i-s+j-1}, \gamma_i)) - \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \gamma_i)) - 1 = d(\sigma_{s-j+1}(\alpha, \gamma)) - 1$$

for all  $j = 1, \dots, s - 1$ , and

$$f_{j+1} = \sum_{i=1}^{n+s-j} d(\text{lcm}(\alpha_{i-s+j}, \gamma_i)) - \sum_{i=1}^{n+s-j-1} d(\text{lcm}(\alpha_{i-s+j+1}, \gamma_i)) - 1 = d(\sigma_{s-j}(\alpha, \gamma)) - 1$$

for all  $j = 1, \dots, s - 1$ . Thus, by Lemma 4.1, we get  $f_j \geq f_{j+1}$ ,  $j = 1, \dots, s - 1$ , as wanted.

Let the condition (iii) from Theorem 3.1 be valid, i.e., let (3.22) be satisfied.

From (3.21), we have

$$(4.80) \quad \sum_{i=j}^x w_i = \max_{k=0, \dots, y-s} \left[ \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-k+1}, \gamma_i)) - \sum_{i=k+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i) \right]$$

for all  $j = 1, \dots, x$ . Let us denote by

$$F(j, k) := \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-k+1}, \gamma_i)) - \sum_{i=k+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i)$$

for all  $j = 1, \dots, x$ , and  $k = 0, \dots, y - s$ .

For any  $j = 1, \dots, x$ , let  $k_j \in \{0, \dots, y - s\}$  be such that

$$(4.81) \quad \sum_{i=j}^x w_i = \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-k_j+1}, \gamma_i)) - \sum_{i=k_j+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i) = F(j, k_j),$$

i.e., for every  $j = 1, \dots, x$ , we have

$$(4.82) \quad F(j, k_j) \geq F(j, k), \quad k = 0, \dots, y - s.$$

For every  $j = 1, \dots, x - 1$ , we have

$$(4.83) \quad \begin{aligned} w_j &= \sum_{i=j}^x w_i - \sum_{i=j+1}^x w_i = F(j, k_j) - F(j+1, k_{j+1}) \\ &\geq F(j, k_{j+1}) - F(j+1, k_{j+1}) = \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j+1-k_{j+1}}, \gamma_i)) - \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-k_{j+1}}, \gamma_i)) \\ &= d(\pi_{-s-j+1-k_{j+1}}(\gamma, \alpha)) - d(\pi_{-s-j-k_{j+1}}(\gamma, \alpha)) = d(\sigma_{-s-j+1-k_{j+1}}(\gamma, \alpha)). \end{aligned}$$

Also, for every  $j = 1, \dots, x - 2$ , we have

$$\begin{aligned} w_{j+1} &= \sum_{i=j+1}^x w_i - \sum_{i=j+2}^x w_i = F(j+1, k_{j+1}) - F(j+2, k_{j+2}) \\ &\leq F(j+1, k_{j+1}) - F(j+2, k_{j+1}) \\ &= \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-k_{j+1}}, \gamma_i)) - \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-j-1-k_{j+1}}, \gamma_i)) \\ &= d(\sigma_{-s-j-k_{j+1}}(\gamma, \alpha)). \end{aligned}$$

Moreover, by condition (3.22) for  $k = k_x$ , we have

$$\begin{aligned} w_x &= \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-x-k_x+1}, \gamma_i)) - \sum_{i=k_x+1}^{y-s} \bar{r}_i - \sum_{i=1}^{n+s} d(\gamma_i) \\ &\leq \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-x-k_x+1}, \gamma_i)) - \sum_{i=1}^{n+s} d(\text{lcm}(\alpha_{i-s-x-k_x}, \gamma_i)) \\ &= d(\sigma_{-s-x+1-k_x}(\gamma, \alpha)). \end{aligned}$$

Therefore, for all  $j = 1, \dots, x-1$ , we have

$$(4.84) \quad w_{j+1} \leq d(\sigma_{-s-j-k_{j+1}}(\gamma, \alpha)).$$

By applying Lemma 4.1, (4.83) and (4.84) give  $w_j \geq w_{j+1}$  for all  $j = 1, \dots, x-1$ , as wanted.

**Acknowledgements.** We thank the referee for improvements of the manuscript.

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