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GROUP INVERSES OF MATRICES WITH PATH GRAPHS*

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Abstract. A simple formula for the group inverse of a 2×2 block matrix with a bipartite digraph is given in terms of the block matrices. This formula is used to give a graph-theoretic description of the group inverse of an irreducible tridiagonal matrix of odd order with zero diagonal (which is singular). Relations between the zero/nonzero structures of the group inverse and the Moore-Penrose inverse of such matrices are given. An extension of the graph-theoretic description of the group inverse to singular matrices with tree graphs is conjectured.

Key words. Group inverse, Tridiagonal matrix, Tree graph, Moore-Penrose inverse, Bipartite digraph.

AMS subject classifications. 15A09, 05C50.

1. Introduction. For a real $n \times n$ matrix A, the *group inverse*, if it exists, is the unique matrix $A^\#$ satisfying the matrix equations $AA^\# = A^\#A$, $AA^\#A = A$ and $A^\#AA^\# = A^\#$. If A is invertible, then $A^\# = A^{-1}$. It is well-known that $A^\#$ exists if and only if rank $A =$ rank A^2. For more detailed expositions on the group inverse and its properties, see [3], [7].

We present a new formula in Section 2 for the group inverse of a 2×2 block matrix with bipartite form as in (1.1) below. We use this formula to give a graph-theoretic description of the entries of the group inverse of an irreducible tridiagonal matrix of order $2k+1$ with zero diagonal (which has a path graph and is singular). This description, given in Section 3, is proved using a graph-theoretic characterization of the usual inverse of a nonsingular tridiagonal matrix of order k (see e.g. [11]). In Section 4, we relate our results to the zero/nonzero structure of another type of generalized inverse, the Moore-Penrose inverse. We conclude in Section 5 with a conjecture, which extends our graph-theoretic description of the entries of the group inverse to a matrix with a tree graph.

Generalized inverses of banded matrices, including tridiagonal matrices, are considered in [2] where the focus is on the rank of submatrices of the generalized inverse. Campbell and Meyer [7, page 139] investigate the Drazin inverse (which is a generalization of the group inverse) for a 2×2 block matrix. Recently, special cases of this problem that have been studied are listed in [10] and some new formulas are derived.

We first introduce some graph-theoretic notation. There is a one-to-one correspon-

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dence between $n \times n$ matrices $A = (a_{ij})$ and digraphs $D(A) = (V, E)$ having vertex set $V = \{1, \cdots, n\}$ and arc set E, where $(i, j) \in E$ if and only if $a_{ij} \neq 0$. For $q \geq 1$, a sequence $(i_1, i_2, i_3, \cdots, i_q, i_{q+1})$ of distinct vertices with arcs $(i_1, i_2), (i_2, i_3), \cdots, (i_q, i_{q+1})$ all in E is called a *path of length q* from i_1 to i_{q+1} in $D(A)$. For $q \geq 2$, a sequence $(i_1, i_2, i_3, \cdots, i_q, i_1)$ with $i_1, i_2, \cdots, i_q$ distinct and arcs $(i_1, i_2), \cdots (i_q, i_1)$ in E is called a *q-cycle* (a *cycle of length q*) in $D(A)$. A digraph is called a (directed) *tree graph* if it is strongly connected and all of its cycles have length 2. If the digraph $D(A)$ of a matrix A is a tree graph, then all of the diagonal entries of A are necessarily zero. Since a tree graph is bipartite, its vertices can be labeled so that its associated matrix has the form

$$A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}, \tag{1.1}$$

where $B \in \mathbb{R}^{p \times (n-p)}$, $C \in \mathbb{R}^{(n-p) \times p}$ and $p \leq \frac{n}{2}$.

A particular example of a tree graph is a *path graph* on n vertices $i_1, i_2, \cdots, i_n$ which consists of the path $p = (i_1, i_2, \cdots, i_n)$ from i_1 to i_n and its reversal (i.e., the path obtained by reversing all of the arcs in p). If, for $k \geq 1$, a path graph on $n = 2k + 1$ vertices consists of the path $(k+1, 1, k+2, 2, \cdots, 2k, k, 2k+1)$ and its reversal, then we call this the *bipartite path graph* on $n = 2k + 1$ vertices.

Consider a tree graph $D(A)$, with A as in (1.1). For every pair of distinct vertices i_1 and i_{q+1}, there is a unique path $(i_1, i_2, \cdots, i_q, i_{q+1})$ from i_1 to i_{q+1}. For this path, the product $a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_q,i_{q+1}}$ is called the *path product* and is denoted by $P_A[i_1 \to i_{q+1}]$. All of the cycles in $D(A)$ are 2-cycles and a product $a_{i_1,i_2} a_{i_2,i_1} a_{i_3,i_4} a_{i_4,i_3} \cdots a_{i_{r-1},i_r} a_{i_r,i_{r-1}}$ corresponding to a set $\{(i_1, i_2, i_1), (i_3, i_4, i_3), \cdots, (i_{r-1}, i_r, i_{r-1})\}$ of $r/2$ disjoint 2-cycles in $D(A)$ is called a *matching* in $D(A)$ of size r. If this set of 2-cycles has maximal cardinality, then the matching is a *maximal matching* and the number r is called the *term rank* of A. The sum of all maximal matchings in $D(A)$ is denoted by Δ_A. The notation $\gamma[i_1, i_{q+1}]$ denotes the sum of all maximal matchings in the path subgraph of $D(A)$ on the vertices $i_1, \cdots, i_{q+1}$, and we set $\gamma[i_w, i_w] = 1$. Also, $\gamma(i_1, i_{q+1})$ denotes the sum of all maximal matchings *not* on the path subgraph of $D(A)$ on the vertices $i_1, \cdots, i_{q+1}$. If there are no such maximal matchings, then $\gamma(i_1, i_{q+1}) = 1$. It follows from these definitions that $\gamma[i_1, i_{q+1}] = \gamma[i_{q+1}, i_1]$ and $\gamma(i_1, i_{q+1}) = \gamma(i_{q+1}, i_1)$. If $D(A)$ is the path graph on vertices $i_1, \cdots, i_n$, then $\Delta_A = \gamma[i_1, i_n]$.

For a tree graph $D(A)$, the matrix A is nearly reducible, so the term rank of A is equal to the rank of A [4, Theorem 4.5]. The following proposition shows that a necessary and sufficient condition for $A^\#$ to exist is that the sum of all maximal matchings in $D(A)$ is nonzero, i.e. $\Delta_A \neq 0$. An analogous result for an arbitrary complex $n \times n$ matrix is given in [6, Lemma 2.2]. Our proof uses the fact that the group inverse of A exists if and only if rank $A =$ rank A^2, or equivalently, the geometric and algebraic multiplicities of the eigenvalue 0 are equal [8, Exercise 17, page 141].

PROPOSITION 1.1. *Let A be an $n \times n$ matrix with a tree graph $D(A)$. Then the group*

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inverse $A^{\#}$ *exists if and only if* $\Delta_A \neq 0$.

Proof. Note that since $D(A)$ is a tree graph, A has zero diagonal. Let $p(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x^{n-1} + c_n$ be the characteristic polynomial of A. The coefficient c_t of x^{n-t} equals $(-1)^t$ times the sum of the determinants of the principal submatrices of A of order t (see [5]). Thus, $c_t = 0$ if t is odd; for t even, c_t is equal to $(-1)^{t/2}$ times the sum of all matchings in $D(A)$ of size t. Let r be the term rank, and thus the rank, of A. The order of the largest nonsingular submatrix in A is then r, and there is no nonsingular submatrix of larger order. Assume that $\Delta_A \neq 0$. Then the coefficient $(-1)^r\Delta_A$ of x^{n-r} in $p(x)$ is nonzero, and all coefficients c_t of x^{n-t} for $t > r$ are zero. Thus, the algebraic multiplicity of the eigenvalue 0 is $n-r$, which equals $n-$rank A, the geometric multiplicity of 0. By the preceding discussion, rank $A =$ rank A^2 and hence $A^{\#}$ exists. Conversely, if $\Delta_A = 0$, then $p(x) = x^s q(x)$, where $s > n-r$ and $q(x)$ is a polynomial. This implies that the algebraic multiplicity of the eigenvalue 0 is strictly greater than its geometric multiplicity; thus rank $A \neq$ rank A^2 and $A^{\#}$ does not exist. □

2. Group Inverses of Matrices with Bipartite Digraphs. In the following theorem, A has a bipartite digraph, but it is not necessarily a tree graph. Our proof of the theorem uses the next result.

LEMMA 2.1. *Let* $B \in \mathbb{R}^{p\times(n-p)}, C \in \mathbb{R}^{(n-p)\times p}$. *If rank* $B =$ *rank* $C =$ *rank* $BC =$ *rank* CB, *then rank* $(BC)^2 =$ *rank* BC, *i.e.,* $(BC)^{\#}$ *exists. Furthermore,* $BC(BC)^{\#}B = B$ *and* $C(BC)^{\#}BC = C$.

Proof. Let rank $B =$ rank $C =$ rank $BC =$ rank $CB = m$. A rank inequality of Frobenius (see [8, page 13])

$$\operatorname{rank} BC + \operatorname{rank} CB \ \leq \ \operatorname{rank} C + \operatorname{rank} BCB$$

implies that $\operatorname{rank} BCB \geq m$. But clearly $\operatorname{rank} BCB \leq m$, hence equality holds. Similarly, $\operatorname{rank} CBC = m$. Now using the Frobenius inequality again gives

$$\operatorname{rank} BCB + \operatorname{rank} CBC \ \leq \ \operatorname{rank} CB + \operatorname{rank} BCBC.$$

By a similar argument as above, $\operatorname{rank}(BC)^2 = m$. Thus, $\operatorname{rank}(BC)^2 = \operatorname{rank} BC$, i.e., $(BC)^{\#}$ exists.

For the second part, the equality $BC(BC)^{\#}BC = BC$ implies that $BC(BC)^{\#}x = x$ for all vectors x in $\mathcal{R}(BC)$, the range of BC. Now, $\mathcal{R}(BC) \subseteq \mathcal{R}(B)$ so the assumption rank $BC =$ rank B implies that $\mathcal{R}(BC) = \mathcal{R}(B)$. Thus, $BC(BC)^{\#}x = x$ for all x in $\mathcal{R}(B)$ and therefore, $BC(BC)^{\#}B = B$. Similarly, $(BC)^T(BC)^{T\#}y = y$ for all y in $\mathcal{R}((BC)^T)$ and the rank assumptions imply that $\mathcal{R}((BC)^T) = \mathcal{R}(C^T)$. Thus, $y^T(BC)^{\#}(BC) = y^T$ for all y in $\mathcal{R}(C^T)$ and therefore, $C(BC)^{\#}(BC) = C$. □

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THEOREM 2.2. *Let* $A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, *where* $B \in \mathbb{R}^{p\times(n-p)}$, $C \in \mathbb{R}^{(n-p)\times p}$ *and* $p \le \frac{n}{2}$. *Then the group inverse* $A^\#$ *of* A *exists if and only if rank* $B =$ *rank* $C =$ *rank* $BC =$ *rank* CB. *If* $A^\#$ *exists, then*

$$A^\# = \begin{bmatrix} 0 & (BC)^\# B \\ C(BC)^\# & 0 \end{bmatrix}. \tag{2.1}$$

Proof. If rank $B =$ rank $C =$ rank $BC =$ rank CB, then rank $B +$ rank $C =$ rank $BC+$ rank CB, which implies that rank $A =$ rank A^2. Thus $A^\#$ exists. Conversely, if $A^\#$ exists and rank $B \neq$ rank C, then without loss of generality suppose that rank $B <$ rank C. Then rank $A^2 =$ rank $BC+$ rank $CB \le 2$ rank $B <$ rank $B+$ rank $C =$ rank A, which contradicts the existence of $A^\#$. Thus, rank $B =$ rank C, and by a similar argument, rank $BC =$ rank CB. Hence rank $A =$ rank A^2 implies that rank $B +$ rank $C =$ rank $BC+$ rank CB and therefore rank $B =$ rank $C =$ rank $BC =$ rank CB.

For the second part, $(BC)^\#$ exists by Lemma 2.1. Denoting the right hand side of (2.1) by G, we need only show that $AG = GA$, $AGA = A$ and $GAG = G$ to prove that $G = A^\#$. Since $BC(BC)^\# = (BC)^\# BC$, it follows that
$AG = \begin{bmatrix} BC(BC)^\# & 0 \\ 0 & C(BC)^\# B \end{bmatrix} = \begin{bmatrix} (BC)^\# BC & 0 \\ 0 & C(BC)^\# B \end{bmatrix} = GA$. Using the equalities established in Lemma 2.1,
$AGA = \begin{bmatrix} 0 & BC(BC)^\# B \\ C(BC)^\# BC & 0 \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = A$, and
$GAG = \begin{bmatrix} 0 & (BC)^\# BC(BC)^\# B \\ C(BC)^\# BC(BC)^\# & 0 \end{bmatrix} = \begin{bmatrix} 0 & (BC)^\# B \\ C(BC)^\# & 0 \end{bmatrix} = G$. ∎

If rank $BC =$ rank $CB =$ rank $B =$ rank $C = p$, then the $p \times p$ matrix BC is invertible and we obtain the following result.

COROLLARY 2.3. *Using the notation of Theorem 2.2, if rank* $BC =$ *rank* $CB =$ *rank* $B =$ *rank* $C = p$, *then the group inverse* $A^\#$ *exists and is given by*

$$A^\# = \begin{bmatrix} 0 & (BC)^{-1} B \\ C(BC)^{-1} & 0 \end{bmatrix}.$$

We note that in [10], formulas for the more general Drazin inverse of certain 2×2 block matrices are given. However, the conditions there are not in general satisfied by a matrix of form (1.1).

The following example has BC singular but satisfying the conditions of Theorem 2.2.

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EXAMPLE 2.4. If

$$A = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & a_{14} & a_{15} & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & 0 \\ 0 & 0 & 0 & 0 & a_{35} & 0 \\ \hline a_{41} & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & 0 & 0 \\ a_{61} & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix},$$

then

$$BC = \begin{bmatrix} a_{14}a_{41}+a_{15}a_{51}+a_{16}a_{61} & a_{15}a_{52} & a_{15}a_{53} \\ a_{25}a_{51} & a_{25}a_{52} & a_{25}a_{53} \\ a_{35}a_{51} & a_{35}a_{52} & a_{35}a_{53} \end{bmatrix}$$

and

$$CB = \begin{bmatrix} a_{41}a_{14} & a_{41}a_{15} & a_{41}a_{16} \\ a_{51}a_{14} & a_{51}a_{15}+a_{52}a_{25}+a_{53}a_{35} & a_{51}a_{16} \\ a_{61}a_{14} & a_{61}a_{15} & a_{61}a_{16} \end{bmatrix}.$$

Note that $D(A)$ is a tree graph.

Here, $\Delta_A = a_{14}a_{41}a_{25}a_{52} + a_{14}a_{41}a_{35}a_{53} + a_{16}a_{61}a_{25}a_{52} + a_{16}a_{61}a_{35}a_{53} = (a_{14}a_{41}+a_{16}a_{61})(a_{25}a_{52}+a_{35}a_{53})$, the sum of maximal matchings in $D(A)$. If $\Delta_A \neq 0$, then the matrices B,C,BC and CB all have rank 2 and by Theorem 2.2, $A^{\#}$ exists and is given by (2.1). Using Algorithm 7.2.1 in [7] and Maple,

$$(BC)^{\#} = \frac{1}{\Delta_A}\begin{bmatrix} a_{25}a_{52}+a_{35}a_{53} & -a_{15}a_{52} & -a_{15}a_{53} \\ -a_{25}a_{51} & \frac{a_{25}a_{52}(a_{14}a_{41}+a_{15}a_{51}+a_{16}a_{61})}{a_{25}a_{52}+a_{35}a_{53}} & \frac{a_{25}a_{53}(a_{14}a_{41}+a_{15}a_{51}+a_{16}a_{61})}{a_{25}a_{52}+a_{35}a_{53}} \\ -a_{35}a_{51} & \frac{a_{35}a_{52}(a_{14}a_{41}+a_{15}a_{51}+a_{16}a_{61})}{a_{25}a_{52}+a_{35}a_{53}} & \frac{a_{35}a_{53}(a_{14}a_{41}+a_{15}a_{51}+a_{16}a_{61})}{a_{25}a_{52}+a_{35}a_{53}} \end{bmatrix}.$$

It follows that if $\Delta_A \neq 0$, then from (2.1),

$$A^{\#} = \frac{1}{\Delta_A}\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix},$$

where

$$R = \begin{bmatrix} a_{14}(a_{25}a_{52}+a_{35}a_{53}) & 0 & a_{16}(a_{25}a_{52}+a_{35}a_{53}) \\ -a_{25}a_{51}a_{14} & a_{25}(a_{14}a_{41}+a_{16}a_{61}) & -a_{25}a_{51}a_{16} \\ -a_{35}a_{51}a_{14} & a_{35}(a_{14}a_{41}+a_{16}a_{61}) & -a_{35}a_{51}a_{16} \end{bmatrix}$$

and

$$S = \begin{bmatrix} a_{41}(a_{25}a_{52}+a_{35}a_{53}) & -a_{41}a_{15}a_{52} & -a_{41}a_{15}a_{53} \\ 0 & a_{52}(a_{14}a_{41}+a_{16}a_{61}) & a_{53}(a_{14}a_{41}+a_{16}a_{61}) \\ a_{61}(a_{25}a_{52}+a_{35}a_{53}) & -a_{61}a_{15}a_{52} & -a_{61}a_{15}a_{53} \end{bmatrix}.$$

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3. $A^{\#}$ for a Matrix with a Path Graph. Let $k \geq 1$. For the path graph $D(A)$ on $n = 2k$ vertices, A is nonsingular and $A^{\#} = A^{-1}$ (and a graph-theoretic description of the entries of A^{-1} is known; see Theorem 3.5 below). So we consider the path graph $D(A)$ with an odd number of vertices, for which A is singular. For $n = 2k+1$, if $D(A)$ is the bipartite path graph, then its associated matrix A is as in (1.1) with

$$B = \begin{bmatrix} a_{1,k+1} & a_{1,k+2} & 0 & 0 & \cdots & 0 \\ 0 & a_{2,k+2} & a_{2,k+3} & 0 & \cdots & 0 \\ 0 & 0 & a_{3,k+3} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_{k,2k} & a_{k,2k+1} \end{bmatrix} \in \mathbb{R}^{k\times(k+1)} \tag{3.1}$$

and

$$C = \begin{bmatrix} a_{k+1,1} & 0 & 0 & \cdots & 0 \\ a_{k+2,1} & a_{k+2,2} & 0 & \cdots & 0 \\ 0 & a_{k+3,2} & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_{2k,k-1} & a_{2k,k} \\ 0 & 0 & \cdots & 0 & a_{2k+1,k} \end{bmatrix} \in \mathbb{R}^{(k+1)\times k}, \tag{3.2}$$

where each specified entry a_{ij} is nonzero. Then rank $B =$ rank $C = k$, and the entries of the $k \times k$ tridiagonal matrix BC are as follows:

$$\begin{array}{llll} (BC)_{ii} & = & a_{i,k+i}a_{k+i,i} + a_{i,k+i+1}a_{k+i+1,i} & \text{if } 1 \leq i \leq k \\ (BC)_{i,i+1} & = & a_{i,k+i+1}a_{k+i+1,i+1} & \text{if } 1 \leq i \leq k-1 \\ (BC)_{i+1,i} & = & a_{i+1,k+i+1}a_{k+i+1,i} & \text{if } 1 \leq i \leq k-1 \\ (BC)_{ij} & = & 0 & \text{otherwise.} \end{array} \tag{3.3}$$

In Proposition 3.2 below, it is proved that the determinant of the matrix BC is equal to the sum of maximal matchings in $D(A)$. The following simple observations are used in the succeeding proofs.

LEMMA 3.1. *Let* $A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *with* B, C *as in* (3.1) *and* (3.2)*, respectively, i.e.,* $D(A)$ *is the bipartite path graph on* $2k+1$ *vertices. In* $D(A)$ *and for* $1 \leq j \leq k+1$*, the following relations hold.*

$$\gamma[k+j, k+j+1] = \gamma[k+j, j] + \gamma[j, k+j+1], \; j \neq k+1. \tag{3.4}$$

$$P_A[j \to j+1]P_A[j+1 \to j] = \gamma[j, k+j+1]\gamma[k+j+1, j+1], \; j \neq k+1. \tag{3.5}$$

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$$\gamma[k+1,k+j] \;=\; \gamma[j-1,k+j]\gamma[k+1,k+j-1] \;+\; \gamma[k+1,j-1],\; j\neq 1. \tag{3.6}$$

$$\gamma[k+1,j] \;=\; \gamma[j,k+j]\gamma[k+1,j-1],\; j\neq 1,k+1. \tag{3.7}$$

$$\gamma(i,j) \;=\; \gamma[k+1,k+i]\gamma[k+j+1,2k+1],\; 1\leq i<j\leq k. \tag{3.8}$$

In the following, $BC[j;\ell]$ denotes the principal submatrix of BC in rows and columns $j,\cdots,\ell$.

PROPOSITION 3.2. *For* $k\geq 1$, *let* $D(A)$ *be the bipartite path graph on* $2k+1$ *vertices, i.e.,* $A=\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *with* B,C *as in* (3.1) *and* (3.2), *respectively. Then for* $1\leq t\leq k$, $\det BC[1;t]=\gamma[k+1,k+t+1]$.

Proof. We use induction on t. First note, from (3.3), that the $k\times k$ matrix BC can be written as

$$\begin{bmatrix} \gamma[k+1,k+2] & P_A[1\to 2] & 0 & \cdots & 0 \\ P_A[2\to 1] & \gamma[k+2,k+3] & P_A[2\to 3] & \ddots & \vdots \\ 0 & P_A[3\to 2] & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \gamma[2k-1,2k] & P_A[k-1\to k] \\ 0 & \cdots & 0 & P_A[k\to k-1] & \gamma[2k,2k+1] \end{bmatrix}. \tag{3.9}$$

If $t=1$, then $\det BC[1;1]=\gamma[k+1,k+2]=\gamma[k+1,k+t+1]$ as desired.

Now suppose that for $2\leq g\leq k$ the result is true for all $t\leq g-1$; thus, for example,

$$\det BC[1;g-1] \;=\; \gamma[k+1,k+g] \tag{3.10}$$

and

$$\det BC[1;g-2] \;=\; \gamma[k+1,k+g-1]. \tag{3.11}$$

(Note that $BC[1;0]$ is vacuous and $\det BC[1;0]=1$.) Letting $t=g$ and expanding the deter-

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minant about the last row of $BC[1;g]$,

$$\begin{aligned}
\det BC[1;g] &= \gamma[k+g,k+g+1]\det BC[1;g-1] \\
&\quad - P_A[g-1\to g]P_A[g\to g-1]\det BC[1;g-2] \\
&= (\gamma[k+g,g]+\gamma[g,k+g+1])\gamma[k+1,k+g] \\
&\quad - \gamma[g-1,k+g]\gamma[k+g,g]\gamma[k+1,k+g-1] \text{ by } (3.4),\ (3.5),\ (3.10) \\
&\quad \text{and } (3.11) \\
&= \gamma[g,k+g+1]\gamma[k+1,k+g]\ +\ \gamma[g,k+g]\gamma[k+1,g-1] \text{ by } (3.6) \\
&= \gamma[g,k+g+1]\gamma[k+1,k+g]\ +\ \gamma[k+1,g] \text{ by } (3.7) \\
&= \gamma[k+1,k+g+1] \text{ by } (3.6). \qquad \square
\end{aligned}$$

COROLLARY 3.3. *For $k \geq 1$, let $D(A)$ be the bipartite path graph on $2k+1$ vertices, i.e., $A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ with B,C as in* (3.1) *and* (3.2), *respectively. Then* $\det BC = \gamma[k+1, 2k+1] = \Delta_A$.

In the following, $W(i)$ (respectively $W(i;), W(;j)$) denotes the submatrix obtained from a matrix W by deleting both row and column i (respectively row i, column j).

COROLLARY 3.4. *For $k \geq 1$, let $D(A)$ be the bipartite path graph on $2k+1$ vertices, i.e., $A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ with B,C as in* (3.1) *and* (3.2), *respectively. For $1 \leq i \leq k$, let $D(A(i))$ be the associated digraph obtained by deleting vertex i from $D(A)$. Then $B(i;)C(;i) = BC(i)$,*

$$\begin{aligned}
\det BC(1) &= \gamma[k+2,2k+1], \\
\det BC(k) &= \gamma[k+1,2k]
\end{aligned}$$

and

$$\det BC(i) \ =\ \gamma[k+1,k+i]\gamma[k+i+1,2k+1],\ i \neq 1,k.$$

Proof. These results follow from the structure of B and C, and the fact that $D(A(1))$, $D(A(k))$ can be re-labeled to be bipartite path graphs on $2k-1$ vertices (along with one isolated vertex), while $D(A(i))$ for $i \neq 1,k$ consists of two disjoint path graphs that can be re-labeled to be bipartite path graphs on $2i-1$ and $2(k-i)+1$ vertices. $\square$

For $\Delta_A \neq 0$, Proposition 3.6 below gives the entries of $(BC)^{-1}$ in terms of path products and matchings in $D(A)$. The proof uses the following theorem, stated for tree graphs in [9] and for general digraphs in [11], which we restate here for digraphs $D(W)$ with tridiagonal W.

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THEOREM 3.5. [9, 11] *Let W be an $n \times n$ nonsingular tridiagonal matrix with digraph $D(W)$, and let $W^{-1} = (\omega_{ij})$. Then*

$$\omega_{ii} = \frac{\det W(i)}{\det W}, \tag{3.12}$$

and

$$\omega_{ij} = \frac{1}{\det W}(-1)^{\ell} P_W[i \to j] \det W(i, \cdots, j), \tag{3.13}$$

where ℓ is the length of the path from i to j, $W(i)$ is the matrix obtained from W by deleting row and column i, and $W(i, \cdots, j)$ is the matrix obtained from W by deleting rows and columns corresponding to the vertices on the path from i to j.

In the next two results, we set $P_A[i \to i] = 1$ and $\gamma(i,i) = \gamma[k+1, k+i]\gamma[k+i+1, 2k+1]$.

PROPOSITION 3.6. *Let $A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ with B, C as in* (3.1) *and* (3.2)*, respectively, and assume that $\Delta_A \neq 0$. Then $(BC)^{-1} = (\beta_{ij})$ exists and is given by*

$$\beta_{ij} = \frac{1}{\Delta_A}(-1)^{i+j} P_A[i \to j]\gamma(i, j). \tag{3.14}$$

Proof. From Corollary 3.3, $\det BC = \Delta_A$ and the assumption $\Delta_A \neq 0$ implies that $(BC)^{-1}$ exists. We apply Theorem 3.5 to the tridiagonal matrix BC as in (3.9). Let $1 \leq i, j \leq k$.

If $i = j$, then by Corollary 3.4,

$$\beta_{11} = \frac{\gamma[k+2, 2k+1]}{\Delta_A}, \quad \beta_{kk} = \frac{\gamma[k+1, 2k]}{\Delta_A},$$

and

$$\beta_{ii} = \frac{\gamma[k+1, k+i]\gamma[k+i+1, 2k+1]}{\Delta_A}, \quad \text{for } i \neq 1 \text{ or } k,$$

which agree with (3.14).

If $i < j$, with $i \neq 1$ and $j \neq k$, then removing the vertices on the path $(i, \cdots, j)$ in $D(A)$ results in two disjoint path graphs on vertices $k+1, \cdots, k+i$ and $k+j+1, \cdots, 2k+1$, respectively. As these can be re-labeled to be bipartite path graphs, Proposition 3.2 gives

$$\begin{aligned} \det BC(i, \cdots, j) &= \det BC[1; i-1] \det BC[j+1; k] \\ &= \gamma[k+1, k+i]\gamma[k+j+1, 2k+1]. \end{aligned}$$

If $i = 1$, then $\det BC(i, \cdots, j) = \det BC[j+1; k] = \gamma[k+j+1, 2k+1]$; if $j = k$, then $\det BC(i, \cdots, j) = \det BC[1; i-1] = \gamma[k+1, k+i]$. For all $i < j$, the (i, j) entry β_{ij} of $(BC)^{-1}$ is

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computed, using Theorem 3.5, with the path product in (3.13) taken from the digraph $D(BC)$. From (3.9), the path product $P_{BC}[i \to j]$ is given by the product $P_A[i \to i+1]P_A[i+1 \to i+2]\cdots P_A[j-1 \to j]$ of $j-i$ path products in the path graph $D(A)$. This path product is equal to $P_A[i \to j]$. It follows from (3.13) and the above that

$$\begin{aligned}\beta_{ij} &= \frac{1}{\Delta_A}(-1)^{j-i}P_A[i \to j]\gamma[k+1,k+i]\gamma[k+j+1,2k+1] \\ &= \frac{1}{\Delta_A}(-1)^{i+j}P_A[i \to j]\gamma(i,j) \text{ by } (3.8).\end{aligned}$$

The proof for the case $i > j$ can be obtained by switching the roles of i and j in the above argument, completing the proof for $i \neq j$. $\square$

The next theorem is the main result of this section.

THEOREM 3.7. *Let* $A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *be a matrix of order* $2k+1$ *with* B,C *as in* (3.1) *and* (3.2), *respectively. Assume that* $\Delta_A \neq 0$. *Then the group inverse* $A^\# = (\alpha_{ij})$ *exists and*

$$\alpha_{ij} = \begin{cases} \frac{1}{\Delta_A}(-1)^s P_A[i \to j]\gamma(i,j) & \text{if the path in } D(A) \text{ from } i \text{ to } j \text{ is of length} \\ & 2s+1 \text{ with } s \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Proof. The assumption $\Delta_A \neq 0$ together with Corollary 3.3 imply that rank $BC = k$. In addition, CB is a tridiagonal matrix of order $k+1$ with a nonzero superdiagonal. Thus, rank $CB \geq k$ and since rank $CB \leq$ rank $B = k$, it follows that rank $CB = k$. Hence, rank$B =$ rank$C =$ rank$BC =$ rank$CB = k$, and by Corollary 2.3, the group inverse $A^\#$ exists with entries α_{ij} given by

$$\alpha_{ij} = \begin{cases} ((BC)^{-1}B)_{i,j-k} & \text{if } (i,j) \in \{1,\cdots,k\} \times \{k+1,\cdots,2k+1\}, \\ (C(BC)^{-1})_{i-k,j} & \text{if } (i,j) \in \{k+1,\cdots,2k+1\} \times \{1,\cdots,k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

Let $(i,j) \in \{1,\cdots,2k+1\} \times \{1,\cdots,2k+1\}$. Note that $D(A)$ is the bipartite path graph on $2k+1$ vertices. The path from i to j is of even length if and only if (i,j) is in $\{1,\cdots,k\} \times \{1,\cdots,k\}$ or $\{k+1,\cdots,2k+1\} \times \{k+1,\cdots,2k+1\}$. It follows from (3.16) that $\alpha_{ij} = 0$ if the path from i to j is of even length or if $i = j$. Now assume that the path from i to j is of odd length. Then either $(i,j) \in \{1,\cdots,k\} \times \{k+1,\cdots,2k+1\}$ or $(i,j) \in \{k+1,\cdots,2k+1\} \times \{1,\cdots,k\}$.

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Suppose that $(i,j) \in \{1,\cdots,k\} \times \{k+1,\cdots,2k+1\}$, and set $j' = j-k$. Then from (3.16) and (3.14),

$$\alpha_{ij} = ((BC)^{-1}B)_{ij'} = \frac{1}{\Delta_A}\sum_{m=1}^{k}(-1)^{i+m}P_A[i \to m]\gamma(i,m)a_{mj}.$$

Hence for $j = k+1$,

$$\begin{aligned}\alpha_{i,k+1} &= \frac{1}{\Delta_A}(-1)^{i+1}P_A[i \to 1]\gamma(i,1)a_{1,k+1}\\ &= \frac{1}{\Delta_A}(-1)^{i+1}P_A[i \to k+1]\gamma(i,k+1).\end{aligned}$$

Since $(-1)^{i+1} = (-1)^{i-1}$ and the path in $D(A)$ from i to $k+1$ has length $2(i-1)+1$, the theorem is true for $j = k+1$. Similarly, the theorem is true for $j = 2k+1$, so suppose that $j \neq k+1, 2k+1$. Then

$$\alpha_{ij} = \frac{1}{\Delta_A}(-1)^{i+j'}(P_A[i \to j']\gamma(i,j')a_{j'j} - P_A[i \to j'-1]\gamma(i,j'-1)a_{j'-1,j}).$$

Suppose that $1 \leq i < j' = j-k \leq k$. Then

$$\begin{aligned}\alpha_{ij} &= \frac{1}{\Delta_A}(-1)^{i+j'}P_A[i \to j](\gamma(i,j')\gamma[j',j] - \gamma(i,j'-1))\\ &= \frac{1}{\Delta_A}(-1)^{j'-i-1}P_A[i \to j]\gamma(i,j).\end{aligned}$$

Since the path in $D(A)$ from i to j has length $2(j'-i-1)+1$, the theorem is true for all such (i,j). Now suppose that $2 \leq i, j' \leq k$ and $i \geq j' = j-k$. Then

$$\begin{aligned}\alpha_{ij} &= \frac{1}{\Delta_A}(-1)^{i+j'}P_A[i \to j](\gamma(i,j') - \gamma(i,j'-1)\gamma[j'-1,j])\\ &= \frac{1}{\Delta_A}(-1)^{i-j'}P_A[i \to j]\gamma(i,j).\end{aligned}$$

Since the path in $D(A)$ from i to j has length $2(i-j')+1$, the theorem is true for all such (i,j), and thus for all $(i,j) \in \{1,\cdots,k\} \times \{k+1,\cdots,2k+1\}$.

The proof for $(i,j) \in \{k+1,\cdots,2k+1\} \times \{1,\cdots,k\}$ is similar. $\square$

The next two results follow since an irreducible tridiagonal matrix with zero diagonal is permutationally similar to the matrix in Theorem 3.7.

COROLLARY 3.8. *Let A be an irreducible tridiagonal matrix of order $2k+1$ with zero diagonal and a path graph $D(A)$ on vertices $1,\cdots,2k+1$. Assume that $\Delta_A \neq 0$. Then the group inverse $A^{\#}$ exists and its entries are given by* (3.15).

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COROLLARY 3.9. *If in addition to the assumptions of Corollary 3.8, A is nonnegative, then $A^\#$ is sign determined. Specifically, $A^\# = (\alpha_{ij})$ has a diagonally-striped sign pattern with*

$$\begin{aligned} \alpha_{ij} &= 0 && \text{if } i+j \text{ is even} \\ \alpha_{i,i\pm t} &> 0 && \text{for } t = 1,5,9,\cdots \\ \alpha_{i,i\pm t} &< 0 && \text{for } t = 3,7,11,\cdots, \end{aligned}$$

where $1 \le i \le n$ and $1 \le i \pm t \le n$.

4. Relation of $A^\#$ with $A^\dagger$ for Tridiagonal Matrices. It is well-known (see e.g. [3], [7]) that if A is symmetric and $A^\#$ exists, then $A^\# = A^\dagger$, the Moore-Penrose inverse of A. To explore the relation between these two inverses for irreducible tridiagonal matrices with zero diagonal (which are combinatorially symmetric), we use the following notation from [4]. Let $U = \{u_1,\cdots,u_n\}$ and $V = \{v_1,\cdots,v_n\}$ be disjoint sets. For an $n\times n$ matrix $A = (a_{ij})$, $B(A)$ is the bipartite graph with vertices $U \cup V$ and edges $\{(u_i,v_j):\ u_i \in U, v_j \in V, a_{ij} \neq 0\}$. For any $h \ge 1$ and any bipartite graph B, $M_h(B)$ denotes the family of subsets of h distinct edges of B, no two of which are adjacent.

THEOREM 4.1. *Let $k \ge 1$ and $A = (a_{ij}) \in \mathbb{R}^{2k+1\times 2k+1}$ be an irreducible tridiagonal matrix with zero diagonal and assume that $\Delta_A \neq 0$. Let $A^\# = (\alpha_{ij})$, $A^\dagger = (\mu_{ij})$ and $1 \le i,j \le 2k+1$.*
(i) *If the path from i to j in $D(A)$ is of even length or if $i = j$, then $\alpha_{ij} = \mu_{ij} = 0$.*
(ii) *If $\alpha_{ij} \neq 0$, then $\mu_{ij} \neq 0$.*
(iii) *If $\gamma(i,j) \neq 0$, then $\alpha_{ij} \neq 0$ if and only if $\mu_{ij} \neq 0$.*

Proof. By Corollary 3.8 and [4, Corollary 2.7], $\alpha_{ii} = \mu_{ii} = 0$ for all i. Let $1 \le i < j \le 2k+1$. By Corollary 3.8,

$$\alpha_{ij} = \frac{1}{\Delta_A}(-1)^s a_{i,i+1}a_{i+1,i+2}a_{i+2,i+3}\cdots a_{j-2,j-1}a_{j-1,j}\gamma(i,j) \tag{4.1}$$

if the path from i to j in $D(A)$ is of length $2s+1$ with $s \ge 0$, and $\alpha_{ij} = 0$ otherwise. According to [4, Corollary 2.7], $\mu_{ji} \neq 0$ if and only if $B(A)$ contains a path p from u_i to v_j

$$p:\quad u_i \to v_{i+1} \to u_{i+2} \to v_{i+3} \to \cdots \to v_{j-2} \to u_{j-1} \to v_j$$

of length $2s+1$ with $s \ge 0$, and $M_{r-s-1}(B(A))$ has at least one element with $r-s-1$ edges none of which are adjacent to p, where $r = 2k$ is the rank of A. Note that by the theorem assumptions on A, if a path p from u_i to v_j in $B(A)$ of length $2s+1$, with $s \ge 0$, exists, then the latter condition on $M_{r-s-1}(B(A))$ and the path p always holds. Furthermore, by [4, Corollary 2.7], if such a path exists, then μ_{ji} has the same sign as

$$(-1)^s a_{i,i+1}a_{i+2,i+1}a_{i+2,i+3}\cdots a_{j-1,j-2}a_{j-1,j}. \tag{4.2}$$

Since A is an irreducible tridiagonal matrix with zero diagonal, it is combinatorially symmetric (i.e., $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$). Thus, there is a path of length $2s+1$ from i to

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j in $D(A)$ if and only if there is a path of length $2s+1$ from u_j to v_i in $B(A)$. If no such path of odd length exists, then $\alpha_{ij} = \mu_{ij} = 0$, completing the proof of (i). If $\alpha_{ij} \neq 0$, then by (4.1), the path from i to j in $D(A)$ is of length $2s+1$ with $s \geq 0$. Thus, using (4.1), (4.2) and by combinatorial symmetry, $\mu_{ij} \neq 0$, proving (ii) and one direction of (iii). Lastly, if $\gamma(i, j) \neq 0$ and $\mu_{ij} \neq 0$, then $\alpha_{ij} \neq 0$ by a similar argument. This completes the proof of (iii) and hence the theorem for $i \leq j$. The proof for $i > j$ is similar. □

The following example illustrates that the condition $\gamma(i, j) \neq 0$ in (iii) above is necessary.

EXAMPLE 4.2. Consider the 5×5 tridiagonal matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

having

$$A^{\dagger} = \frac{1}{3}\begin{bmatrix} 0 & 2 & 0 & -1 & 0 \\ 2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & 2 & 0 \end{bmatrix}$$

and

$$A^{\#} = \begin{bmatrix} 0 & 2 & 0 & -1 & 0 \\ 2 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Here the $(4,5)$ and $(5,4)$ entries of $A^{\#}$ are zero since $\gamma(4,5) = 0$, whereas the corresponding entries of $A^{\dagger}$ are nonzero.

Theorem 4.1 shows that for an irreducible tridiagonal matrix A, the nonzero entries of $A^{\#}$ are a subset of the nonzero entries of $A^{\dagger}$. However, this is not in general true for a matrix A with $D(A)$ bipartite, as is shown in the following example.

EXAMPLE 4.3. Consider the following 5×5 matrix A which has $D(A)$ bipartite, but not

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a tree graph:

$$A = \left[\begin{array}{cc|ccc} 0 & 0 & a_{13} & 0 & a_{15} \\ 0 & 0 & 0 & a_{24} & 0 \\ \hline 0 & a_{32} & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 & 0 \\ a_{51} & 0 & 0 & 0 & 0 \end{array}\right].$$

By Corollary 2.3, the $(2,4)$ entry of $A^{\#}$ is $-a_{15}a_{51}/a_{13}a_{32}a_{41}$, whereas by [4, Theorem 2.6], the $(2,4)$ entry of $A^{\dagger}$ is zero since there is no path in $B(A)$ from u_4 to v_2.

5. Conjecture. We conclude with a conjecture and some related remarks. Recall that if $D(A)$ is a tree graph, then all diagonal entries of A are zero.

CONJECTURE 5.1. *Let A be a singular matrix with a tree graph $D(A)$, term rank r and $\Delta_A \neq 0$. Suppose that there exists a path subgraph $p(i,j)$ on vertices $i, i_2, \cdots, i_{2s+1}, j$, where $s \geq 0$. Define*

$$\delta(i,j) = \begin{cases} \gamma(i,j) & \text{if the matrix associated with } D(A) \setminus p(i,j) \\ & \text{has term rank } r-2(s+1), \\ 0 & \text{otherwise.} \end{cases}$$

Then $A^{\#} = (\alpha_{ij})$ exists and its entries are given by

(5.1)

$$\alpha_{ij} = \begin{cases} \dfrac{1}{\Delta_A}(-1)^s P_A[i \to j]\delta(i,j) & \text{if the path in } D(A) \text{ from } i \text{ to } j \text{ is of length } 2s+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $D(A)$ in Example 2.4 has a path of length 1 from vertex 1 to vertex 5. However, the matrix associated with $D(A) \setminus p(1,5)$ has term rank 0, whereas $r-2(s+1) = 4-2 = 2$. Thus, the $(1,5)$ entry of $A^{\#}$ is zero.

EXAMPLE 5.2. For $n \geq 3$, consider an $n \times n$ matrix with a star graph centered at 1, i.e., $A = (a_{ij})$ has $a_{1j}, a_{j1} \neq 0$, for $j = 2, \cdots, n$, and $a_{ij} = 0$ otherwise. Then from (1.1), $BC = \Delta_A$ is a scalar. Assuming that $\Delta_A \neq 0$, Corollary 2.3 gives $A^{\#} = \frac{1}{\Delta_A}A$. Note that for $j \neq 1$, the path from 1 to j is of length $2s+1 = 1$, where $s = 0$; thus $r-2(s+1) = 0$, which is the term rank of the matrix associated with $D(A) \setminus p(1,j)$. Hence $\delta(1,j) = \gamma(1,j) = 1$. This shows that (5.1) holds, and the conjecture is true for matrices having a star graph. Note also that for a matrix A with $D(A)$ a star graph, the above formula for $A^{\#}$ and [4, Corollary 2.7] give that the sign patterns $\mathrm{sgn}(\Delta_A A^{\#})$ and $\mathrm{sgn}((A^{\dagger})^T)$ are identical. If, in addition, A is nonnegative, then $\Delta_A > 0$ and $\mathrm{sgn}(A^{\#}) = \mathrm{sgn}(A) = \mathrm{sgn}((A^{\dagger})^T)$, which is a special case of [1, Theorem 4].

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The existence of $A^{\#}$ in Conjecture 5.1 follows from Proposition 1.1. In addition to matrices A that have a path or a star graph, we have verified with Maple that (5.1) of Conjecture 5.1 holds for all singular matrices with tree graphs of order 7 or less.

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