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SIGN PATTERNS THAT REQUIRE A POSITIVE OR NONNEGATIVE LEFT INVERSE*

IN-JAE KIM† AND BRYAN L. SHADER‡

Abstract. An m by n sign pattern $\mathcal{A}$ is an m by n matrix with entries in $\{+,-,0\}$. The sign pattern $\mathcal{A}$ requires a positive (resp. nonnegative) left inverse provided each real matrix with sign pattern $\mathcal{A}$ has a left inverse with all entries positive (resp. nonnegative). In this paper, necessary and sufficient conditions are given for a sign pattern to require a positive or nonnegative left inverse. It is also shown that for $n \geq 2$, there are no square sign patterns of order n that require a positive (left) inverse, and that an n by n sign pattern requiring a nonnegative (left) inverse is permutationally equivalent to an upper triangular sign pattern with positive main diagonal entries and nonpositive off-diagonal entries.

Key words. Nonnegative left inverse, Positive left inverse, Sign-consistent constrained system, Sign pattern.

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1. Introduction and Preliminaries. An m by n *sign pattern* $\mathcal{A} = [\alpha_{ij}]$ is an m by n matrix with entries in $\{+,-,0\}$. If the sign pattern $\mathcal{A}$ has all entries in $\{+,0\}$ (resp. $\{+\}$), then $\mathcal{A}$ is a *nonnegative* (resp. *positive*) sign pattern. *Nonpositive* and *negative* sign patterns are analogously defined. A *superpattern* of $\mathcal{A}$ is an m by n sign pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = \alpha_{ij}$ whenever $\alpha_{ij} \neq 0$. The *sign pattern class* $Q(\mathcal{A})$ of the m by n sign pattern $\mathcal{A}$ is the set of all m by n real matrices with the sign pattern $\mathcal{A}$, i.e.,

$$Q(\mathcal{A}) = \{A = [a_{ij}] \in \mathbb{R}^{m\times n} \mid \operatorname{sgn}(a_{ij}) = \alpha_{ij} \text{ for all } i, j\}.$$

If $A \in Q(\mathcal{A})$, then A is a *realization* of $\mathcal{A}$.

Let $A = [a_{ij}]$ be an m by n matrix. If A is a realization of a positive (resp. nonnegative) sign pattern, then A is *positive* (resp. *nonnegative*), written $A > 0$ (resp. $A \geq 0$). A *left inverse* of the m by n matrix A is an n by m matrix B such

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†Department of Mathematics and Statistics, Minnesota State University, Mankato, MN 56001, USA (in-jae.kim@mnsu.edu). The research of this author was partially supported by a Faculty Reassigned Time for Research and a Faculty Research Grant from the Minnesota State University, Mankato.

‡Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA (bshader@uwyo.edu).

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that $BA = I_n$, where I_n is the n by n identity matrix. In addition, if $B > 0$, then B is a *positive* left inverse (abbreviated as *PLI*) of A. If $B \geq 0$, then B is a *nonnegative* left inverse (abbreviated as *NLI*) of A. Note that A has a left inverse if and only if $\operatorname{rank} A = n$. This implies that if A has a left inverse, then $m \geq n$. For $m > n$, if A has a left inverse, then there are infinitely many left inverses of A. An m by n sign pattern $\mathcal{A}$ *allows a PLI* (resp. *an NLI*) provided there exists $A \in Q(\mathcal{A})$ with a PLI (resp. an NLI). An m by n sign pattern $\mathcal{A}$ *requires a PLI* (resp. *an NLI*) provided each $A \in Q(\mathcal{A})$ has a PLI (resp. an NLI). It is clear that if $\mathcal{A}$ requires a PLI (NLI), then $\mathcal{A}$ allows a PLI (NLI). Furthermore, if $\mathcal{A}$ is a square sign pattern that requires a PLI (NLI), then each realization of $\mathcal{A}$ is nonsingular, i.e., $\mathcal{A}$ is an SNS-matrix (see [2, page 7]). If P_1 and P_2 are permutation matrices of orders m and n, respectively, then the m by n sign pattern $\mathcal{A}$ allows (resp. requires) a PLI (NLI) if and only if the *permutationally equivalent* sign pattern $P_1\mathcal{A}P_2$ allows (resp. requires) a PLI (NLI). The following observation also shows that the left-inverse nonnegativity and positivity of a sign pattern are invariant under the multiplication by positive diagonal matrices.

OBSERVATION 1.1. *Let A be an m by n matrix, and let D_1 and D_2 be positive diagonal matrices of orders m and n, respectively. Then A has a positive (resp. nonnegative) left inverse if and only if D_1AD_2 has a positive (resp. nonnegative) left inverse.*

Let $\mathcal{A}$ be an n by n sign pattern with a realization of rank n. Then $\mathcal{A}$ is permutationally equivalent to

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1k} \\ O & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2k} \\ \vdots & & \ddots & \vdots \\ O & \cdots & O & \mathcal{A}_{kk} \end{bmatrix}, \tag{1.1}$$

where $k \geq 1$, and $\mathcal{A}_{ii}$ is a fully indecomposable square sign pattern for each $i \in \{1, \ldots, k\}$ (see [1, Theorem 4.2.6]). If $\mathcal{A}$ is fully indecomposable, then $k = 1$. By induction, the following observation can be shown.

OBSERVATION 1.2. *Let $\mathcal{A}$ be an n by n sign pattern of the form* (1.1)*. If $\mathcal{A}$ requires a PLI (resp. NLI), then each $\mathcal{A}_{ii}$ $(i = 1, \ldots, k)$ also requires a PLI (resp. NLI).*

In [4], a characterization of all m by n sign patterns allowing a PLI is given, which generalizes the known result for the square case (see, for example, [2, Chapter 9]). In [4], there are also necessary or sufficient conditions for m by n sign patterns to allow an NLI; however, a complete characterization of such sign patterns remains open. In this paper, we study sign patterns that require a PLI or an NLI. First, the following result shows that m by 1 sign patterns $\mathcal{A}$ requiring a PLI or an NLI share a necessary

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and sufficient condition, the existence of a + entry in $\mathcal{A}$, with m by 1 sign patterns allowing a PLI or an NLI (see [4, Proposition 2.1]).

PROPOSITION 1.3. *For $m \geq 1$, let $\mathcal{A} = [\alpha_1, \ldots, \alpha_m]^T$ be an m by 1 sign pattern. Then the following are equivalent:*

(a) *$\mathcal{A}$ has a + entry.*
(b) *$\mathcal{A}$ requires a PLI.*
(c) *$\mathcal{A}$ requires an NLI.*
(d) *$\mathcal{A}$ allows an NLI.*
(e) *$\mathcal{A}$ allows a PLI.*

Proof. By [4, Proposition 2.1], (a), (d) and (e) are equivalent. Since (b) implies (c), and (c) implies (d), it suffices to show that (a) implies (b).

Suppose that $\alpha_i = +$ and $A = [a_1, \ldots, a_m]^T \in Q(\mathcal{A})$ with $a_i > 0$. Let $B = [b_1, \ldots, b_m]$ such that $b_j = 1$ for each $j \in \{1, \ldots, m\} \setminus \{i\}$ and $b_i = \frac{1}{a_i}(1 + \sum_{j \neq i} |a_j|) > 0$. If $c = 1 + \sum_{j \neq i}(|a_j| + a_j) > 0$, then $\frac{1}{c}B$ is a PLI (and hence NLI) of A. □

For square sign patterns of order $n \geq 2$ requiring a PLI or an NLI, a complete characterization is given in Section 2. It is shown in Section 2 that for $n \geq 2$, no square sign patterns of order n require a PLI (which is in contrast with [2, Theorem 9.2.1]), and that a sign pattern of order n requiring an NLI is, up to permutational equivalence, an upper triangular sign pattern with positive main diagonal entries and nonpositive off-diagonal entries.

For $m \geq n$, let $\mathcal{A}$ be an m by n sign pattern and $\widehat{b}$ an n by 1 sign pattern. The system $x^T\mathcal{A} = \widehat{b}^T$ with $x \geq 0$ and $x \neq 0$ (resp. $x > 0$) is a *constrained* system, and the constrained system is *sign-consistent* provided, for each matrix $A \in Q(\mathcal{A})$ and each vector $b \in Q(\widehat{b})$, there exists an m by 1 nonzero and nonnegative (resp. positive) vector x such that $x^T A = b^T$; see [5]. Note that, for an n by m matrix X with rows x_i^T and an m by n matrix A, $XA = I_n$ implies that $x_i^T A = e_i^T$ for each $i \in \{1, \ldots, n\}$, where e_i is the column matrix with 1 in row i and 0's elsewhere. By this fact, it can be shown that an m by n sign pattern $\mathcal{A}$ requiring an NLI (resp. a PLI) is closely related to a sign-consistent, constrained system $x^T\mathcal{A} = \widehat{b}^T$ with $x \geq 0$ and $x \neq 0$ (resp. $x > 0$). We denote by $\widehat{e_i}$ the column sign pattern with a + entry in row i and 0's elsewhere.

OBSERVATION 1.4. *Let $\mathcal{A}$ be an m by n sign pattern. Then $\mathcal{A}$ requires an NLI (resp. a PLI) if and only if, for each $i = 1, \ldots, n$, the constrained system $x^T\mathcal{A} = \widehat{e_i}^T$ with $x \geq 0$ and $x \neq 0$ (resp. $x > 0$) is sign-consistent.*

Using this observation and results on sign-consistent, constrained systems in [5], in Section 3, we give necessary and sufficient conditions for an m by n sign pattern to

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require an NLI or a PLI.

2. Square Case. Let $\mathcal{I}_n$ denote the n by n sign pattern with +'s on the main diagonal and 0's elsewhere. The following result gives a complete characterization on square sign patterns of order $n \geq 2$ that require a PLI or an NLI.

THEOREM 2.1. *For $n \geq 2$, let $\mathcal{A}$ be an n by n sign pattern. Then the following hold.*

(a) *$\mathcal{A}$ does not require a positive (left) inverse.*
(b) *If $\mathcal{A}$ is fully indecomposable, then $\mathcal{A}$ does not require a nonnegative (left) inverse.*
(c) *Assume that $\mathcal{A}$ is partly decomposable. Then $\mathcal{A}$ requires a nonnegative (left) inverse if and only if $\mathcal{A}$ is permutationally equivalent to $\mathcal{I}_n - \mathcal{N}$ where $\mathcal{N}$ is an n by n strictly upper triangular, nonnegative sign pattern.*

Proof. (a) Suppose that $\mathcal{A}$ requires a positive inverse. By [4, Lemma 2.2], each column of $\mathcal{A}$ has a + and a − entry. Hence, there exists $A \in Q(\mathcal{A})$ each of whose column sums is 0, i.e., $[1, \ldots, 1]A = 0$. This implies that there is a singular matrix in $Q(\mathcal{A})$, which is a contradiction. Thus, the result follows.

(b) Let $\mathcal{A}$ be fully indecomposable. Suppose that $\mathcal{A}$ requires a nonnegative inverse. If $\mathcal{A}$ has a nonnegative column, then, by [4, Proposition 3.3], the nonnegative column has at most one + entry. This contradicts that $\mathcal{A}$ is fully indecomposable. Thus, each column of $\mathcal{A}$ has a + and a − entry. By a similar argument as in the proof of (a), the result follows.

(c) Let $\mathcal{A}$ be partly decomposable. Without loss of generality, we may assume that $\mathcal{A}$ is of the form (1.1) with $k \geq 2$.

First, suppose that $\mathcal{A}$ requires a nonnegative inverse. Then, by Observation 1.2, each $\mathcal{A}_{ii}$ for $i = 1, \ldots, k$ is a fully indecomposable sign pattern that requires a nonnegative inverse. By (b) and Proposition 1.3, for each $i = 1, \ldots, k$, $\mathcal{A}_{ii}$ is a 1 by 1 sign pattern $[+]$. Thus, $\mathcal{A}$ has the form $\mathcal{I}_n - \mathcal{N}$ where $\mathcal{N}$ is strictly upper triangular. Let $A = I_n - N \in Q(\mathcal{A})$ for some $N \in Q(\mathcal{N})$. Then, the inverse of A is

$$(I_n - N)^{-1} = I_n + N + N^2 + N^3 + \cdots + N^{n-1}. \tag{2.1}$$

Assume that, for some $s < t$, $\mathcal{N}$ has (s,t)-entry negative. By emphasizing the (s,t)-entry, (2.1) and the fact that $\mathcal{N}$ is strictly upper triangular imply that there exists a realization A in $Q(\mathcal{A})$ whose inverse has negative (s,t)-entry, which is a contradiction. Hence, the result follows.

Next, assume that $\mathcal{A} = \mathcal{I}_n - \mathcal{N}$ where $\mathcal{N}$ is an n by n strictly upper triangular, nonnegative sign pattern. Let $A = E - N \in Q(\mathcal{A})$ with $E \in Q(\mathcal{I}_n)$ and $N \in Q(\mathcal{N})$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 17, pp. 209-218, April 2008

Then there exists a positive diagonal matrix D such that $DE = I_n$. Note that $DN \in Q(\mathcal{N})$, and hence $DA = I_n - DN \in Q(\mathcal{A})$. By (2.1), DA has a nonnegative inverse. Thus, the result follows by Observation 1.1. ☐

3. General Case. In this section, by using results on sign-consistent, constrained systems in [5], we give necessary and sufficient conditions for a sign pattern to require an NLI or a PLI.

A *signing* is a nonzero, diagonal sign pattern. For an m by n sign pattern $\mathcal{A}$, let S be the set of all signings $\mathcal{D}$ of order n such that each nonzero row of $\mathcal{AD}$ has a $+$ entry. The following result gives a necessary and sufficient condition for a sign pattern to require an NLI in terms of signings with a particular sign.

THEOREM 3.1. *For $m \geq n$, let $\mathcal{A}$ be an m by n sign pattern. Then $\mathcal{A}$ requires an NLI if and only if, for each signing $\mathcal{D}$ with at least one $-$ diagonal entry, $\mathcal{AD}$ has a nonzero and nonpositive row.*

Proof. By [5, Lemma 3.1], it follows that the constrained system $x^T\mathcal{A} = \widehat{e_i}^T$ with $x \geq 0$ and $x \neq 0$ is sign-consistent for each $i \in \{1, \ldots, n\}$ if and only if every signing $\mathcal{D}$ in S is nonnegative. Hence, Observation 1.4 implies that $\mathcal{A}$ requires an NLI if and only if each signing $\mathcal{D} \in S$ is nonnegative. Note that the definition of the set S implies that each signing $\mathcal{D} \in S$ is nonnegative if and only if, for each signing $\mathcal{D}$ with at least one diagonal entry which is $-$, $\mathcal{AD}$ has a nonzero and nonpositive row. Thus, the result follows. ☐

The next result gives a condition on rows of a sign pattern that requires an NLI (and hence a condition on rows of a sign pattern that requires a PLI).

COROLLARY 3.2. *For $m \geq n$, let $\mathcal{A}$ be an m by n sign pattern. If $\mathcal{A}$ requires an NLI (or a PLI), then $\mathcal{A}$ has a nonzero and nonnegative row.*

Proof. The result follows from Theorem 3.1 by taking $\mathcal{D}$ to be the signing with all diagonal entries negative. ☐

Corollary 3.2 does not necessarily hold for a sign pattern that allows an NLI or a PLI (see the sign pattern $\mathcal{B}$ in Remark 3.11). The following corollaries describe sign patterns that have a common necessary and sufficient condition to allow an NLI (see [4, Theorems 3.4 and 3.8]) and to require an NLI.

COROLLARY 3.3. *For $m \geq n \geq 1$, let $\mathcal{A}$ be an m by n nonnegative sign pattern. Then the following are equivalent:*

(a) *$\mathcal{A}$ requires an NLI.*

(b) *$\mathcal{A}$ is permutationally equivalent to a sign pattern $\mathcal{A} = \begin{bmatrix} \mathcal{I}_n \\ \mathcal{T} \end{bmatrix}$ for some* $(m-n)$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 17, pp. 209-218, April 2008

by n nonnegative sign pattern $\mathcal{T}$.

(c) *$\mathcal{A}$ allows an NLI.*

Proof. By [4, Theorem 3.4], (b) and (c) are equivalent. Since (a) implies (c), it suffices to show that (b) implies (a).

Without loss of generality, we may assume that $\mathcal{A} = \begin{bmatrix} \mathcal{I}_n \\ \mathcal{T} \end{bmatrix}$. Let $i \in \{1, \ldots, n\}$ and $\mathcal{D}$ be a signing with the ith diagonal entry negative. Then the ith row of $\mathcal{AD}$ is nonzero and nonpositive. By Theorem 3.1, the result follows. ◻

COROLLARY 3.4. *For $m \geq 2$, let $\mathcal{A}$ be an m by 2 sign pattern such that its first column is nonnegative and its second column has a $+$ and a $-$ entry. Then the following are equivalent:*

(a) *$\mathcal{A}$ requires an NLI.*
(b) *The first column of $\mathcal{A}$ has a $+$ entry and $[0 \;\; +]$ is a row of $\mathcal{A}$.*
(c) *$\mathcal{A}$ allows an NLI.*

Proof. By [4, Theorem 3.8], (b) and (c) are equivalent. Since (a) implies (c), it suffices to show that (b) implies (a).

Let $\mathcal{D} = \mathrm{diag}(\delta_1, \delta_2)$ be a signing. Assume that $\delta_2 = -$. Since $[0\ +]$ is a row of $\mathcal{A}$, $\mathcal{AD}$ has a nonzero and nonpositive row.

Next, assume that $\delta_1 = -$ and $\delta_2 \in \{+, 0\}$. If $[+\ -]$ is a row of $\mathcal{A}$, then $\mathcal{AD}$ has a nonzero and nonpositive row. Otherwise, by the conditions on the columns of $\mathcal{A}$, either both of $[+\ 0]$ and $[0\ -]$ are rows of $\mathcal{A}$, or both of $[+\ +]$ and $[0\ -]$ are rows of $\mathcal{A}$. Since, for each of those cases, $\mathcal{AD}$ has a nonzero and nonpositive row, the result follows by Theorem 3.1. ◻

Let A be an m by n matrix. If there exists an m by 1 vector $y > 0$ satisfying $y^T A = 0$, then y^T is a *positive left nullvector* of A. For $m > n$, an n by m sign pattern $\mathcal{M}$ (with more columns than rows) is an *L^+-matrix* if, for each $M \in Q(\mathcal{M})$, $y = 0$ whenever $y^T M \geq 0$ (see [5, page 5]). The following result gives equivalent conditions for a sign pattern (with more rows than columns) to be the transpose of an L^+-matrix (see [5, Theorem 2.4] in which the results are stated for L^+-matrices).

LEMMA 3.5. *For $m > n$, let $\mathcal{A}$ be an m by n sign pattern. Then the following are equivalent:*

(a) *The transpose of $\mathcal{A}$ is an L^+-matrix.*
(b) *$\mathcal{A}$ requires a positive left nullvector and $\mathcal{A}$ has no zero column.*
(c) *For each signing $\mathcal{D}$ of order n, $\mathcal{AD}$ has a nonzero and nonnegative row.*
(d) *For each signing $\mathcal{D}$ of order n, $\mathcal{AD}$ has a nonzero and nonpositive row.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 17, pp. 209-218, April 2008

(e) *For each n by 1 sign pattern $\widehat{b} \neq 0$, the constrained system $x^T\mathcal{A} = \widehat{b}^T$ with $x \geq 0$ and $x \neq 0$ is sign-consistent.*

By Lemma 3.5 (e) and Observation 1.4, it follows that if the transpose of $\mathcal{A}$ is an L$^+$-matrix, then $\mathcal{A}$ requires an NLI. Moreover, in Corollary 3.10, we prove that if the transpose of $\mathcal{A}$ is an L$^+$-matrix, then $\mathcal{A}$ requires a PLI. The following gives properties of the transpose of an L$^+$-matrix.

Proposition 3.6. *For $m > n$, let $\mathcal{A}$ be an m by n sign pattern. Suppose that the transpose of $\mathcal{A}$ is an L$^+$-matrix. Then*

(a) *$\mathcal{A}$ has both a nonzero, nonnegative row and a nonzero, nonpositive row; and*

(b) *any sign pattern of the form $\begin{bmatrix} \mathcal{A} \\ \mathcal{T} \end{bmatrix}$ obtained by augmenting rows to $\mathcal{A}$ is the transpose of an L$^+$-matrix.*

Proof. (a) The result follows by Lemma 3.5 (c) and (d) with $\mathcal{D} = \mathcal{I}_n$.

(b) Since the transpose of $\mathcal{A}$ is an L$^+$-matrix, by Lemma 3.5 (c), it follows that for each signing $\mathcal{D}$, $\mathcal{A}\mathcal{D}$ has a nonzero and nonnegative row. Thus, for each signing $\mathcal{D}$, $\begin{bmatrix} \mathcal{A} \\ \mathcal{T} \end{bmatrix}\mathcal{D}$ has a nonzero, nonnegative row, and the result follows by Lemma 3.5. ◻

The following result gives a necessary and sufficient condition for a sign pattern to require a PLI, where we consider signings with at least one diagonal entry in $\{-, 0\}$ instead of signings with at least one $-$ entry (which are for sign patterns requiring an NLI).

Theorem 3.7. *For $m \geq n$, let $\mathcal{A}$ be an m by n sign pattern with no zero column. Then the following are equivalent:*

(a) *$\mathcal{A}$ requires a PLI.*

(b) *For each $i = 1, \ldots, n$, the $(m+1)$ by n sign pattern $\begin{bmatrix} \mathcal{A} \\ -\widehat{e_i}^T \end{bmatrix}$ is the transpose of an L$^+$-matrix.*

(c) *For each signing $\mathcal{D}$ with at least one diagonal entry in $\{-, 0\}$, $\mathcal{A}\mathcal{D}$ has a nonzero and nonpositive row.*

Proof. By Observation 1.4, it follows that $\mathcal{A}$ requires a PLI if and only if, for each $i = 1, \ldots, n$, the constrained system $x^T\mathcal{A} = \widehat{e_i}^T$ with $x > 0$ is sign-consistent. Note that the constrained system $x^T\mathcal{A} = \widehat{e_i}^T$ with $x > 0$ is sign-consistent if and only if the constrained system $z^T\begin{bmatrix} \mathcal{A} \\ -\widehat{e_i}^T \end{bmatrix} = 0$ with an $(m+1)$ by 1 vector $z > 0$ is sign-consistent. Thus, $\mathcal{A}$ requires a PLI if and only if, for each $i = 1, \ldots, n$, the sign

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 17, pp. 209-218, April 2008

ELA

pattern $\begin{bmatrix} \mathcal{A} \\ -\widehat{e_i}^T \end{bmatrix}$ requires a positive nullvector. By Lemma 3.5, it follows that (a) and (b) are equivalent.

Next, we show that (b) and (c) are equivalent. Suppose that, for each $i \in \{1,\ldots,n\}$, $\mathcal{B}_i = \begin{bmatrix} \mathcal{A} \\ -\widehat{e_i}^T \end{bmatrix}$ is the transpose of an L$^+$-matrix. Let $\mathcal{D} = \text{diag}(\delta_1,\ldots,\delta_n)$ be a signing with $\delta_j \in \{-,0\}$ for some $j \in \{1,\ldots,n\}$. Since the transpose of $\mathcal{B}_j$ is an L$^+$-matrix, by Lemma 3.5, $\mathcal{B}_j\mathcal{D}$ has a nonzero and nonpositive row. Note that the last row of $\mathcal{B}_j\mathcal{D}$ is nonnegative. Hence, $\mathcal{AD}$ has a nonzero and nonpositive row.

Suppose that, for each signing $\mathcal{D}$ with at least one diagonal entry in $\{-,0\}$, $\mathcal{AD}$ has a nonzero and nonpositive row. If $\mathcal{D} = \text{diag}(\delta_1,\ldots,\delta_n)$ is a signing with $\delta_j \in \{-,0\}$ for some $j \in \{1,\ldots,n\}$, then, by the assumption, some row of $\mathcal{AD}$ (and hence $\mathcal{B}_i\mathcal{D}$ for each $i = 1,\ldots,n$) is nonzero and nonpositive. If $\mathcal{D}$ is a signing with all diagonal entries positive, then the last row of each $\mathcal{B}_i\mathcal{D}$ is nonzero and nonpositive. Thus, for each signing $\mathcal{D}$, $\mathcal{B}_i\mathcal{D}$ has a nonzero and nonpositive row for each $i \in \{1,\ldots,n\}$. The result follows by Lemma 3.5. ∎

Recall that, by Theorem 2.1 (a) and (b), no square sign pattern of order $n \geq 2$ requires a PLI and no fully indecomposable square sign pattern of order $n \geq 2$ requires an NLI. Hence, in the next (unexpected) result on square sign patterns, (a) and (b) follow directly from the negations of Theorem 3.7 (b) and (c), and (c) follows from Theorem 3.1.

Corollary 3.8. *For $n \geq 2$, let $\mathcal{A}$ be an n by n sign pattern. Then the following hold.*

(a) *There exists an index $i \in \{1,\ldots,n\}$ such that the $(n+1)$ by n sign pattern $\begin{bmatrix} \mathcal{A} \\ -\widehat{e_i}^T \end{bmatrix}$ is not the transpose of an L^+-matrix.*
(b) *There exists a signing $\mathcal{D}$ with at least one diagonal entry in $\{-,0\}$ such that every nonzero row of $\mathcal{AD}$ has a $+$ entry.*
(c) *If $\mathcal{A}$ is fully indecomposable, then there exists a signing $\mathcal{D}$ with at least one $-$ diagonal entry such that every nonzero row of $\mathcal{AD}$ has a $+$ entry.*

The following example gives a non-square sign pattern that requires an NLI, but not a PLI.

Example 3.9. Let

$$\mathcal{A} = \begin{bmatrix} + & + \\ - & + \\ + & 0 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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It can be verified that, for each of the signings $\mathcal{D}$ with at least one $-$ entry, namely $\begin{bmatrix} - & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & - \end{bmatrix}$, $\begin{bmatrix} - & 0 \\ 0 & + \end{bmatrix}$, $\begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}$, $\begin{bmatrix} - & 0 \\ 0 & - \end{bmatrix}$, the product $\mathcal{AD}$ has a nonzero and nonpositive row. Hence, by Theorem 3.1, $\mathcal{A}$ requires an NLI. However, for the signing $\mathcal{D} = \begin{bmatrix} 0 & 0 \\ 0 & + \end{bmatrix}$ with a zero diagonal entry, $\mathcal{AD}$ has no nonzero and nonpositive row. Thus, Theorem 3.7 implies that $\mathcal{A}$ does not require a PLI.

Corollary 3.10. *For $m > n$, let $\mathcal{A}$ be an m by n sign pattern with no zero column. If the transpose of $\mathcal{A}$ is an L^+-matrix, then $\mathcal{A}$ requires a PLI. Equivalently, if $\mathcal{A}$ requires a positive left nullvector, then $\mathcal{A}$ requires a PLI.*

Proof. Note that Lemma 3.5 implies that if $\mathcal{A}$ has no zero column, then the transpose of $\mathcal{A}$ is an L^+-matrix if and only if $\mathcal{A}$ requires a positive left nullvector. Since the transpose of $\mathcal{A}$ is an L^+-matrix, Proposition 3.6 (b) implies that, for each $i = 1, \ldots, n$, $\begin{bmatrix} \mathcal{A} \\ -\widehat{e_i}^T \end{bmatrix}$ is the transpose of an L^+-matrix. Hence, the result follows by Theorem 3.7. $\square$

The converse of Corollary 3.10 does not hold. Consider the 3 by 2 sign pattern

$$\mathcal{A} = \begin{bmatrix} - & + \\ + & - \\ 0 & + \end{bmatrix}.$$

Then, by Theorem 3.7 (c), it can be verified that $\mathcal{A}$ requires a PLI. However, since $\mathcal{A}$ does not have any nonzero and nonpositive row, by Proposition 3.6 (a), the transpose of $\mathcal{A}$ is not an L^+-matrix.

An n by $(n+1)$ sign pattern $\mathcal{M}$ is an *S-matrix* provided that the nullspace of each realization of $\mathcal{M}$ is spanned by an $(n+1)$ by 1 positive vector (see [2, page 12]). Since Lemma 3.5 implies that an n by $(n+1)$ S-matrix is an L^+-matrix, by Corollary 3.10, its transpose requires a PLI. Hence, transposes of S-matrices provide a family of sign patterns which require a PLI. Note that S-matrices can be recognized in polynomial-time (see [3]), while the problem of recognizing if a sign pattern is not an L^+-matrix is NP-complete (see [5]).

Remark 3.11. In contrast with [4, Remark 2.11 (i)], a superpattern of a sign pattern requiring a PLI does not necessarily require a PLI. Consider the 3 by 2 sign pattern

$$\mathcal{A} = \begin{bmatrix} + & 0 \\ - & + \\ 0 & - \end{bmatrix}$$

such that the transpose of $\mathcal{A}$ is an L^+-matrix (and also an S-matrix). Then, by

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ELA

Corollary 3.10, $\mathcal{A}$ requires a PLI (and hence, by [4, Remark 2.11 (i)], each superpattern of $\mathcal{A}$ allows a PLI). However, by Corollary 3.2, the following superpattern (without any nonzero and nonnegative row) of $\mathcal{A}$

$$\mathcal{B} = \begin{bmatrix} + & - \\ - & + \\ - & - \end{bmatrix}$$

does not require a PLI.

Recall that there are no square sign patterns of order n with $n \geq 2$ that require a PLI (see Theorem 2.1), but there are square sign patterns of order n with $n \geq 2$ that allow a PLI (see [2, Theorem 9.2.1]). We conclude this section by showing that for $m > n \geq 2$, the set of all m by n sign patterns requiring a PLI (resp. an NLI) is also a proper subset of the set of all m by n sign patterns allowing a PLI (resp. an NLI). For $m > n \geq 2$, let $\mathcal{A} = [\alpha_{ij}]$ be the m by n sign pattern with $\alpha_{ii} = +$ for each $i = 1, \ldots, n$ and $\alpha_{ij} = -$ for all $i \neq j$. Then, by [4, Theorem 2.10], $\mathcal{A}$ allows a PLI (and hence an NLI). However, Corollary 3.2 implies that $\mathcal{A}$ does not require either of an NLI or a PLI (since $\mathcal{A}$ has no nonzero and nonnegative row).

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